Reflection-Induced Perspectivities
Among Triangles

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Abstract. In the plane of a reference triangle $ABC$, let $DEF$ be a triangle and $P$ a point. A line $U$ through $P$ is defined as a successful reflector through $P$ if the reflection $D'E'F'$ of $DEF$ in $U$ is perspective to $ABC$, in the sense that the lines $AD', BE', CF'$ concur. The point of concurrence is the perspector, $P(DEF, P, U)$. The main theorem is that for given $DEF$ and $P$, there are either infinitely many successful reflectors through $P$, or else there are at most four. Examples include choices of $DEF$ and $P$ for which the Steiner axes are the successful reflectors, and also choices for which there are infinitely many successful reflectors, in which case $P$ is called a pivot for $DEF$. In three of the examples, the locus of $P(DEF, P, U)$ as $U$ rotates about $P$, called the pivotal curve, is the isogonal conjugate of a circle.

Key Words: barycentric coordinates, centroid, circumcircle, Euler triangle, ex-touch triangle, Gossard triangle, hexyl triangle, incircle, intouch triangle, isogonal conjugate, Kosnita triangle, medial triangle, pivot, pivotal curve, reflection, successful reflector, tangential triangle

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1. Introduction

With the help of computers, many new relationships in the plane of a triangle have been discovered during the past few years. At first, a newcomer might assume that these discoveries depend on graphics software and that the discovery-making occurs visually. However, much of the new geometry is actually of a much more algebraic character, stemming from representations of points, lines, circles, and other curves using homogeneous coordinates. For
example, almost every special point in the plane of a triangle has simple algebraic representations, and these lend themselves to algebraic criteria – most notably zero determinants for properties such as collinearity of three points, concyclicity of four points, concurrency of three lines, and so on. In that vein, this article introduces what appears to be a new kind of reflection theorem and shows that the axes of the Steiner ellipse play a prominent role.

We begin with a few terms and notation. In homogeneous barycentric coordinates (henceforth barycentrics), a point will be written as $\alpha : \beta : \gamma$. Points include the vertices and centroid of a reference triangle $ABC$, given by $A = 1 : 0 : 0$, $B = 0 : 1 : 0$, $C = 0 : 0 : 1$, $G = 1 : 1 : 1$.

The points $D = AG \cap BC$, $E = BG \cap CA$, $F = CG \cap AB$ are the vertices of the medial triangle. Their anticomplements, $0 : 1 : -1$, $-1 : 0 : 1$, $1 : -1 : 0$, are the vertices of the anticomplementary triangle. The line at infinity, $L^{\infty}$, is given by the equation $\alpha + \beta + \gamma = 0$. These and many other objects in triangle geometry (and triangle algebra, in which the symbols $a, b, c$ represent indeterminates or variables rather than sidelengths of a triangle) are developed elsewhere (e.g., [3], [4], [12], [13]).

2. Main theorem

Suppose that $U$, given by the barycentric equation $u\alpha + v\beta + w\gamma = 0$, is a line other than $L^{\infty}$. The reflection $R(X)$ of a point $X = x : y : z$ in $U$ is the point given by

$$R(X) = f(a, b, c, u, v, w)x + g(a, b, c, u, v, w)(vy + wz) :$$
$$f(b, c, a, v, w, u)y + g(b, c, a, v, w, u)(wz + ux) :$$
$$f(c, a, b, w, u, v)z + g(c, a, b, w, u, v)(ux + vy),$$

where

$$f(a, b, c, u, v, w) = a^2u^2 - b^2v^2 - c^2w^2 + vw(b^2 + c^2 - a^2)$$
$$g(a, b, c, u, v, w) = 2a^2u - v(a^2 + b^2 - c^2) - w(c^2 + a^2 - b^2).$$

The procedure for obtaining this result is straightforward, following these steps: write a barycentric equation for the line $U'$ through $X$ perpendicular ([3, p. 29], [12], [13]) to $U$, find barycentrics for the point $Q = U \cap U'$, and then find $R(X)$ as the translation of $Q$ by the vector $\overrightarrow{XQ}$.

Suppose that $ABC$ and $DEF$ are triangles, and $P = p : q : r$ is a point. A line $U$ through $P$ is a successful reflector through $P$ if the reflection $D'E'F'$ of $DEF$ in $U$ is perspective to $ABC$, in the sense that the lines $AD'$, $BE'$, $CF'$ concur. We shall sometimes say merely that a line is a successful reflector of a triangle $DEF$; in that case $P$ can be any point on the line.

