Journal for Geometry and Graphics
Volume 13 (2009), No. 1, 91–99.

Orientation Parameters and Reconstruction of Space Models from a Single Photo

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Dedicated to Prof. Hellmuth Stachel at the occasion of his 65th birthday

Abstract. Precise measurements in photogrammetry require the use of metric cameras. But they are too expensive and are not always available. Non-metric cameras are always at hand and have flexibility in focusing range. However, a non-metric camera needs to be calibrated. The determination of its orientation parameters (camera position, image plane, principle point in photo and camera focal length) allows the use of the camera in many fields of photogrammetry. Previous methods used to determine the orientation parameters require at least five control points, and the solution is complicated unless the equations are linearized. Moreover, the reconstruction of the space model is determined through a stereo-pair, i.e., two photos of the object from different positions.

In some cases a reflecting surface such as water, mirror, etc. can be detected in a photo. In the present paper it will be shown that if only four known coplanar points with their images appear in a photo, then the above problem of a non-metric camera can be solved, together with the reconstruction of the space object using only one photo. Here we use a mirror plane (in general position) to reflect the control points and the object and all appear in the photo. The new method is simple and direct and needs no linearization of equations.

Key Words: Close range photogrammetry, projective geometry, affine and collinear transformation, orientation parameters

MSC 2000: 51N05

1. Introduction

One of the most important items in photogrammetry is the orientation problem of photographs taken with non-metric cameras. The accuracy of orientation parameters is very important in estimating the accuracy of the photogrammetric measurements. It is well known that we need five control points to determine the orientation parameters of a non-metric camera (see,

ISSN 1433-8157/$ 2.50

e.g., [2] or [1]). Also the three-dimensional measurements in close range photogrammetry need at least two photos (*stereo pairs*) or in some cases multi-image of the object ([4, 3, 5]). Therefore, the main objective of this paper is to determine the orientation parameters of a non-metric camera using a minimum number of control points, i.e., four coplanar control points. Furthermore, we can reconstruct the three coordinates from a single photo. To do so, an additional condition must be fulfilled: A mirror plane γ in general position is used to reflect the control points and the depicted object, and all is visible in the photo.
The work is divided into two main parts:

1. Determination of the position of the image plane π by finding its lines of intersection with both object plane α and its reflected plane $\overline{\alpha}$. The technique of the solution depends on using homogeneous coordinates in both planes α and $\overline{\alpha}$ of the four coplanar points. The rest of the parameters can easily be found; they depend on finding π.
2. The second part deals with the reconstruction of a space model subject to the above conditions, using one image only.

2. Determining the space position of the image plane π

It is required to find the space position of π relative to a Cartesian frame $\tilde{O}(\tilde{X}, \tilde{Y}, \tilde{Z})$. This will be done by finding the lines of intersection t and $\overline{t}$ of π with α and $\overline{\alpha}$, respectively.

The lines t and $\overline{t}$ are determined directly by using homogeneous coordinates in both planes

