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Regular Colour-Preserving Coverings of Low Genus of the Face-Colored Hemicube

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Abstract. In this paper we classify the Δ^{012}-regular hypermaps of characteristic $\chi \geq -2$. We also define and compute the upper and lower irregularity group which are a generalization of the chirality group defined in [8]. From the obtained classification it follows that there is no edge-transitive map of characteristic $\chi \geq -2$ and type 5^{P} (defined in [11], see [13] for the torus case).

Key Words: Maps, hypermaps, embeddings of graphs and hypergraphs, restricted regularity of hypermaps, chirality

MSC 2000: 05C10, 05C25, 20B25, 20F65, 51E30, 57M60

1. Introduction

The *hemicube* is the regular map on the projective plane obtained by identifying the antipodal points of the cube. Labelling (or colouring) its three faces with 0, 1 and 2, we obtain the James representation of a hypermap with one hypervertex (the face labelled 0), one hyperedge (the face labelled 1), one hyperface (the face labelled 2) and four flags (the vertices of the underlying graph).

The free product $\Delta = C_2 * C_2 * C_2$ generated by R_0, R_1, R_2 acts on the set of vertices of the face-labelled hemicube (the flags): for every vertex v of the hemicube, vR_i is the vertex adjacent to v such that the edge $\{v, vR_i\}$ is not incident to the face labelled i. The subgroup Δ^{012} of Δ generated by $R_1R_0R_2$, $R_2R_1R_0$ and $R_0R_2R_1$, is a normal subgroup of index 4 which acts trivially on the vertices of the hemicube. Moreover, the quotient Δ/Δ^{012} describes the face-labelled hemicube as a regular Δ^{012}-marked hypermap [5]. For this reason we denote the hemicube by $\mathcal{T}_{\Delta^{012}}$ and call it the trivial Δ^{012}-regular hypermap. Every label-preserving covering of $\mathcal{T}_{\Delta^{012}}$ corresponds to a subgroup of Δ^{012}, and a regular covering to

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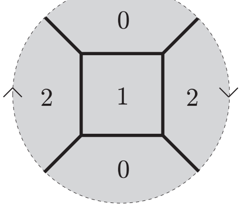


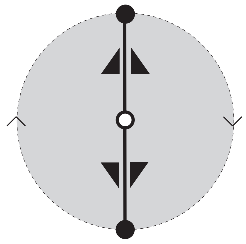


Figure 1: The James representation (left) and the Walsh representation (right) of the hemicube

a normal subgroup. Regular coverings of the face-labelled hemicube have been not totally absent in the literature, for example in [13] there is an indirect classification of the "just" regular coverings of genus one (see [5] for more details), where "just" means that the covering to any other regular hypermap properly covered by $\mathcal{T}_{\Delta^{012}}$ is not regular.

In this paper we classify the regular label-preserving coverings of the face-labelled hemicube of characteristic $\chi \geq -2$. In other words, we classify the Δ^{012}-regular hypermaps, or the hypermaps of characteristic $\chi \geq -2$ that regularly cover $\mathcal{T}_{\Delta^{012}}$. By an exhaustive use of the techniques presented in this paper, the classification can be extended further down, to lower characteristic bounds. However, these values already give a representative picture, not only of the techniques, but also of the kind of classification we expect to find. As a natural extension of the chirality group defined in [8], we introduce a group that measures the quality of the irregularity of restricted-regular hypermaps [5].

This paper is organised as follows: the rest of Chapter 1 introduces restrictedly-marked hypermaps; Chapter 2 defines upper and lower irregularity groups and their respective irregularity indices; Chapter 3 gives the classification of the Δ^{012}-regular hypermaps of characteristic $\chi \geq -2$; and Chapter 4 gives the calculations of the upper and lower irregularity groups.

Restrictedly-marked hypermaps

Hypermaps are trivalent face-labelled maps where each vertex is incident to three faces carrying different labels, and coverings are label-preserving map coverings. Algebraically, a hypermap is a four-tuple $\mathcal{H} = (F; r_0, r_1, r_2)$ consisting of a non-empty finite set F, called the set of *flags*, and three permutations r_0, r_1, r_2 of F satisfying ${r_0}^2 = {r_1}^2 = {r_2}^2 = 1$ and generating a permutation group $Mon(\mathcal{H}) = \langle r_0, r_1, r_2 \rangle$, called the *monodromy group* of $\mathcal{H}$, acting transitively on F. The transitivity of the action implies that $|Mon(\mathcal{H})| \geq |F|$, and equality occurs if and only if the automorphism group of $\mathcal{H}$, $Aut(\mathcal{H})$, acts transitively on F, that is, if $\mathcal{H}$ is *regular*. Hypermaps correspond to finite transitive permutation representations $\rho : \Delta \to Mon(\mathcal{H})$, $R_i \mapsto r_i$, where $\Delta = C_2 * C_2 * C_2$ is the free product generated by the reflections on the sides of a triangle with zero internal angles on the hyperbolic plane. Any normal subgroup Θ of Δ of finite index acts on the set of flags with orbits of the same size.

A *fundamental subgroup* or *hypermap subgroup* for $\mathcal{H}$ is the stabilizer of a flag of $\mathcal{H}$ by the action of Δ. Fundamental subgroups are unique up to conjugation in Δ. Every hypermap is completely described by a fundamental subgroup. The hypermap $\mathcal{H}$ is Θ-*conservative* if its fundamental subgroup H is a subgroup of Θ and it is Θ-*regular* if the automorphism group of $\mathcal{H}$ acts transitively on each Θ-orbit, or equivalently, if H is a normal subgroup of Θ. We say that $\mathcal{H}$ is *just*-Θ-*regular* if it is Θ-regular but not Π-regular for any normal subgroup Π in Δ that properly contains Θ. By the Kurosh Subgroup Theorem [10], Θ freely decomposes uniquely (up to a permutation of the factors) in a free product $C_2 * \cdots * C_2 * C_\infty * \cdots * C_\infty = \langle X_1, \ldots, X_m \mid {X_i}^2 = 1,\, i = 1 \ldots s \rangle$, for some $0 \le s \le m$.

A Θ-conservative hypermap is described by a $(m+1)$-tuple $\mathcal{Q} = (\Omega, x_1, \ldots, x_m)$ where Ω is a finite set, and $x_1, \ldots, x_m$ are permutations acting transitively on Ω. In particular, $\mathcal{Q}$ is a marked transitive permutation group [12], and the assignment $X_i \mapsto x_i$, $i = 1, \ldots, m$, induces an epimorphism. Such an $(m+1)$-tuple is called a Θ-*marked hypermap*; homomorphisms and automorphisms of Θ-marked hypermaps are just homomorphisms and automorphisms of marked transitive permutation groups. For more details, such as the characteristic formula, the Θ-type and so on, we refer the reader to [5].