A standard criterion for three lines $L_1, L_2, L_3$ given by

$$l_1\alpha + l_2\beta + l_3\gamma = 0,$$  
$$l_4\alpha + l_5\beta + l_6\gamma = 0,$$  
$$l_7\alpha + l_8\beta + l_9\gamma = 0,$$
to concur ([3, p. 28-29], [12], [13]) is that $\Delta = 0$, where

$$\Delta = \begin{vmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{vmatrix}. \tag{4}$$

We shall refer to $\Delta$ as a concurrence determinant of the three lines, and to the equation $\Delta = 0$ as a concurrence equation.

**Theorem 1** Suppose $ABC$ and $DEF$ are triangles, and $P = p : q : r$ is a point. Either there are infinitely many successful reflectors through $P$, or else there are at most four successful reflectors through $P$.

**Proof:** The line at infinity is given parametrically by

$$x_t = 1 + t \quad y_t = -t \quad z_t = -1,$$

where $t = t(a, b, c)$ is homogeneous of degree 0 in $a, b, c$. Let $U_t$ be the line $ua + v\beta + w\gamma = 0$ of the points $P$ and $x_t : y_t : z_t$, so that

$$u = -q + rt \quad v = p + r + rt \quad w = -q - pt - qt.$$

Represent the vertices of $DEF$ and $D'\alpha'F'\alpha = D_tE_tF_t$ as (the rows of) matrices

$$\begin{pmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{pmatrix},$$

respectively, so that

$$u_1 = f(a, b, c, u, v, w)d_1 + g(a, b, c, u, v, w)(vd_2 + wd_3)$$
$$u_2 = f(b, c, a, v, w, u)d_2 + g(b, c, a, v, w, u)(wd_3 + ud_1)$$
$$u_3 = f(c, a, b, w, u, v)d_3 + g(c, a, b, w, u, v)(ud_1 + vd_2)$$
$$u_4 = f(a, b, c, u, v, w)d_4 + g(a, b, c, u, v, w)(vd_5 + wd_6)$$
$$u_5 = f(b, c, a, v, w, u)d_5 + g(b, c, a, v, w, u)(ud_6 + vd_5)$$
$$u_6 = f(c, a, b, w, u, v)d_6 + g(c, a, b, w, u, v)(ud_5 + vd_5)$$
$$u_7 = f(a, b, c, u, v, w)d_7 + g(a, b, c, u, v, w)(vd_8 + wd_9)$$
$$u_8 = f(b, c, a, v, w, u)d_8 + g(b, c, a, v, w, u)(wd_9 + ud_7)$$
$$u_9 = f(c, a, b, w, u, v)d_9 + g(c, a, b, w, u, v)(ud_7 + vd_8).$$

We are interested in values of $t$ for which the lines $AD_t, BE_t, CF_t$ concur. Write these lines, respectively, as in (1). The concurrence equation $\Delta = 0$, with $\Delta$ as in (4), can now be written as

$$u_2u_6u_7 - u_3u_4u_8 = 0. \tag{5}$$

The left-hand side is a polynomial $P(t)$ formally of degree 6 in $t$. If for a particular choice of $P$ all the coefficients of powers of $t$ are zero, then for all $t$, the line joining the points $P$ and $x_t : y_t : z_t$ is a successful reflector through $P$. For the rest of this proof, assume that not all the coefficients are zero. Then, in spite of its hundreds of terms, the polynomial $P(t)$ is found by computer to have

$$c^2t^2 + (b^2 + c^2 - a^2)t + b^2$$

as a factor. The discriminant is $-16\sigma^2$, where $\sigma$ denotes the area of $ABC$. Thus, two of the roots of $P(t)$ are nonreal, so that $P(t)$ has at most four real roots. Consequently, there are at most four successful reflectors through $P$. \qed
3. Steiner axes

The isotomic conjugate of $\mathcal{L}^\infty$ is the Steiner ellipse, $\mathcal{E}$, also called the Steiner circumellipse in order to distinguish it from the Steiner inellipse; see [12] for details. A barycentric equation for $\mathcal{E}$ is

$$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0.$$  

This ellipse meets the circumcircle (which is the isogonal conjugate of $\mathcal{L}^\infty$) in these four points: $A$, $B$, $C$, and the Steiner point, indexed as $X_{99}$ in the Encyclopedia of Triangle Centers [4]. Barycentric equations for the major and minor axes of the Steiner ellipse are given in terms of the roots $\pm W$ of

$$W^2 = a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2$$  

and the functions

$$s_a = s(a, b, c) = (b^2 - c^2)(b^4 + c^4 - a^2b^2 - a^2c^2)$$
$$t_a = t(a, b, c) = (b^2 - c^2)(a^2 - b^2 - c^2)$$

as follows:

$$(s_a + t_aW)\alpha + (s_b + t_bW)\beta + (s_c + t_cW)\gamma = 0,$$  

$$(s_a - t_aW)\alpha + (s_b - t_bW)\beta + (s_c - t_cW)\gamma = 0,$$

respectively.

