

3D-Images in Photography?

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Abstract. Photography is geometrically considered a typical example of a perspective, i.e., a central projection of 3-space onto a plane. Although in principle true, this cannot explain some quite common facts that each photographer is confronted with: depth of field (and the fact that it varies with the size of the scene), circles of confusion, etc. A much more correct explanation is that the real 3-space in front of the lens is converted via the lens system into a collinear virtual 3-space behind the lens (more precisely, the transformation is an elation). The photographer depicts an ideal cross section of this virtual space by optimizing the distance of the focus plane which corresponds to the sensor plane in virtual space. In this paper, we will illuminate artifacts of photography that cannot be explained by means of the simplified projection.

Key Words: Projective Geometry, collineation, geometrical optics, diffraction blur, depth of field

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1. Introduction

When a geometer or mathematician is asked for a typical example of a central projection of 3-space onto 2-space, he or she will immediately come up with photography. In fact, this seems so obvious that we do not even think any further whether this is really true or not. The lens center is the center of projection, the sensor plane (chip plane, film plane) is the projection plane. Of course we have some restrictions: We can only depict points within a certain viewing frustum in front of the lens. Straight lines appear as straight lines, thus the projection is linear. One can even reconstruct space out of a sufficient number of “perfect” photographic images.

The emphasis lies on the word “perfect”. What is a perfect photo? For a mathematician, the image has to be completely undistorted and “sharp”. The first condition is easier to fulfill: the lens system has to be of high quality. The second condition, however, is the major subject of this paper:



Figure 1: The closer to the lens we are, the more we have to be aware of “confusions”

It is not possible to depict 3-space absolutely sharp into 2-space by means of photography. This is especially true for macro photography, where objects are close to the lens center ([5]).

Figure 1 shall illustrate the problem. It is of utmost importance to depict eyes and antennas of the beetle as sharp as possible — the rest of the scene is of minor importance for the viewer. It is even an advantage that the background is completely confused.

Each good photographer is aware of this problem: It is a challenge to be able to handle this, and, what is more, to even take advantage of it (Fig. 5).

2. The lens equation

If we only consider “infinite sharp images”, photography works as follows: The lens system — it usually consists of several combined lenses — has a well defined center C . We intersect the light ray PC through an arbitrary space point P within the viewing frustum with the sensor plane and get the projected point P^c . Together with some additional information (e.g., known lengths and angles) we can even reconstruct 3-space by means of 2D-images.

A conventional photographic lens collects rays that emanate from a space point P through a relatively large circular aperture and converges these rays to a point of focus P^* on the other side of the lens — in the ideal case on the film.

Mathematically, we explain the generation of an image point by means of the lens equation

$$\frac{1}{f} = \frac{1}{d} + \frac{1}{d^*} \quad (1)$$

Here f is the focal length and d and d^* are the oriented distances of the space point P and the virtual image point P^* from the main plane (Fig. 2). Thus we have

$$\overrightarrow{CP^*} = \frac{f}{f-d} \cdot \overrightarrow{CP} \quad (2)$$

This figure can be found in every school book. It shows that points P that lie in one “principal plane” orthogonally to the optical axis have image points P^* in a corresponding principal plane.