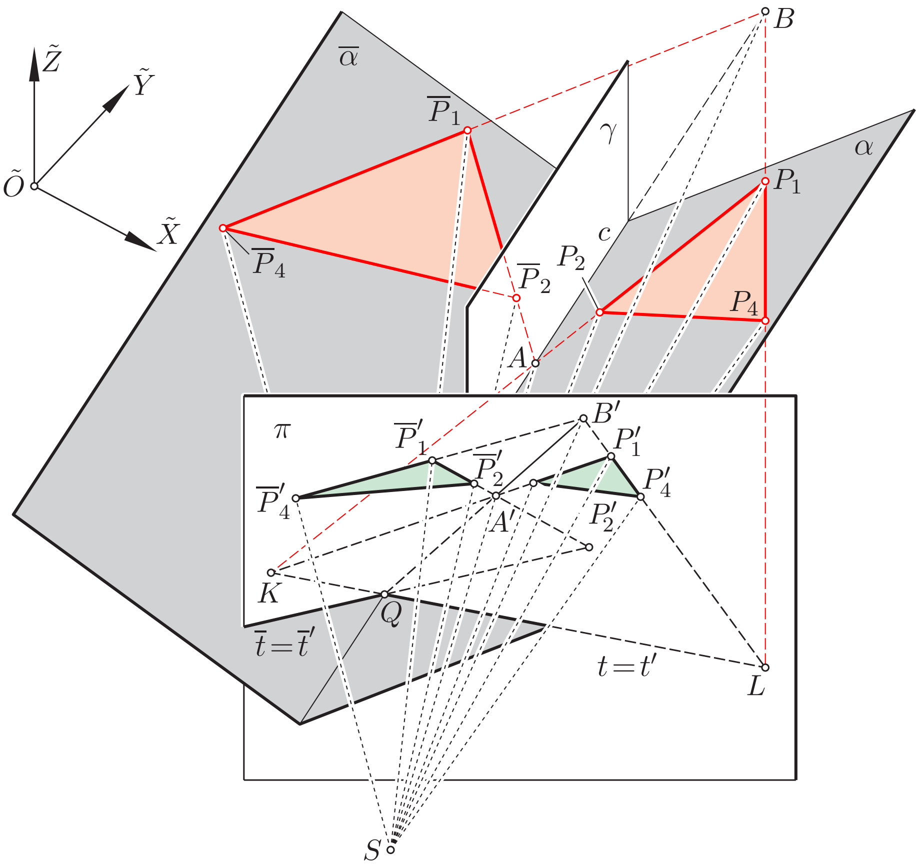

Figure 1: The control points P_i, $i = 1, \dots, 4$, their mirrors $\overline{P}_i$ and the corresponding image points P'_i, $\overline{P}'_i$

α and $\overline{\alpha}$ as will be discussed in the following sections. To determine the trace t of plane α in π, the following steps are carried out:

1. The correspondence
$$P_i \in \alpha \;\mapsto\; P_i' \in \pi, \quad i = 1, 2, 3, 4,$$
induced by the projection, will be determined first.
2. The vanishing lines $v \in \alpha$ and $u' \in \pi$ are computed.
3. A pair of congruent corresponding lines $t \in \alpha$ and $t' \in \pi$ is determined. t should be the trace of α in π.

2.1. Three-dimensional coordinate transformations

It is required to transform the coordinates of a point P_i from the frame $\tilde{O}(\tilde{X}, \tilde{Y}, \tilde{Z})$ to the Cartesian frame $O(X, Y, Z)$, where O coincides with P_1 and the X-axis passes through P_2 (see Fig. 2). This transformation can be expressed in terms of six independent transformation parameters: three rotation angles and three components of the translation vector. It reads