A Θ-conservative hypermap $\mathcal{H}$ can be represented in two forms: the usual Δ-form $\mathcal{H} = (F; r_0, r_1, r_2)$ and the Δ^{012}-form $\mathcal{Q} = (F'; x, y, z)$, where $|F'| = |F|/4$. In the first form we use the terms *regular* and Δ^{012}-*regular* with the meaning as explained above, while in the second we use the local terminology Δ-*regular* and *regular*, respectively.

2. Irregularity

Let $\mathcal{H}$ be a hypermap with fundamental subgroup $H \le \Delta$. Then $\mathcal{H}$ is Δ-regular if and only if H is normal in Δ. The *closure cover* $\mathcal{H}^{\vartriangle}$ of $\mathcal{H}$ is the hypermap whose fundamental subgroup is the normal closure $H^{\vartriangle}$ of H in Δ; it is the largest Δ-regular hypermap covered by $\mathcal{H}$. The *covering core* $\mathcal{H}_{\vartriangle}$ of $\mathcal{H}$ is the hypermap whose fundamental subgroup is the core $H_{\vartriangle}$ of H in Δ; it is the smallest Δ-regular hypermap covering $\mathcal{H}$. Note that $H_{\vartriangle}$ is always a normal subgroup of H but H may be not normal in $H^{\vartriangle}$. In fact H is normal in $H^{\vartriangle}$ if and only if H is normal in Θ, for some normal subgroup Θ of Δ. We have a group

$$\Upsilon_{\vartriangle}(\mathcal{H}) = H/H_{\vartriangle}$$

called the "chirality group" in [4, 8, 2, 3], when $\mathcal{H}$ is Δ^+-regular (that is, *orientably-regular*). In this paper we call it the *lower irregularity group* of $\mathcal{H}$, and its size is called the *lower irregularity index* and denoted by $\iota_{\vartriangle}(\mathcal{H})$. The *upper irregularity index* is the index $|H^{\vartriangle} : H|$ and is denoted by $\iota^{\vartriangle}(\mathcal{H})$. When $\mathcal{H}$ is Θ-regular, for some normal subgroup Θ of Δ, then $H \triangleleft H^{\Delta}$ and we have another group, the *upper irregularity group*

$$\Upsilon^{\vartriangle}(\mathcal{H}) = H^{\vartriangle}/H\,.$$

Upper and lower irregularity indices may be not equal. We say that $\mathcal{H}$ is *balanced* if its upper and lower irregularity indices are equal, and *unbalanced* otherwise. Balanced hypermaps with $\Upsilon^{\vartriangle} \cong \Upsilon_{\vartriangle}$ are called *fully-balanced*. When $\mathcal{H}$ is balanced, we say *irregularity index* instead of upper or lower irregularity index, for short.

In [8] the authors showed that all Δ^+-regular hypermaps are fully-balanced. This result extends similarly to Θ-regular hypermaps when Θ has index 2 in Δ (see following lemma). However, if Θ has not index 2 in Δ, then $\mathcal{H}$ can be unbalanced (see Chapter 3 for examples).

Lemma 1 *Let Θ be a normal subgroup of index 2 in Δ. If $\mathcal{H}$ is Θ-regular, then $\mathcal{H}$ is fully-balanced.*

Computing the irregularity groups

In general, it may not be easy to compute the upper irregularity index. However, if $\mathcal{H}$ is Θ-regular, for some normal subgroup Θ of Δ of finite index, then the upper irregularity group

H^{Δ}/H is a normal subgroup of its Θ-monodromy group Θ/H, and hence, it can be obtained from a presentation of Θ/H (Theorem 3).

Lemma 2 *Let Θ be a normal subgroup of index n in Δ and $T = \{b_1, b_2, \ldots, b_n\}$ be a transversal for Θ in Δ. If H is normal in Θ, then*

(1) *H is normal in Δ if and only if $H^{b_i} = H$, for $i = 1, \ldots, n$.*

(2) *$H^{\Delta} = \langle H^{b_1}, H^{b_2}, \ldots, H^{b_n}\rangle = H^{b_1}H^{b_2}\cdots H^{b_n}$.*

(3) *$H_{\Delta} = \bigcap_{i=1}^{n} H^{b_i}$.*

The hypermap $\mathcal{H}$, being Θ-regular, is described by a regular Θ-marked hypermap

$$\mathcal{Q} = (G; x_1, \ldots, x_m) \cong (\Theta/H; HX_1, \ldots, HX_m),$$

where $\Theta = \langle X_1, \ldots, X_m\rangle$ is a free product of the form $C_2 * \cdots * C_2 * C_\infty * \cdots * C_\infty$, with a finite number of factors, H is a fundamental subgroup for $\mathcal{Q}$, and G is the group generated by $x_1, \ldots, x_m$. The following theorem extends Theorem 1 of [4].

Theorem 3 *If G has presentation $\langle X_1, \ldots, X_m \mid R\rangle$ and $T = \{b_1, b_2, \ldots, b_n\}$ is a transversal for Θ in Δ, then $\Upsilon^{\Delta}(\mathcal{H}) = \langle R^{b_1}, \ldots, R^{b_n}\rangle^{G}$.*

Proof: It is a consequence of Lemma 2 and von Dyck's theorem. □

By Theorem 20 of [5] each b-image of $\mathcal{H}^{\Theta} \cong \mathcal{Q}$ is isomorphic to the regular Θ-marked hypermap

$$\mathcal{Q}^b = (\Theta/H; H{X_1}^{b^{-1}}, \ldots, H{X_m}^{b^{-1}}) = (G; {x_1}^{b^{-1}}, \ldots, {x_m}^{b^{-1}}),$$

where ${x_i}^{b^{-1}}$ is the image of ${X_i}^{b^{-1}}$ by the epimorphism ρ extending the assignment $X_i \mapsto x_i$, $i = 1, ..., m$. Hence, the set of flags of $\mathcal{H}$ can be regarded as n distinct copies of G, that is, we can set $F = G \times T$, where $F_b^{\Theta} = G \times \{b\}$. Identifying the n-tuple $({x_i}^{b_1^{-1}}, \ldots, {x_i}^{b_n^{-1}}) \in G^n$ with the permutation

$$\sigma_{x_i}\colon\ G \times T \to G \times T, \quad (g, b) \mapsto (g{x_i}^{b^{-1}}, b),$$

we see that Θ/H_Δ acts on F as $\langle \sigma_{x_1}, \ldots, \sigma_{x_n}\rangle$ acts on $G \times T$, that is, the function $H_\Delta X_i \mapsto \sigma_{x_i}$, $i = 1, \ldots, m$, extends to an isomorphism from Θ/H_Δ to $\Sigma = \langle \sigma_{x_1}, \ldots, \sigma_{x_m}\rangle \le G^n$. The group $\Sigma \le G^n$ is called the *monodromy product* [1], or the *parallel product* [14] of the groups $\langle x_1^{b^{-1}}, \ldots, x_m^{b^{-1}}\rangle = G$ with $b \in T$.