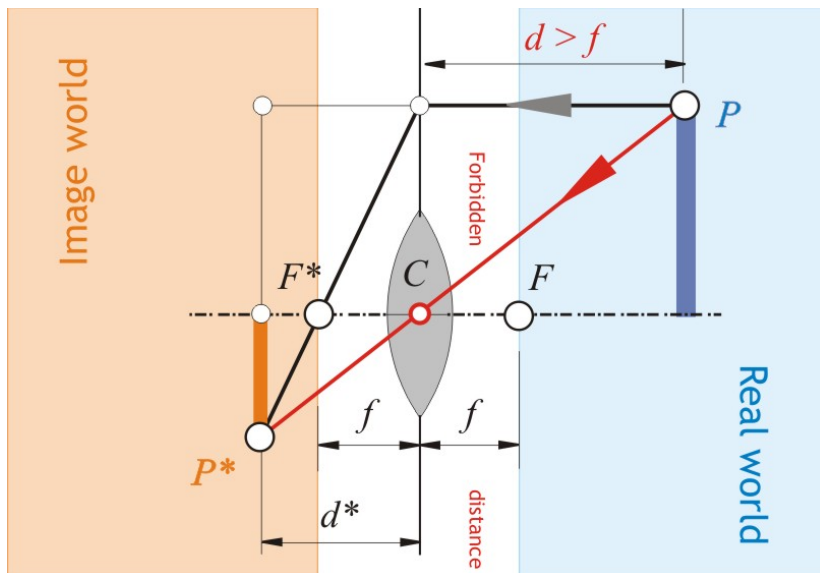


Figure 2: The Gaussian collineation. The closer to the focus F , the larger the image gets. For $d < f$ the image point P^* is on the “wrong side”

3. A spatial collineation

Figure 2 also shows that points with different normal distance d have a different normal distance d^* after the light rays have passed the lens system. Thus, space points cannot only be “projected” onto a single plane. The relationship is a spatial collineation (sometimes named after C.F. GAUSS [6]). This can be proven via Eq. (2) or can be seen geometrically as follows:

We have to show that a straight line in real space corresponds to a straight line in image space. Consider a general line g in real space, skew to the optical axis. When we apply the described construction to all the points P of g , the orbit $g^* = \{P^*\}$ will be in the connection plane $\varepsilon_1 = gC$. All light rays parallel to the optical axis through $\{P\}$ fulfill a plane ε_2 that is transformed into a plane through F^* and the intersection line of ε_2 and the main plane. Thus $g^* = \varepsilon_1 \cap \varepsilon_2$ is a straight line again.

The center of the collineation is the lens center C , the fixed plane is the plane through C . A collineation where center and fixed plane coincide is called *elation* ([1, 2]).

3.1. Objects with large distance

We now consider a special case which is common in “normal” photography: For space points with distance d far larger than the focal length f , $1/d$ is negligible and we have $d^* \approx f$. Therefore, the focal plane through point F^* is suitable as an image plane, in which most of the image points “almost” lie in (Fig. 3).

Unfortunately, most geometry books do not consider the general case (to my knowledge, only in [4] the elation is taken into account), in contrary to newer publications in Computer Graphics (e.g., [7]):

3.2. Objects closer to the lens center

When the object of interest is closer, e.g., as close as in Fig. 4, the focal plane is not suitable any more. Instead, we have to choose a new principal plane in a way that the corresponding

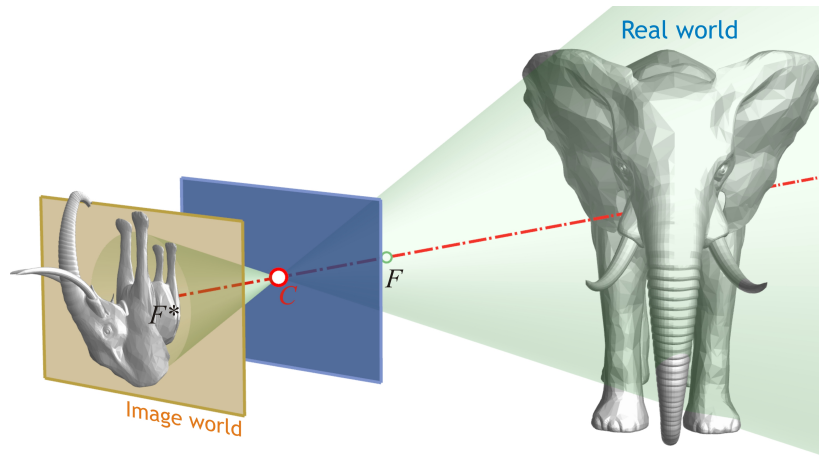


Figure 3: Objects far away from the lens system have image points that are almost in one focal plane ($d \gg f \Rightarrow 1/d \approx 0$)

plane intersects the real object such that the major points are as close to this plane as possible.