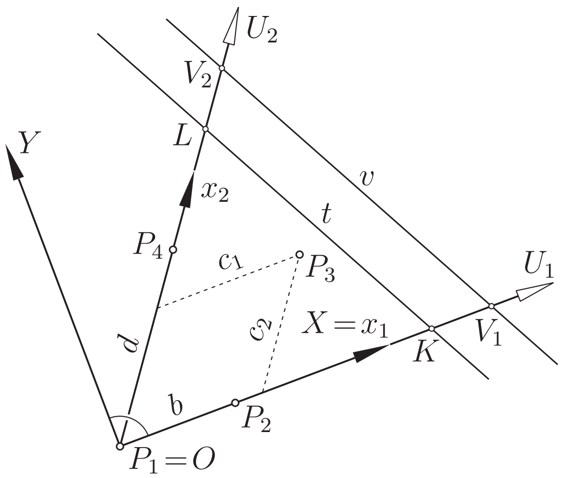

Figure 2: Coordinates $(x_0 : x_1 : x_2)$ in α

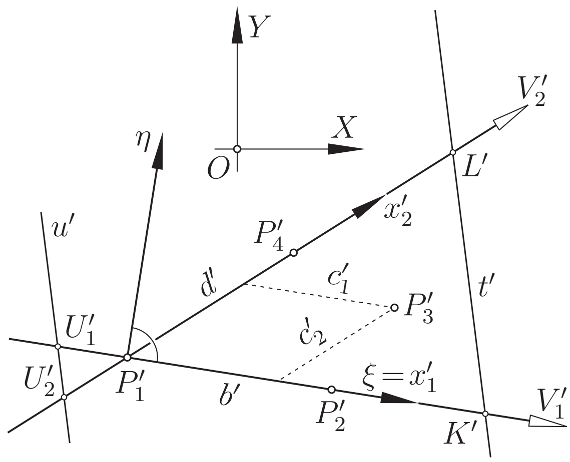

Figure 3: Coordinates $(x_0' : x_1' : x_2')$ in π

$$\boldsymbol{x} = \boldsymbol{t} + R\,\tilde{\boldsymbol{x}} \tag{1}$$

where $\boldsymbol{t}$ is a translation vector and R an orthogonal 3×3 rotation matrix. In homogeneous coordinates the above equation can be rewritten as

$$\begin{pmatrix} 1 \\ X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{t} & R \end{pmatrix} \begin{pmatrix} 1 \\ \tilde{X} \\ \tilde{Y} \\ \tilde{Z} \end{pmatrix} \quad \text{or} \quad \boldsymbol{x} = A\,\tilde{\boldsymbol{x}}\,. \tag{2}$$

The inverse transformation reads

$$\tilde{\boldsymbol{x}} = A^{-1}\boldsymbol{x}, \quad A^{-1} = \begin{pmatrix} 1 & 0 \\ -R^T\boldsymbol{t} & R^T \end{pmatrix}. \tag{3}$$

To determine the entries of A, it is necessary to know the coordinates of at least three pairs of corresponding points in both coordinate systems, and these can be calculated.

2.2. Homogeneous coordinate systems

In the object plane α a homogeneous coordinate system is chosen with origin at P_1, the x_1- and x_2-axis passing through P_1 and P_4, respectively (Fig. 2); the x_0-axis lies at infinity. The homogeneous coordinates of a given point are three ratios $(x_0 : x_1 : x_2)$, not all zeros, which can be arbitrarily multiplied by any non-zero factor. From Fig. 2 we extract the homogeneous coordinates of our control points:

$$P_1(1:0:0), \quad P_2(1:b:0), \quad P_3(1:c_1:c_2). \quad P_4(1:0:d)$$

Because of $x_0 = 1$ the entries b, d, c_1, and c_2 are metric lengths, which can be calculated from the space coordinates or measured directly.

A similar homogeneous coordinate system is chosen in the image plane π as shown in Fig. 3. The coordinates $(x'_0 : x'_1 : x'_2)$ of the image points are

$$P'_1(1:0:0), \quad P'_2(1:b':0), \quad P'_3(1:c'_1:c'_2), \quad P'_4(1:0:d'),$$

where like before b', d', c'_1, and c'_2 are metric lengths.

2.3. Transformation equations between π and α

Figures 2 and 3 show the four pairs of corresponding points

$$P_i(x_0 : x_1 : x_2) \;\mapsto\; P'_i(x'_0 : x'_1 : x'_2), \; i = 1, \dots 4,$$

with their homogeneous coordinates as given before. The equations of this linear transformation can be expressed in the form

$$\begin{aligned} x'_0 &= a_{00}x_0 + a_{01}x_1 + a_{02}x_2 \\ x'_1 &= a_{10}x_0 + a_{11}x_1 + a_{12}x_2 \\ x'_2 &= a_{20}x_0 + a_{21}x_1 + a_{22}x_2 \end{aligned} \tag{4}$$

where a_{ij} are constant coefficients.