Theorem 4 *If G has presentation $\langle X_1, \ldots, X_m \mid R\rangle$ and $T = \{b_1, b_2, \ldots, b_n\}$ is a transversal for Θ in Δ, then $\Upsilon_{\Delta}(\mathcal{H}) = \langle R_\sigma\rangle^{\Sigma}$, where $R_\sigma = R(\sigma_{x_1}, \ldots, \sigma_{x_m}) \subseteq \Sigma$.*

Proof: The canonical epimorphism $\Theta/H_\Delta \to \Theta/H$ has kernel $\Upsilon_{\Delta}(\mathcal{H}) = H/H_\Delta$. Hence, from $G \cong \Theta/H$ we have that $\Upsilon_{\Delta}(\mathcal{H}) = \langle R\rangle^{\Theta/H_\Delta}$. The statement now follows from the fact that the function $H_\Delta X_i \mapsto \sigma_{x_i}$, $i = 1, \ldots, m$, extends to an isomorphism from Θ/H_Δ to Σ. □

We now extend a result established in [8] for the even word subgroup Δ^+ to all normal subgroups of Δ of index 2.

Theorem 5 *Let Θ be a normal subgroup of Δ of index 2. The irregularity index of a Θ-regular hypermap can be any positive integer.*

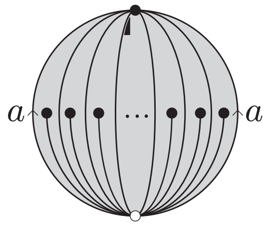


Figure 2: $\Delta^{\hat{0}}$-regular hypermap with irregularity index n

Proof: The triangle group Δ has 7 normal subgroups of index 2, namely Δ^+, $\Delta^{\hat{0}}$, $\Delta^{\hat{1}}$, $\Delta^{\hat{2}}$, Δ^0, Δ^1 and Δ^2 (for more details see [7]). In [8] it was proved that the irregularity index (called chirality index) of a Δ^+-regular hypermap can be any positive integer. It is shown in [6] that the regular $\Delta^{\hat{0}}$-marked hypermap $(D_n; a, b, c, d)$ on the sphere (Fig. 2), where $D_n = \langle a, b\rangle$ is a dihedral group of order n and $c = d = a$, has upper irregularity group C_n and irregularity index n.
By swapping cell labels (that is by "duality"), the same is true for $\Delta^{\hat{\imath}}$-regular hypermaps, where $i = 1, 2$. Hence we only need to show the result for Δ^i-regular hypermaps. Up to a cell labelling, let us prove it for $\Theta = \Delta^0$. Our goal is to find a Δ^0-regular hypermap $\mathcal{H}$ with upper irregularity group $\Upsilon^{\Delta}(\mathcal{H})$ isomorphic to a cyclic group of order n, for each n. The subgroup Δ^0 of Δ is generated by $A = R_0$, $B = {R_0}^{R_1}$ and $C = R_1R_2$. By the Reidemeister-Schreirer Rewriting process, Δ^0 is isomorphic to the free product $C_2 * C_2 * C_\infty = \langle A, B, C\rangle$.

Let $n = 2m$ be an even positive integer and consider $G = C_2 \times D_n$, where $C_2 = \langle \alpha \mid \alpha^2 = 1\rangle$ is a cyclic group of order 2 and D_n is a dihedral group of order $2n$ with presentation $\langle \beta, \gamma \mid \beta^2 = \gamma^n = (\beta\gamma)^2 = 1\rangle$. Consider the Δ^0-marked hypermap $\mathcal{Q} = (G; a, b, c)$, where $G = C_2 \times D_n$ and $a = (\alpha, 1)$, $b = (1, \beta)$ and $c = (1, \gamma)$. By Theorem 22 of [5], $\mathcal{Q}$ is regular. The group G has presentation

$$\langle A, B, C \mid A^2, B^2, C^n, (BC)^2, [A, B], [A, C]\rangle\,,$$

so $R(A, B, C) = \{A^2, B^2, C^n, (BC)^2, [A, B], [A, C]\}$. As $T = \{1, R_1\}$ is a transversal for Δ^0 in Δ and $A^{R_1} = B$, $B^{R_1} = A$ and $C^{R_1} = C^{-1}$, by Theorem 3, we have

$$\begin{aligned}\Upsilon^{\Delta} &= \langle R(A^{R_1}, B^{R_1}, C^{R_1})\rangle^G = \langle R(B, A, C^{-1})\rangle^G \\ &= \langle B^2, A^2, C^{-n}, (AC^{-1})^2, [B, A], [B, C^{-1}]\rangle^G = \langle C^2\rangle = C_m\,.\end{aligned}$$

Since $\mathcal{H}$ is fully-balanced (Lemma 1), its irregularity index $\iota = \iota_{\Delta} = \iota^{\Delta}$ is m. $\square$

In the proof of Theorem 5 we give an example of a family of orientable Δ^0-regular hypermaps of Δ^0-type $(n, 4, 4)$ on a surface of genus $g = n - 2$ ($n = 2m$) with irregularity index $\iota = \frac{n}{2} = \frac{g+2}{2}$. Another such family of examples, all lying in the torus, can be found in line 7 of Table 3, setting $m = 1$. One gets a family $(T^1_{(1,n)}; x, y, y^{-1})$ of Δ^0-regular hypermaps of Δ^0-type $(2, 4, 4)$ with irregularity group $\Upsilon^{\Delta} = \Upsilon_{\Delta} = C_n$.

3. Δ^{012}-regular hypermaps of characteristic $\chi \geq -2$

The classification is done on regular Δ^{012}-marked hypermaps, up to duality (relabelling of 0,1,2 cells) and a b-image (choice of a flag in $\mathcal{T}_{\Delta^{012}}$ as root-flag). The advantage of using the

Δ^{012}-marked form over the Δ-marked form (usual algebraic form of a hypermap) is that the monodromy group of the former is at least 4 times smaller than that of the later.

Because Δ^{012} is generated by $X = R_1R_0R_2$, $Y = R_2R_1R_0$ and $Z = R_0R_2R_1$, the trivial Δ^{012}-hypermap is the regular hypermap $\mathcal{T}_{\Delta^{012}} = (V_4; a, b, ab)$, where $V_4 = \langle a, b \mid a^2, b^2, (ab)^2 \rangle \cong C_2 \times C_2$ is the vierergruppe, i.e., $\mathcal{T}_{\Delta^{012}}$ is the Walsh representation of the face-labelled hemicube (see Fig. 1, page 2). As seen earlier, $\mathcal{T}_{\Delta^{012}}$ is a hypermap on the projective plane, of type $(2, 2, 2)$, with 1 hypervertex, 1 hyperedge and 1 hyperface. According to the definition of Θ-type given in [5], the Δ^{012}-type of a Δ^{012}-regular hypermap is (k, ℓ, m), with k, ℓ and m even.