Figure 4 illustrates that there is a difference between the virtual space point P^* and the projected point P^c . One might now say that this does not matter for the image: The light ray through P is identical with the light ray through P^* , and its intersection with the sensor plane (= image plane) leads to the same result. A physicist, however, would tell you that a single light ray is not enough to create an image point. Instead, we need a whole light bundle that is refracted by the lens system: Light rays parallel to the optical axis, e.g., are refracted through F^* , and such a ray will hit the image plane somewhere else and not in P^c .

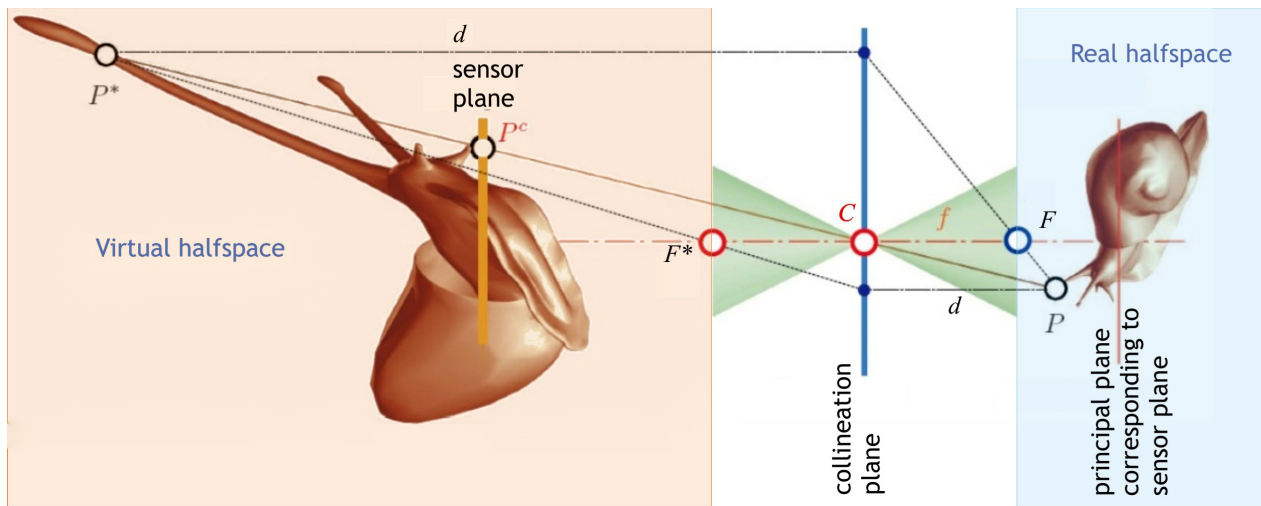


Figure 4: The Gaussian collineation, 3-dimensionally interpreted.
To each space object, we get a collinear virtual object.

4. DOF and COC

In order to allow more light to enter the lens system, cameras have a built in circular aperture. The smaller the diameter of the aperture, the more all light rays are not only bundled in P^* ,

but run more or less through P^c as well. Therefore we can say: *Photography with a negligibly small diameter of the aperture (i.e., a pinhole camera) comes very close to an ordinary central projection.*

There are two facts, however, that inhibit closing the aperture that much. First, we would need extremely much light (which could be provided by means of artificial light) and, second, light has the remarkable property that it behaves like a wave when it passes a very thin hole: The rays are scattered and change direction. This produces the so-called *diffraction blur*¹.

Besides these two “hard” facts, a good photographer loves to play with the depth of field (DOF, i.e., the distance range of acceptable sharp focus in front of the lens): Shallow DOF allows the subject to stand out from the background (Fig. 5).



Figure 5: Shallow depth of field (DOF) allows the subjects to stand out from the background

According to Fig. 4, the range of acceptable sharp focus depends on three parameters:

1. The aperture diameter,
2. the focal length f (the smaller f , the more $d \gg f$ holds),
3. the distance from the lens: Due to the properties of the collineation, the distance of the virtual space point P^* and the image point P^c increases dramatically when our object comes close to the “forbidden space” (where distance to the center gets close to f).

Figure 6 is the key figure of this paper. It shows how the light rays emanating from a space point P form an oblique circular cone through the aperture and are bundled as a second circular cone through the virtual image point P^* . The intersection of the refracted cone with the sensor plane is called circle of confusion (COC) or blur disk².