Substituting the coordinates of an object point in the right hand side of eq. (4) and those of the corresponding image point multiplied by an unknown factor in the left hand side, we get three equations. In total, we get 12 equations in 9 coefficients a_{ij} and the 4 unknown factors. Since only the ratios between the factors are of interest, one of them can be arbitrarily chosen. The solution of these equations is simple and yields

$$\left.\begin{array}{ll} a_{10} = a_{20} = a_{21} = a_{12} = 0 & a_{22} = b'd'c_1c'_2(bd - c_1d - c_2b) \\ a_{00} = bc_1c_2d(b'd' - c_1d' - c'_2b') & a_{01} = c_2[b'c_1d(c'_2 - d') - bc'_1d'(c_2 - d)] \\ a_{11} = b'd'c'_1c_2(bd - c_1d - c_2b) & a_{02} = c_1[b'c'_2d(b - c_1) - bc_2d'(b' - c'_1). \end{array}\right\} \tag{5}$$

Similar formulas are found for the inverse transformation

$$\begin{aligned} x_0 &= a'_{00}x'_0 + a'_{01}x'_1 + a'_{02}x'_2 \\ x_1 &= a'_{11}x'_1 \\ x_2 &= a'_{22}x'_2 \end{aligned} \tag{6}$$

with coefficients

$$a'_{00} = \frac{c'_7}{c_7}\,a_{11}, \quad a'_{01} = -\frac{c'_2}{c_2}\,a_{01}, \quad a'_{02} = -\frac{c'_1}{c_1}\,a_{02} \quad a'_{11} = \frac{c'_2}{c_2}\,a_{00}, \quad a'_{22} = \frac{c'_1}{c_1}\,a_{00}.$$

2.4. The vanishing lines

The vanishing line v in α corresponds to the line at infinity in π. It can be determined using the points at infinity of both x_1' and x_2' axes, whose coordinates are $(0:1:0)$ and $(0:0:1)$, respectively. Substituting in eq. (6), we get the vanishing points

$$V_1(a'_{01}:a'_{11}:0), \quad V_2(a'_{02}:0:a'_{22}). \tag{7}$$

The vanishing line v joins both V_1 and V_2. Similarly, the vanishing line u' in π is spanned by the two vanishing points

$$U'_1(a_{01}:a_{11}:0), \quad U'_2(a_{02}:0:a_{22}) \tag{8}$$

2.5. The trace t

As shown in Fig. 1, the trace t can be considered as belonging to both α and π. Let t' denote it in π when separated from α. The restriction of our correspondance to t and t' must be a congruence. It is well known that they are parallel to the vanishing lines in space (e.g., [7]).

Let $K(1:k:0)$ and $L(1:0:l)$ be the points of intersection of t with the x_1- and x_2-axis, and let their corresponding points be $K'(1:k':0)$ and $L'(1:0:l')$, respectively. The distances $\overline{KL}$ and $\overline{K'L'}$ must be equal. Since KL is parallel to v (Fig. 2), then $\overline{P_1L}/\overline{P_1K} = \overline{P_1V_2}/\overline{P_1V_1}$, hence

$$\frac{\overline{P_1L}}{\overline{P_1K}} = \frac{l}{k} = \frac{a'_{22}}{a'_{02}} \Big/ \frac{a'_{11}}{a'_{01}} = \frac{a_{01}}{a_{02}}. \tag{9}$$

Similarly, for the corresponding points $K'(1:k':0)$ and $L'(1:0:l')$ we obtain

$$\frac{l'}{k'} = \frac{a'_{01}}{a'_{02}}.$$

Substituting the coordinates of L and K into (4) (taking (5) into account) we get the coordinates of K' and L'as follows:

$$K'(a_{00}+a_{01}k:a_{11}k:0), \quad L'(a_{00}+a_{02}l:0:a_{22}l) \tag{10}$$

hence

$$k' = \frac{a_{11}k}{a_{00}+a_{01}k}, \quad l' = \frac{a_{22}l}{a_{00}+a_{02}l}. \tag{11}$$