Let $\mathcal{H} = (F; r_0, r_1, r_2)$ be a Δ^{012}-regular hypermap with fundamental subgroup H. Then $\mathcal{Q} = (G; x, y, z)$, where $G = \Delta^{012}/H$, $x = r_1r_0r_2$, $y = r_2r_1r_0$ and $z = r_0r_2r_1$, is a Δ^{012}-marked hypermap with Δ-form $\mathcal{H}$. The hypermap $\mathcal{T}_{\Delta^{012}}$ may be thought as a Δ^{012}-slice of $\mathcal{H}$ (see [5]). Each b-image $\mathcal{Q}^b$ has also Δ-form $\mathcal{H}$. The characteristic of $\mathcal{Q}$ is the characteristic of $\mathcal{H}$ which is given by the formula

$$\chi(\mathcal{Q}) = |G| \left(\frac{2}{k} + \frac{2}{\ell} + \frac{2}{m} - 2 \right),$$

where (k, ℓ, m) is the Δ^{012}-type of $\mathcal{H}$. Since $(k, \ell, m) = (|r_2r_1|, |r_0r_2|, |r_1r_0|)$, $yz = (r_2r_1)^2$, $zx = (r_0r_2)^2$ and $xy = (r_1r_0)^2$, we have that $|yz| = k/2$, $|zx| = \ell/2$, $|xy| = m/2$ and hence

$$\chi(\mathcal{Q}) = |G| \left(\frac{1}{|yz|} + \frac{1}{|zx|} + \frac{1}{|xy|} - 2 \right).$$

Positive characteristic

Up to duality, the Δ^{012}-type of a Δ^{012}-regular hypermap of positive characteristic is $(2, 2, 2n)$, for some $n \in \mathbb{N}$. Thus $z^{-1} = y = x$ and G is a cyclic group generated by x. From $|xy| = |x^2| = n$, we get $|G| = 2n$ and $\chi = 2$, if n is even, and $|G| = n$ and $\chi = 1$, if n is odd. Hence the Δ^{012}-regular hypermaps on the sphere are given by the Δ^{012}-marked hypermaps $(C_{4n}; x, x, x^{-1})$ and the Δ^{012}-regular hypermaps on the projective plane are given by the Δ^{012}-marked hypermaps $(C_{2n+1}; x, x, x^{-1})$, where C_m is the cyclic group of order m generated by x.

Zero characteristic

Up to duality, the Δ^{012}-type of a Δ^{012}-regular hypermap of zero characteristic is $(2, 4, 4)$. Thus $z = y^{-1}$, $(y^{-1}x)^2 = 1$ and $(xy)^2 = 1$. From these equalities we get (using induction twice) the following commuting rule:

$$y^\beta x^\alpha = x^{(-1)^\beta \alpha} y^{(-1)^\alpha \beta}, \qquad \alpha, \beta \in \mathbb{N}. \tag{1}$$

This shows that $Q = \langle x, y \mid x^p, y^q, (y^{-1}x)^2, (xy)^2 \rangle$ is finite. If one of p or q is odd, then $Q = C_2$ or Q is dihedral (see case below). If both p and q are even, because $H = \langle x^2, y^2 \rangle$ is an abelian normal subgroup of size $\frac{pq}{4}$ and factor Q into V_4, then Q has size pq. Since G is a factor of Q, G has presentation $\langle x, y \mid x^p, y^q, (y^{-1}x)^2, (xy)^2, R \rangle$ for some (possibly empty) set R of relators in x, y. Without loss of generality, we may assume that p and q are the orders of x and y. Note that $y^{-1}x$ and xy are involutions of G.

Case p or q odd: If p is odd, then $y^2 = 1$ and so $x^y = x^{-1}$. Thus $\langle x \rangle \lhd G$. If $y = 1$, then $Q = C_2 = \langle x \rangle$ for otherwise $y \notin \langle x \rangle$, because x has odd order, and $G = \langle x \rangle \rtimes \langle y \rangle = D_m = \langle x, y \mid x^m, y^2, (xy)^2 \rangle$. Similarly, if n is odd, then $Q = C_2 = \langle y \rangle$ or $G = D_n = \langle x, y \mid x^2, y^n, (xy)^2 \rangle$.

Case p and q even: Let $p = 2m$ and $q = 2n$. By (1) we may assume that every relator in R is of the form $x^a y^b$ where $0 < a < p$ and $0 < b < q$, if $R \neq \varnothing$. If a is odd, then $x^a y^b = 1$ implies $y^b = (x^{-a})^y = y^{-b+2}$ and $y^b = (x^{-a})^{y^2} = y^{b+4}$. Thus $(b, q) = (1, 2)$ or $(3, 4)$ giving $y^b = y^{\pm 1}$. Then $x^a = (y^{\mp 1})^x = x^{-a-2}$ and $x^a = (y^{\mp 1})^{x^2} = x^{a-4}$. Therefore we also have $(a, p) = (1, 2)$ or $(3, 4)$ giving $x^a = x^{\pm 1}$. We conclude that $x^a y^b = 1$ implies $xy = 1$ or $y^{-1}x = 1$, contradicting the assumption that both $y^{-1}x$ and xy have order 2. Similarly, assuming that b is odd we get the same contradiction. Hence a and b must be even. From $1 = (x^a y^b)^y = x^{-a} y^b$ we get $x^{2a} = y^{2b} = 1$. Hence $a = m$ and $b = n$, that is, $R = \{x^m y^n\}$, if $R \neq \varnothing$.

We conclude that the Δ^{012}-regular hypermaps with zero characteristic are given by the Δ^{012}-marked hypermaps $(C_2; a, 1, 1)$, $(C_2; 1, a, a)$, $(D_{2n+1}; a, b, b)$, $(D_{2n+1}; b, a, a^{-1})$, $(T^1_{(m,n)}; x, y, y^{-1})$ and $(T^2_{(m,n)}; x, y, y^{-1})$, where $C_2 = \langle a \rangle$, $D_{2n+1} = \langle a, b \mid a^{2n+1}, b^2, (ab)^2 \rangle$ is the dihedral group of order $2(2n + 1)$, $T^1_{(m,n)} = \langle x, y \mid x^{2m}, y^{2n}, (y^{-1}x)^2, (xy)^2 \rangle$ is a group of size $4mn$ and $T^2_{(m,n)} = \langle x, y \mid x^{4m}, y^{4n}, (y^{-1}x)^2, (xy)^2, x^{2m}y^{2n} \rangle$ is a group of size $8mn$. The last two families are hypermaps on the torus while the other four are on the Klein bottle.