Instead of using formulas ([7])³, one can now immediately understand that the diameter of the COC is proportional to the diameter of the aperture and that it heavily depends on the distance of P (Fig. 7).

¹<http://www.cambridgeincolour.com/tutorials/depth-of-field.htm>

²http://en.wikipedia.org/wiki/Circle_of_confusion

³<http://www.timledlie.org/cs/graphics/finalproj/finalproj.html>

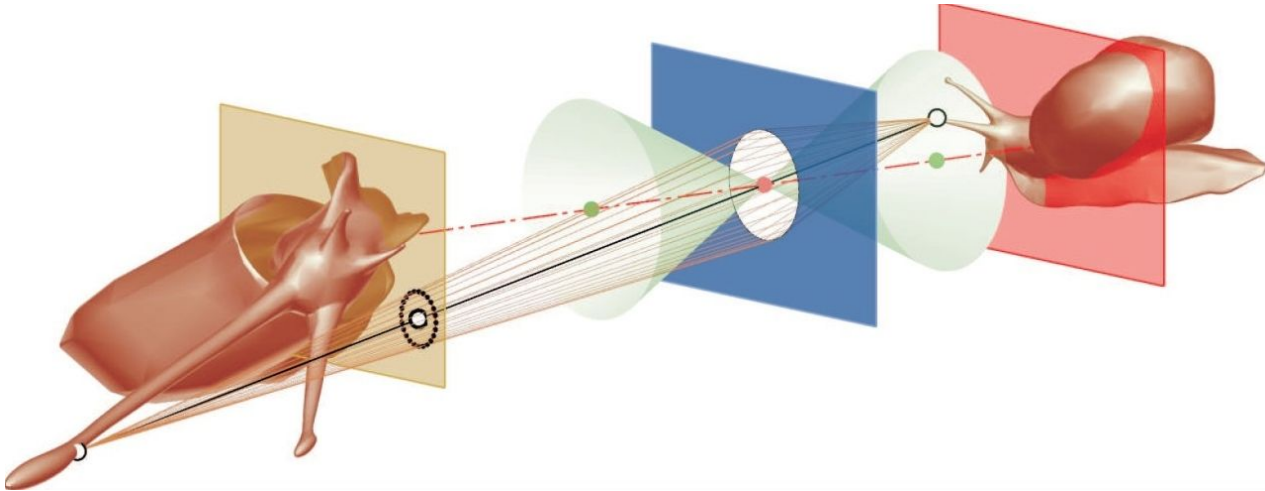


Figure 6: The diameters of the aperture and the circle of confusion (COC) correspond linearly

Figure 7 left shows a camera with professional macro-equipment (Canon MP-E 65mm Macro Lens plus MT-24EX Macro Twin Lite) that allows magnifications up to 5:1 ([5])⁴. The lens center can be 20–30 cm in front of the sensor plane. You may imagine the extremely distorted virtual 3d-image (Fig. 7, lower image) “inside the tube”.