$\overline{LK}^2 = \overline{L'K'}^2$ implies

$$l^2 + k^2 - 2lk\cos\theta = l'^2 + k'^2 - 2l'k'\cos\theta' \tag{12}$$

where θ and θ' are the angles subtended by the axes pairs. Dividing (12) by k^2 and substituting from (9), (10) and (11), we get after several reductions

$$(a_{00}+a_{01}k)^2 = \frac{a_{11}^2 a_{02}^2}{a'_{02}} \cdot \frac{{a'_{01}}^2 - 2a'_{01}a'_{02}\cos\theta' + {a'_{02}}^2}{a_{01}^2 - 2a_{01}a_{02}\cos\theta + a_{02}^2} = \delta^2. \tag{13}$$

It can be shown that the right hand side of (13) is positive and hence it is set to δ^2.

Solving eq. (13) yields two values for k. The practical value of k is chosen such that for a positive photo, t has a position similar to that shown in Fig. 3, in which

$$\overline{P_1K} < \overline{P_1V_1} \quad \text{or} \quad k < \left|\frac{a'_{11}}{a'_{01}}\right| = \left|\frac{a_{00}}{a_{01}}\right|$$

since from (11) $k = \dfrac{-a_{00} \pm \delta}{a_{01}}$.

Determining k, we can calculate the values of l, k' and l' from (9) and (11). The homogeneous coordinates of K, L, K' and L' are therefore known.

2.6. Relation between homogeneous and Cartesian coordinates

Now, after determining the homogeneous coordinates of points K and L, it is required to determine the space coordinates of these points relative to the frame $\tilde{O}(\tilde{X}, \tilde{Y}, \tilde{Z})$.

1. In plane α let $x_0 = 1$ for any finite point. Then the transformation from the homogeneous coordinates $(1 : x_1 : x_2)$ to Cartesian coordinates (X, Y) are as follows:

$$X = x_1 + Y \cot\theta, \quad Y = x_2 \sin\theta, \tag{14}$$

and conversely

$$x_1 = X - Y \cot\theta, \quad x_2 = Y / \sin\theta. \tag{15}$$

From eq. (14) the space coordinates of points $K(1 : k : 0)$ and $L(1 : 0 : l)$ are found as

$$\begin{aligned} X_K &= k, & Y_K &= 0, \\ X_L &= l \sin\theta \cot\theta = l\cos\theta, & Y_L &= l \sin\theta. \end{aligned} \tag{16}$$

The space coordinates of points K and L relative to $\tilde{O}(\tilde{X}, \tilde{Y}, \tilde{Z})$ can be determined from eq. (3).

2. In the image plane π similar formulas can be deduced. Relations between the homogeneous coordinates $(1 : x'_1 : x'_2)$ and the Cartesian coordinates (ξ, η) are as follows (see Fig. 3)

$$\eta = x'_2 \sin\theta', \quad \xi = x'_1 + \eta \cot\theta' \tag{17}$$

and the relation between (ξ, η) and (x, y) are as follows

$$\begin{aligned} y &= \eta \cos\psi + \xi \sin\psi + y_1 \\ x &= -\eta \sin\psi + \xi \cos\psi + x_1 \end{aligned} \tag{18}$$

where ψ is the angle between the x-axis and the line $P'_1 P'_2$.
And conversely

$$\begin{aligned} \xi &= (y - y_1)\sin\psi + (x - x_1)\cos\psi, & x'_1 &= \xi - \eta \cot\theta, \\ \eta &= (y - y_1)\cos\psi - (x - x_1)\sin\psi, & x'_2 &= \eta / \sin\theta'. \end{aligned} \tag{19}$$

In a similar manner the intersection line $\overline{t}$ of $\overline{\alpha}$ and π can be determined, if the mirror plane γ is known in space. This can be found as follows:

2.7. Determination of the mirror plane

The position of the mirror plane γ in space can be determined according to the following procedures (note Fig. 1):

- Determination of the line c of intersection between α and the mirror plane γ:
  1. In the image plane π, find the intersection point A' between the line connecting the image point P'_1 and P'_2 and the line connecting their corresponding reflected images $\overline{P}'_1$ and $\overline{P}'_2$.