Negative characteristic

As

$$|G| = \frac{-\chi(\mathcal{Q})}{2 - \left(\frac{1}{|yz|} + \frac{1}{|zx|} + \frac{1}{|xy|} \right)},$$

if $\chi(\mathcal{Q}) < 0$ then $|G| \leq \dfrac{-\chi(\mathcal{Q})}{2 - M}$, where M is the maximum value in $\left\{ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \mid a, b, c \in \mathbb{N} \right\}$ $\cap \,]0, 2[$. One can easily see that $M = \frac{11}{6}$ and therefore $|G| \leq -6\,\chi(\mathcal{Q})$, or equivalently, if we denote the genus of $\mathcal{Q}$ by g,

$$|G| \leq \begin{cases} 12(g-1) & \text{if } \mathcal{Q} \text{ is orientable and } \chi(\mathcal{Q}) < 0, \\ 6(g-2) & \text{if } \mathcal{Q} \text{ is non-orientable and } \chi(\mathcal{Q}) < 0. \end{cases}$$

- **Characteristic** -1

From $\chi(\mathcal{Q}) = -1$ we have that

$$\frac{1}{2 - \left(\frac{1}{|yz|} + \frac{1}{|zx|} + \frac{1}{|xy|} \right)} = |G| \leq 6,$$

where $|yz|$, $|zx|$ and $|xy|$ divide $|G|$. The following Table 1 lists the possible solutions $k = 2|yz|$, $\ell = 2|zx|$, $m = 2|xy|$ and $|G|$ of this equality with $k \leq \ell \leq m$. It also displays the numbers

Table 1: Δ^{012}-type of Δ^{012}-regular hypermaps of characteristic -1

	k	ℓ	m	V	E	F	$\lvert G\rvert$	G	#
1	4	4	4	1	1	1	2	C_2	0
2	2	6	6	3	1	1	3	C_3	2
3	2	4	6	6	3	2	6	C_6 or D_3	0

$V = 2|G|/k$, $E = 2|G|/\ell$, $F = 2|G|/m$, of hypervertices, hyperedges and hyperfaces of the regular Δ^{012}-marked hypermap $\mathcal{Q}$ of Δ^{012}-type (k, ℓ, m), and the possible groups G. The last column shows the number of non-isomorphic Δ^{012}-marked hypermaps for each solution.

line 1: $G = C_2$ is clearly not possible.

line 2: $G = \langle zx \rangle = \langle xy \rangle = C_3$. As $yz = 1$, we have $y^{-1}x = zx = (xy)^\epsilon$, where $\epsilon = \pm 1$. If $\epsilon = 1$, then $y^2 = 1$, so $y = z = 1$, since G is abelian, and $\langle x \rangle = C_3$. Else, if $\epsilon = -1$, then $x = 1$ and $\langle y \rangle = C_3$. This gives two regular Δ^{012}-marked hypermaps $(C_3; x, 1, 1)$ and $(C_3; 1, y, y^{-1})$, both non-orientable of genus 3.

line 3: $G = \langle x, y \rangle$ since $yz = 1$, and $G = \langle y^{-1}x, xy \rangle$ since $|y^{-1}x| = |zx| = 2$ and $|xy| = 3$. Yet $|G|$ cannot be 6, because $x^2 = (xy)(y^{-1}x)$ cannot have order 2 nor 6 in such a group.

- **Characteristic** -2

$\chi(\mathcal{Q}) = -2$ gives

$$\frac{2}{2 - \left(\frac{1}{|yz|} + \frac{1}{|zx|} + \frac{1}{|xy|} \right)} = |G| \leq 12$$

and, as above, the possibilities are listed in the following table.

Table 2: Δ^{012}-type of Δ^{012}-regular hypermaps of characteristic -2

	k	ℓ	m	V	E	F	$\|G\|$	G	#
1	2	8	8	4	1	1	4	C_4	4
2	4	4	4	2	2	2	4	C_4 or V_4	$1+2$
3	2	4	12	6	3	1	6	C_6	2
4	2	6	6	6	2	2	6	C_6 or D_3	$2+1$
5	2	4	8	8	4	2	8	C_8, $C_4 \times C_2$, D_4, Q_4	$1+0+0+0$
6	2	4	6	12	6	4	12	$C_3 \rtimes C_4$, C_{12}, $C_6 \times C_2$, D_6, A_4	$0+1+0+0+1$

Since $k = 2$ is equivalent to $z = y^{-1}$, for all items not on **line 2** we have $G = \langle x, y \rangle = \langle x, z \rangle$. A detailed description of the entries in the last two columns of Table 2 now follows.

line 1: Clearly $a = xy$ generates G. Thus $(x, y) = (a^i, a^j)$, for some $i, j \in \{0, 1, 2, 3\}$. In addition, $a = a^{i+j}$, so $i + j \equiv 1 \mod 4$. Since $4 = |zx| = |y^{-1}x|$, we get $i - j \equiv \pm 1 \mod 4$, and hence $(i, j) = (0, 1), (1, 0), (2, 3)$ or $(3, 2)$. Consequently, the regular Δ^{012}-marked hypermaps of Δ^{012}-type $(2, 8, 8)$ are $(C_4; 1, a, a^{-1})$, $(C_4; a, 1, 1)$, $(C_4; a^2, a^3, a)$ and $(C_4; a^3, a^2, a^2) \cong (C_4; a, a^2, a^2)$.

line 2: If $G = C_4 = \langle a \rangle$, then $(x, y, z) = (a^i, a^j, a^h)$, for some $i, j, h \in \{0, 1, 2, 3\}$ satisfying $j + h \equiv h + i \equiv i + j \equiv 2 \mod 4$. It follows that $i = j = h = \pm 1$. This gives the two isomorphic Δ^{012}-marked hypermaps $(C_4; a, a, a)$ and $(C_4; a^{-1}, a^{-1}, a^{-1})$. If $G = V_4$, one of the following conditions holds: (1) $1 \in \{x, y, z\}$, (2) $V_4 = \{1, x, y, z\}$, or (3) $|\{x, y, z\}| = 2$. All are possible except item (3) that does not have the right Δ^{012}-type. So this case produces the two Δ^{012}-marked hypermaps $(V_4; x, y, 1)$ and $(V_4; x, y, xy)$.

line 3: $G = \langle xy \rangle = C_6 = \langle a \rangle$. Then $(x, y) = (a^i, a^j)$, for some $i, j \in \{0, 1, 2, 3, 4, 5\}$ such that $i + j \equiv 1 \mod 6$ and $i - j \equiv 3 \mod 6$. This system of congruencies has two solutions mod 6:

$(i,j)=(2,5)$ and $(i,j)=(5,2)$, giving the two regular Δ^{012}-marked hypermaps $(C_6;a^2,a^5,a)$ and $(C_6;a^5,a^2,a^4)\cong(C_6;a,a^4,a^2)$.

line 4: (i) $G=C_6=\langle a\rangle$. By a similar argument as used before, this case produces the two regular Δ^{012}-marked hypermaps $(C_6;a^5,a^3,a^3)\cong(C_6;a,a^3,a^3)$ and $(C_6;a^3,a,a^5)\cong(C_6;a^3,a^5,a)$.
(ii) $G=D_3=\langle a,b\mid a^3,b^2,(ab)^2\rangle$. According to the Δ^{012}-type, we have $y^{-1}x=zx, xy\in\{a,a^2\}$. Since there is an automorphism of D_3 mapping a to a^2 and b to b, we just need to consider two cases: (1) $(y^{-1}x,xy)=(a,a)$. This contradicts $\langle x,y\rangle=D_3$. (2) $(y^{-1}x,xy)=(a,a^2)$. This gives one regular Δ^{012}-marked hypermap $(D_3;b,a^2b,a^2b)\cong(D_3;ab,b,b)\cong(D_3;a^2b,ab,ab)$.