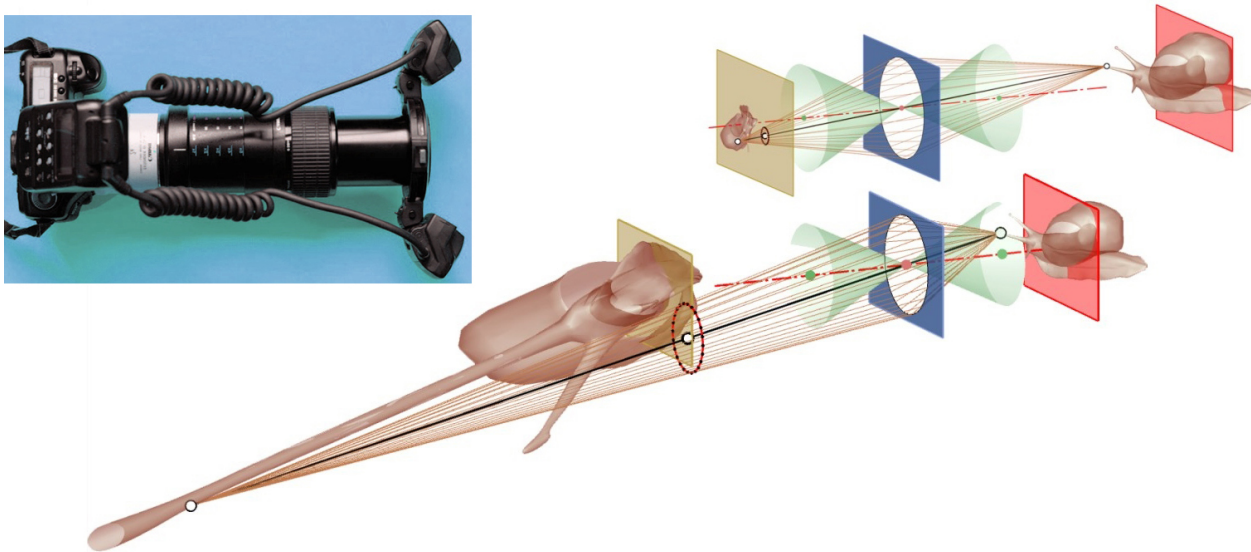


Figure 7: The size of the COC (blur disk) heavily depends on the distance of the space point.
Upper left: A macro lens that allows the large virtual image to fit in ...

5. The limits

As we mentioned earlier, we cannot use an arbitrary small aperture in order to simulate a pinhole camera, since this produces diffraction blur⁵. Thus, there is an equilibrium between

⁴<http://www.vividlight.com/articles/2914.htm>

⁵<http://www.huecandela.com/hue-x/pin-pdf/Prober-%20Wellman.pdf>



Figure 8: Left image: With such a short distance, it is not possible to depict an ant sharper than this. If we close the aperture further, diffraction blur will confuse the sharp areas of the photo. Right image: The only way out is to keep as many points of interest as possible in a principal plane

geometrical blur due to aperture and diffraction blur, called optimum aperture⁶.

Professional macro lenses like the Canon MP-E 65mm Macro Lens therefore do not even allow to close the aperture further and come with tables that explain how to reopen aperture with increasing magnification.

Even the use of smaller focal lengths f is a “dead end”. Fig. 9 shows the collinear virtual images of a snail, one time transformed by a wide-angle lens (left), the other time with a tele-lens (right). The tele-lens produces a 3D-image with different distributions of the normal distances d^* . This helps to keep the COC approximately as small as with the wide-angle lens. Furthermore, there is a physical limit when using a wide-angle lens: Parts of the objects will touch the lens.

The only way out of the dilemma is to use some geometric knowledge: As long as points of interest can be kept in a plane (and at least three points can always be used for the definition of a plane), one can try to keep the COCs within limits (Fig. 8 right).

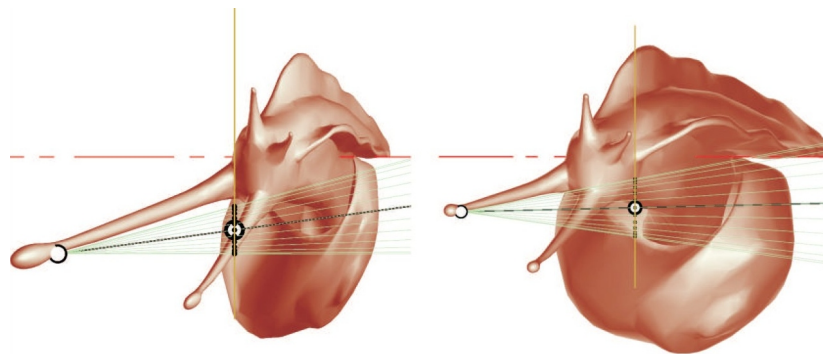


Figure 9: The decrease of the focal length f does not diminish the diameter of the COC

⁶<http://www.bobatkins.com/photography/technical/diffraction.html>

6. Conclusion

Photography is a beautiful example of how to explain non-trivial geometrical relationships. Students are highly motivated and will understand things that happen daily when they take pictures with their digital camera. Even for advanced photographers the above geometrical considerations help to comprehend otherwise hard-to-understand behaviors of optics.

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