2. The homogeneous coordinates of point A' can be calculated from (19). Then, for the corresponding point A in plane α (trace point of line P_1P_2 in the mirror plane γ) the space coordinates can be calculated by virtue of (6).
3. In the similar manner, the trace point B of line P_1P_4 in γ can be determined.
4. The line c connecting A and B is the collineation axis between α and $\overline{\alpha}$, the common line of planes $\alpha, \overline{\alpha}$ and γ.

- To determine the mirror plane γ, we need only to know the space coordinates of any further point C (which does not necessarily appear in the image plane). Let the equation of γ in vector form be

$$\boldsymbol{n} \cdot \boldsymbol{x} = d, \quad \|\boldsymbol{n}\| = 1, \tag{20}$$

where $\boldsymbol{x}$ is the position vector of any point $X \in \gamma$; $\boldsymbol{n}$ is the unit vector perpendicular to γ and d the signed distance γ from the origin. They can be expressed as

$$\boldsymbol{n} = \frac{(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{b} \times \boldsymbol{c}) + (\boldsymbol{c} \times \boldsymbol{a})}{\|(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{b} \times \boldsymbol{c}) + (\boldsymbol{c} \times \boldsymbol{a})\|}, \quad d = \boldsymbol{a} \cdot \boldsymbol{n} \tag{21}$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ denote the position vectors of the points A, B, C, respectively.

2.8. Determination of the reflected plane $\overline{\alpha}$

After determining the mirror plane in space, the space coordinates of the reflected point $\overline{P}_i$ of P_i can be determined. Let the vector equation of line g passing through P_i perpendicular to γ be $g\colon\ \boldsymbol{x} = \boldsymbol{p}_i + \lambda_i \boldsymbol{n}$, $\lambda_i \in \mathbb{R}$. At the point of intersection between g and γ the parameter λ_i fulfills

$$\lambda_i = d - \boldsymbol{n} \cdot \boldsymbol{p}_i\,.$$

Hence the position vector of $\overline{P}_i$ is

$$\overline{\boldsymbol{p}}_i = \boldsymbol{p}_i + 2(d - \boldsymbol{n} \cdot \boldsymbol{p}_i)\boldsymbol{n}\,. \tag{22}$$

2.9. Equation of image plane π

Lines t and $\overline{t}$ should intersect at one point Q, the trace point of c in the image plane π. Then, the plane containing them is the image plane π. Let the equation of plane π be in the form

$$\pi\colon\ \boldsymbol{m} \cdot \boldsymbol{x} = d_1, \quad \|\boldsymbol{m}\| = 1. \tag{23}$$

Here $\boldsymbol{m}$ is the normal vector of π and $\boldsymbol{m} = \boldsymbol{t} \times \overline{\boldsymbol{t}}$ where $\boldsymbol{t}$ amd $\overline{\boldsymbol{t}}$ are unit vectors along the lines t and $\overline{t}$, resp., and d_1 gives the oriented distance of π from the origin and is equal to

$$d_1 = \boldsymbol{q} \cdot \boldsymbol{m}\,.$$

$\boldsymbol{q}$ is the position vector of point Q.

Due to measurement errors the two lines $t, \overline{t}$ might be skew. In this case, we determined the oriented line m, which intersects both t and $\overline{t}$ perpendicularly and find the points of intersection $M, \overline{M}$ of m with t and $\overline{t}$, respectively. Then the corrected position of π passes through the mid point between M and $\overline{M}$, the *corrected* position of Q.

3. Determination of the rest of orientation parameters

The other orientation parameters depend on knowing the position of the image plane π in space.