line 5: Since xy has order 4, G can only be C_8, $C_4\times C_2$, D_4 or Q_4.
(i) $G=C_8=\langle a\rangle$. Similarly to the cyclic cases, this gives one Δ^{012}-marked hypermap $(C_8;a^3,a^7,a)\cong(C_8;a^7,a^3,a^5)$.
(ii) $G=C_4\times C_2=\langle a,b\mid a^4,b^2,[a,b]\rangle$. Then $\{x,y\}=\{a,b\}$ or $\{a^{-1},b\}$, but both cases give a Δ^{012}-type contradiction.
(iii) $G=D_4=\langle a,b\mid a^4,b^2,(ab)^2\rangle$. One of x or y must be an involution and the other must belong to the cyclic subgroup $\langle a\rangle$. This prevents xy having order 4.
(iv) $G=Q_8=\langle a,b\mid a^2=b^2,a^b=a^{-1}\rangle$. Since a^2 is the unique element of order 2, $y^{-1}x=zx=a^2$. Then $Q_8=\langle x,y\rangle=\langle y^{-1}x,x\rangle=\langle a^2,x\rangle$ which is not possible because a^2 is a non-generating element. Hence $G\neq Q_8$.

line 6: Since G is generated by a pair of elements one of which has order 2, G cannot be $C_3\rtimes C_4=\langle a,b\mid a^3,b^4,a^ba\rangle$, because b^2 is the only element of order 2 and it belongs to the Frattini subgroup. Then G is C_{12}, $C_6\times C_2$, D_6 or A_4.
(i) $G=C_{12}=\langle a\rangle$. In this case we get a regular Δ^{012}-marked hypermap: $(C_{12};a^{11},a^5,a^7)\cong(C_{12};a^5,a^{11},a)\cong(C_{12};a,a^7,a^5)\cong(C_{12};a^7,a,a^{11})$.
(ii) $G=C_6\times C_2=\langle a,b\mid a^6,b^2,[a,b]\rangle$. Here we must have $x=a^ib$ and $y=a^jb$, for some $i,j\in\{0,\dots,5\}$ satisfying $i-j\equiv 3 \mod 6$ and $i+j\equiv\pm 2 \mod 6$, which is impossible.
(iii) $G=D_6=\langle a,b\mid a^6,b^2,(ab)^2\rangle$. The argument of (ii) applies to this case as well.
(iv) $G=A_4$. The elements of order 2 generate V_4 and since $y^{-1}x$ has order 2 this implies that both x and y must have order 3. Since $x^3=y^3=(y^{-1}x)^2=1$ determines A_4, there is only one regular Δ^{012}-marked hypermap (up to isomorphism), namely $(A_4;x,y,y^{-1})$.

4. The upper and lower irregularity group

To compute the irregularity groups of the Δ^{012}-regular hypermaps obtained in previous chapter we use Theorems 3 and 4. The set $T=\{1,R_0,R_2,R_2R_0\}$ is a transversal of $\Delta^{012}=\langle X,Y,Z\rangle$, where $X=R_1R_0R_2$, $Y=R_2R_1R_0$ and $Z=R_0R_2R_1$. In order to use Theorems 3 and 4, we need to determine the conjugates X^b, Y^b, Z^b and $X^{b^{-1}}$, $Y^{b^{-1}}$, $Z^{b^{-1}}$, for each $b\in T$, $b\neq 1$. These are listed in the following table.

A	A^{R_0}	A^{R_2}	$A^{R_2R_0}$	$A^{R_0R_2}$
X	$(ZXY)^{-1}$	Y	Z	Z^X
Y	Z	X	$(ZXY)^{-1}$	$(YZX)^{-1}$
Z	Y	$(YZX)^{-1}$	$X^{Z^{-1}}$	X

If $G=\langle X,Y,Z\mid R\rangle$, where R is a set of words in X, Y, Z and $x=\langle R\rangle^{\Theta}X$, $y=\langle R\rangle^{\Theta}Y$ and $z=\langle R\rangle^{\Theta}Z$, then, by Theorem 3, $\Upsilon^{\Delta}=\langle S\rangle^G$, where $S=R((zxy)^{-1},z,y)\cup R(y,x,(yzx)^{-1})\cup$

$R(z,(zxy)^{-1},zxz^{-1})$ and, by Theorem 4, $\Upsilon_{\Delta}=\langle R_\sigma\rangle^{\Sigma}$, where Σ is the group generated by the permutations $\sigma_x=(x,(zxy)^{-1},y,z^x)$, $\sigma_y=(y,z,x,(yzx)^{-1})$, $\sigma_z=(z,y,(yzx)^{-1},x)$ and

$$R_\sigma=R(\sigma_x,\sigma_y,\sigma_z)=\left(R(x,y,z),\ R\left((zxy)^{-1},z,y\right),\ R\left(y,x,(yzx)^{-1}\right),\ R\left(z^x,(yzx)^{-1},x\right)\right).$$

The upper and lower irregularity groups of the Δ^{012}-regular hypermaps determined in Chapter 3 can be seen in Table 3, where the symbols "+" and "−" in the fourth column mean "orientable" and "non-orientable", respectively.

Table 3: Upper and lower irregularity groups. Here $k=\dfrac{mn}{(m,n)^2}$.

χ	$\mathcal{Q}$	R	$\pm$	Υ^{Δ}	Υ_{Δ}	just-Π-reg.
2	$(C_{4n};x,x,x^{-1})$	$X^{4n},Y^{-1}X,YZ$	$+$	1	1	$\Pi=\Delta$
1	$(C_{2n+1};x,x,x^{-1})$	$X^{2n+1},Y^{-1}X,YZ$	$-$	1	1	$\Pi=\Delta$
0	$(C_2;x,1,1)$	X^2,Y,Z	$-$	C_2	C_2	$\Pi=\Delta^0$
0	$(C_2;1,y,y)$	X,Y^2,YZ	$-$	C_2	C_2	$\Pi=\Delta^0$
0	$(D_{2n+1};x,y,y)$	$X^{2n+1},Y^2,(XY)^2,YZ$	$-$	D_{2n+1}	D_{2n+1}	$\Pi=\Delta^0$
0	$(D_{2n+1};x,y,y^{-1})$	$X^2,Y^{2n+1},(XY)^2,YZ$	$-$	D_{2n+1}	D_{2n+1}	$\Pi=\Delta^0$
0	$(T^1_{(m,n)};x,y,y^{-1})$	$X^{2m},Y^{2n},(Y^{-1}X)^2,(XY)^2,YZ$	$+$	C_k	C_k	$\Pi=\Delta^0$
0	$(T^2_{(m,n)};x,y,y^{-1})$	$X^{4m},Y^{4n},(Y^{-1}X)^2,(XY)^2,$ $X^{2m}Y^{2n},YZ$	$+$	$C_{(2,k)k}$	$C_{(2,k)k}$	$\Pi=\Delta^0$
-1	$(C_3;x,1,1)$	X^3,Y,Z	$-$	C_3	C_3	$\Pi=\Delta^0$
-1	$(C_3;1,y,y^{-1})$	X,Y^3,YZ	$-$	C_3	C_3	$\Pi=\Delta^0$
-2	$(C_4;1,y,y^{-1})$	X,Y^4,YZ	$-$	C_4	C_4	$\Pi=\Delta^0$
-2	$(C_4;x,1,1)$	X^4,Y,Z	$-$	C_4	C_4	$\Pi=\Delta^0$
-2	$(C_4;y^2,y,y^{-1})$	XZ^2,Y^4,YZ	$-$	C_4	C_4	$\Pi=\Delta^0$
-2	$(C_4;x,x^2,x^2)$	X^4,X^2Y,YZ	$-$	C_4	C_4	$\Pi=\Delta^0$
-2	$(C_4;x,x,x)$	$X^4,Y^{-1}X,ZX^{-1}$	$+$	1	1	$\Pi=\Delta$
-2	$(V_4;x,y,1)$	$X^2,Y^2,(XY)^2,Z$	$-$	V_4	C_2	$\Pi=\Delta^{012}$
-2	$(V_4;x,y,xy)$	$X^2,Y^2,(XY)^2,XYZ$	$-$	V_4	C_2	$\Pi=\Delta^{012}$
-2	$(C_6;y^4,y,y^{-1})$	XY^2,Y^6,YZ	$-$	C_2	C_2	$\Pi=\Delta^0$
-2	$(C_6;x,x^4,x^2)$	X^6,X^2Y,YZ	$-$	C_2	C_2	$\Pi=\Delta^0$
-2	$(C_6;x,x^3,x^3)$	X^6,X^3Y,YZ	$+$	C_3	C_3	$\Pi=\Delta^0$
-2	$(C_6;y^3,y,y^{-1})$	XY^3,Y^6,YZ	$+$	C_3	C_3	$\Pi=\Delta^0$
-2	$(D_3;x,y,y)$	$X^2,Y^2,(XY)^3,YZ$	$+$	1	1	$\Pi=\Delta$
-2	$(C_8;y^5,y,y^{-1})$	XY^3,Y^8,YZ	$+$	1	1	$\Pi=\Delta$
-2	$(C_{12};x,x^{-5},x^5)$	X^{12},X^5Y,YZ	$+$	1	1	$\Pi=\Delta$
-2	$(A_4;x,y,y^{-1})$	$X^3,Y^3,(Y^{-1}X)^2,YZ$	$-$	1	1	$\Pi=\Delta$

Since Δ^{012}-regular hypermaps are uniform and uniform hypermaps on the sphere and on the projective plane are regular, every Δ^{012}-regular hypermap of positive characteristic is regular (first two lines of Table 3). For negative characteristic the calculations are rather trivial. For illustrative purposes, we compute the upper irregularity group of 3 hypermaps of characteristic zero (I, II, III) and the lower irregularity group of an unbalanced hypermap (IV).

The just-Π-regularity appearing in last column (see Introduction for definition) is inspected while computing the upper irregularity group. When Π has index 2 the hypermaps are necessarily fully-balanced (by Lemma 1) and henceforth one needs only to compute the upper (or lower) irregularity group.

(I) For $G = D_{2n+1} = \langle X, Y, Z \mid X^{2n+1}, Y^2, (XY)^2, YZ\rangle$ we have $R(X^{R_2}, Y^{R_2}, Z^{R_2})_{|G} = R(y, x, x^{-1}) = \{y, x^2, 1\}$. Hence $G = \langle y, x^2\rangle^G \subset \Upsilon^{\Delta}$, so $\Upsilon^{\Delta} = D_{2n+1}$. As $R(X^{R_0}, Y^{R_0}, Z^{R_0})_{|G} = R((zxy)^{-1}, z, y) = \{1\}$ the hypermap is just Δ^0-regular. Hence $\Upsilon_{\Delta} = \Upsilon^{\Delta}$.

(II) For $G = T^1_{(n,m)}$ we have $\Upsilon^{\Delta} = \langle S\rangle^G = \langle x^{2n}, y^{2m}\rangle^G = \langle x^{2n}, y^{2m}\rangle = \langle x^{2n}\rangle \times \langle y^{2m}\rangle = C_{\frac{n}{(n,m)}} \times C_{\frac{m}{(n,m)}} = C_{\frac{nm}{(n,m)^2}}$, since $(x^2)^y = x^{-2}$, $(y^2)^x = y^{-2}$ and $\left(\frac{2n}{(2n,2m)}, \frac{2m}{(2n,2m)}\right) = \left(\frac{n}{(n,m)}, \frac{m}{(n,m)}\right) = 1$.

(III) For $G = T^2_{(n,m)}$ we have $\Upsilon^{\Delta} = \langle x^{4n}, y^{4m}, x^{2n}y^{2m}\rangle^G = \langle x^{4n}, y^{4m}, x^{2n}y^{2m}\rangle$. Since

$$\left(|x^{4n}|, |y^{4m}|\right) = \left(\frac{4m}{(4n,4m)}, \frac{4n}{(4n,4m)}\right) = \left(\frac{m}{(n,m)}, \frac{n}{(n,m)}\right) = 1\,,$$

$\Upsilon^{\Delta} = \langle x^{4n}y^{4m}, x^{2n}y^{2m}\rangle$ and since $x^{4n}y^{4m} = (x^{2n}y^{2m})^2$, then $\Upsilon^{\Delta} = \langle x^{2n}y^{2m}\rangle$ is a cyclic group of order $t = |x^{2n}y^{2m}|$. Set $a = x^{2n}$ and $b = y^{2m}$. Then $(ab)^t = 1 \Leftrightarrow a^t b^t = 1$, i.e., $a^t = b^t = 1$ or $\{a^t, b^t\} = \{x^{2m}, y^{2n}\}$. In the first case, the least positive integer t for which $a^t = b^t = 1$ is

$$t = lcm\,(|a|, |b|) = lcm\left(\frac{4m}{(4m,2n)}, \frac{4n}{(4n,2m)}\right) = 2\frac{mn}{(m,n)^2}\,.$$

In the second case, $a^t = x^{2m}$, $b^t = y^{2n}$ or $a^t = y^{2n}$, $b^t = x^{2m}$, but both imply $nt \equiv m \mod 2m$ and $mt \equiv n \mod 2n$. Dividing by (m,n) we get

$$vt \equiv u \mod 2u \qquad \text{and} \qquad ut \equiv v \mod 2v\,, \tag{2}$$

where $u = \dfrac{m}{(m,n)}$ and $v = \dfrac{n}{(m,n)}$. The first congruence implies that $vt = (2\alpha+1)u$, for some positive integer α. Therefore u divides vt. Since $(u,v) = 1$, then u divides t, i.e.. $t \equiv 0 \mod u$. So the first congruence has no solution if v is even and has the minimal solution $t = u$ if v is odd. Similarly, the second congruence has no solution if u is even and has the minimal solution $t = v$ if u is odd. Hence (2) has solutions if and only if u and v are both odd (i.e., uv odd), in which case the minimal solution is $t = uv = \dfrac{mn}{(m,n)^2} = k$. Thus $|ab| = k$ if k is odd and $|ab| = lcm\left(\dfrac{2n}{(2n,m)}, \dfrac{2m}{(2m,n)}\right) = 2k$ if k is even. Hence $|ab| = (2,k)k$.