1. Position of the center S: The center can be determined as the point of intersection between the rays connecting any two control points P_i with their respective images P_i'. The space coordinates of P_i' can be easily calculated using its relative position to the trace t' in the image plane π.
2. The angle ω between π and α is found as the angle between their respective normal vectors $\boldsymbol{m}$ and $\overline{\boldsymbol{m}}$ by
$$\cos\omega = \frac{\boldsymbol{m}\cdot\overline{\boldsymbol{m}}}{\|\boldsymbol{m}\|\,\|\overline{\boldsymbol{m}}\|}\,.$$
3. The focal length is
$$f = \boldsymbol{s}\cdot\boldsymbol{m} - d_1,$$
where $\boldsymbol{s}$ is the position vector of the center S.

4. Reconstruction of space models

The space coordinates of any point P, which is depicted together with its reflected point $\overline{P}$ in the image plane π, can be determined as follows (see Fig. 4):

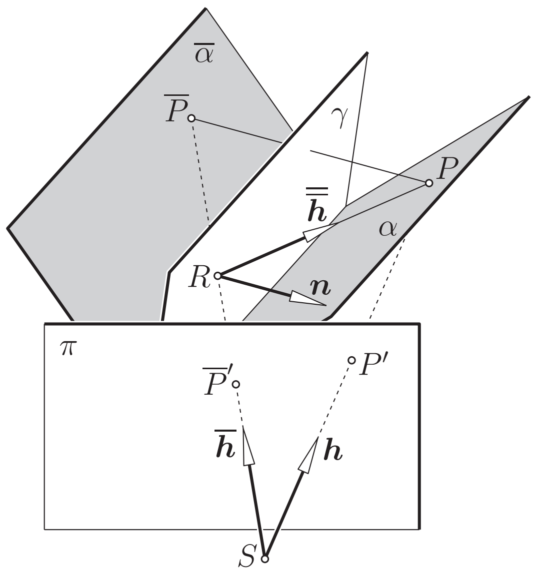

Figure 4: Reconstructing the space coordinates of any point P from P' and $\overline{P}'$

1. The image of point P is P'. The reflection in γ maps P onto $\overline{P}$; its image is denoted by $\overline{P}'$. After determining the position of π, the space coordinates of P' can be calculated. The vector equation of the line h joining S and P' is given by
$$h\colon\ \boldsymbol{x} = \boldsymbol{s} + \lambda\boldsymbol{h}, \quad \lambda\in\mathbb{R},$$
where $\boldsymbol{h}$ is a unit vector along h.
2. In the similar manner the space coordinates of $\overline{P}'$ can be determined. Let the vector equation of line $\overline{h}$ joining S and $\overline{P}'$ be $\overline{h}\colon\ \boldsymbol{x} = \boldsymbol{s} + \lambda\overline{\boldsymbol{h}}$.

3. The position vector of the point R of intersection between $\overline{h}$ and γ is

$$\boldsymbol{r} = \boldsymbol{s} + \left(\frac{d - \boldsymbol{n}\cdot\boldsymbol{s}}{\boldsymbol{n}\cdot\overline{\boldsymbol{h}}}\right)\overline{\boldsymbol{h}}. \tag{24}$$

4. The angle between the two vectors $\boldsymbol{n}$ and $\overline{\boldsymbol{h}}$ can be determined, then the reflected line $\overline{\overline{h}}$ can be found. The space point P lies at the point of intersection between $\overline{\overline{h}}$ and h.

5. Conclusion

In this paper, a new mathematical method is derived to determine the orientation parameters of a non-metric camera using only four control points. The main advantage of this method is that the orientation parameters are determined directly without linearizing the equations, and no complicated techniques are needed. Also, three dimensions measuring from a single photo has been developed. Here, a mirror plane is used to reflect the control points and the object and all appear in the same photo.

A practical use of this method is when a reflecting surface such as water, mirror, etc. can be detected in a photo.

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Received October 10, 2008; final form January 28, 2009