(IV) For $G = V_4 = \langle X, Y, Z \mid X^2, Y^2, (XY)^2, Z\rangle$ we have $R_\sigma = \{\sigma_x^2, \sigma_y^2, (\sigma_x\sigma_y)^2, \sigma_z\}$, where $\sigma_x = (x, (zxy)^{-1}, y, z^x) = (x, xy, y, 1)$, $\sigma_y = (y, z, x, (yzx)^{-1}) = (y, 1, x, xy)$ and $\sigma_z = (z, y, (yzx)^{-1}, x) = (1, y, xy, x)$. Hence $\Upsilon_{\Delta} = \langle R_\sigma\rangle^{\Sigma} = \langle \sigma_x^2, \sigma_y^2, (\sigma_x\sigma_y)^2, \sigma_z\rangle = \langle\sigma_z\rangle = C_2$.

Every hypermap of Table 3 which contains one of YZ, ZX or XY in the set R of relators is a map or a dual of a map. Since, as remarked in [5], just-Δ^{012}-regular maps correspond to edge-transitive maps of type 5^{P} (see [11, 13]), and since there is no just-Δ^{012}-regular map in Table 3, we get the following consequence of the classification given in Table 3.

Proposition 6 *There is no edge-transitive map of type 5^{P} and characteristic $\chi \geq -2$.*

Comments and remarks

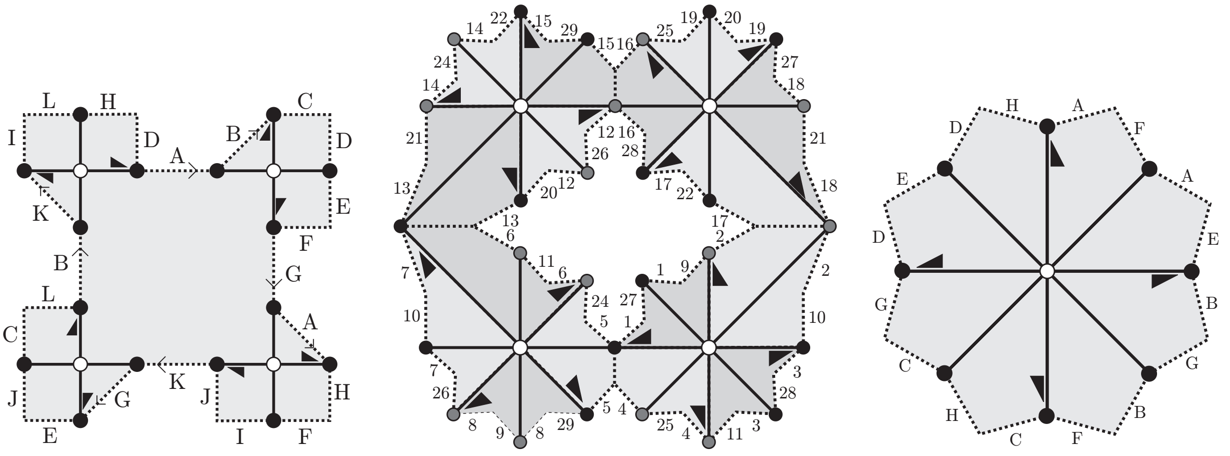
In Table 3 one notices that if $\mathcal{Q}$ is a Δ^{012}-regular hypermap of characteristic $\chi \geq -2$, then

(1) $\iota_\Delta \leq \iota^\Delta$;

(2) whenever $\mathcal{Q}$ is balanced, $\mathcal{Q}$ is also fully-balanced;

(3) whenever $\mathcal{Q}$ is fully-balanced, $\mathcal{Q}$ is not just-Δ^{012}-regular;

(4) all maps are fully-balanced;



(a) (b) (c)

Figure 3: (a) A (just) Δ^{012}-regular hypermap with $\iota_\Delta > \iota^\Delta > 1$;
(b) A balanced (just) Δ^{012}-regular hypermap which is not fully-balanced;
(c) A just Δ^{012}-regular hypermap which is fully-balanced.

None of the above generalizes. The regular Δ^{012}-marked hypermap $(D_4; x, y, xy)$ of characteristic -6, where $D_4 = \langle x, y \mid x^2, y^2, (xy)^4 \rangle$ has $\iota_\Delta > \iota^\Delta > 1$ (Fig. 3(a)); the regular Δ^{012}-marked hypermap $(M; x, y, 1)$ of characteristic -24, where M is the metacyclic group $\langle x, y \mid x^4 = 1, y^4 = x^2, x^y = x^3 \rangle$, is balanced but not fully-balanced (Fig. 3(b)); the regular Δ^{012}-marked hypermap $(C_4; x, x^2, 1)$ of characteristic -4 is just-Δ^{012}-regular and fully-balanced (Fig. 3(c)); and finally, the regular Δ^{012}-marked map $(M; x, y, x^{-1})$ of characteristic -10, where M is the metacyclic group $\langle x, y \mid x^5 = y^4 = 1, x^y = x^2 \rangle$ is unbalanced (Fig. 4).

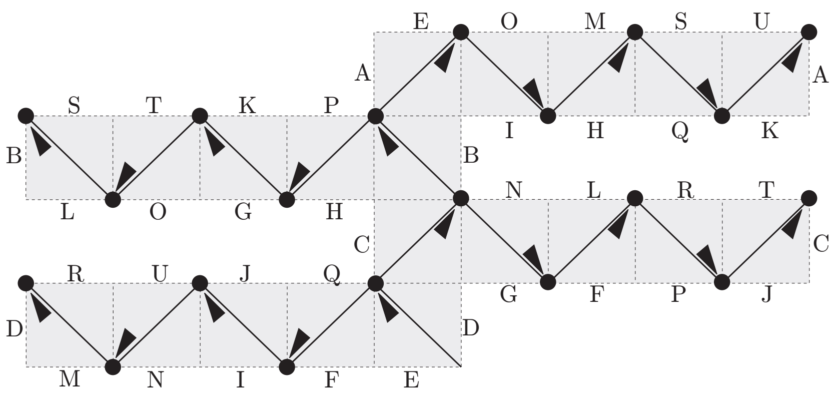


Figure 4: An example of an unbalanced Δ^{012}-regular map

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