![Figure 1: Steiner major axis as a successful reflector](image)

Our colleague Paul Yiu, having seen a preliminary draft of Theorem 1 and an example involving the medial and anticomplementary triangles, noted that the reflectors are the Steiner axes, as proved in Theorems 2 and 3.

**Theorem 2** The medial triangle has exactly two successful reflectors through $G$. They are the Steiner axes of $ABC$.

**Proof:** A computer-assisted proof is straightforward using barycentrics. Let $U$ be an arbitrary line through $G$, and let $D'F'E'$ be the reflection in $U$ of the medial triangle $DEF$. Let $X = x : y : z$ be a point on $U$, so that $U$ is given by

$$u\alpha + v\beta + w\gamma = 0,$$
where \( u = y - z, v = z - x, w = x - y \). The concurrence determinant factors as \( C_1C_2^2 \), where

\[
C_1 = (b^2 - c^2)(x^2 + 2yz) + (c^2 - a^2)(y^2 + 2zx) + (a^2 - b^2)(z^2 + 2xy),
\]
\[
C_2 = f(a, b, c)x^2 + f(b, c, a)y^2 + f(c, a, b)z^2 + g(a, b, c)yz + g(b, c, a)zx + g(c, a, b)xy,
\]

where

\[
f(a, b, c) = a^2 - 2b^2 - 2c^2,
\]
\[
g(a, b, c) = 5a^2 - b^2 - c^2.
\]

Now, multiplying together the left-hand sides of (7) and (8), and substituting for \( W^2 \) as in (6), we find that the product, after cancellations, is \( C_1 \). That is, the conic \( C_1 = 0 \) is the union of the Steiner axes. Thus, regarding (4), for any \( X \) on either Steiner axis, we have \( \Delta = 0 \). The only other way to have \( \Delta = 0 \) is for \( X \) to lie on the conic \( C_2 = 0 \); however, this conic has center \( G \) and also passes through \( G \), so that it is entirely imaginary except for the single point \( G \).

The next theorem concerns the anticomplementary triangle. We digress for a moment to clarify just which triangle this is, because in recent literature, it is sometimes called the antimedial triangle or the precevian triangle of the centroid. It is the triangle whose medial triangle is the reference triangle \( ABC \). For example, its \( A \)-sideline is the line through \( A \) parallel to line \( BC \). The traditional name refers to the fact that the vertices are the anticomplements of \( A, B, C \). Correspondingly, the medial triangle is sometimes called the complementary triangle.

**Theorem 3** The anticomplementary triangle has exactly two successful reflectors through \( G \). They are the Steiner axes of \( ABC \).

**Proof:** A proof is nearly identical to that of Theorem 2.

The next theorem reveals a distinctive property of the 1st Brocard triangle [8]. The proof depends on the fact that two lines are parallel if and only if they meet in the same point on \( L^\infty \).

**Theorem 4** Every line parallel to the Steiner major axis or the Steiner minor axis is a successful reflector of the 1st Brocard triangle.

**Proof:** The vertices of the 1st Brocard triangle are given by

\[
\]

Let \( ua + vb + wc = 0 \) be an arbitrary line \( U \) and \( D', E', F' \) the reflections of \( D, E, F \) in \( U \). The concurrence determinant for the lines \( AD', BE', CF' \) factors as \( \hat{C}_1\hat{C}_2\hat{C}_3 \), where the conics \( \hat{C}_2 = 0 \) and \( \hat{C}_3 = 0 \) are imaginary except for the point \( G \). Regarding the first factor, however, we have

\[
\hat{C}_1 = (b^2 - c^2)(u^2 + 2vw) + (c^2 - a^2)(v^2 + 2wu) + (a^2 - b^2)(w^2 + 2uv),
\]

which is simply (9) with \( u, v, w \) substituted for \( x, y, z \). Thus, as in the proof of Theorem 2, the equation \( \hat{C}_1 = 0 \) holds if and only if the point \( u : v : w \) satisfies (7) or (8). One point
that satisfies (7) is the isogonal conjugate, $X_{1379}^{-1}$, of the point $X_{1379}$ of intersection of the Brocard axis and the circumcircle. Since $X_{1379}^{-1}$ is on $\mathcal{L}^\infty$, every line meeting $\mathcal{L}^\infty$ in $X_{1379}^{-1}$ is a successful reflector of $DEF$. One member of this family of parallel lines is the Steiner major axis.

Likewise, a point that satisfies (8) is the isogonal conjugate, $X_{1380}^{-1}$, of the other point of intersection of the Brocard axis and the circumcircle. Since $X_{1380}^{-1}$ is on $\mathcal{L}^\infty$, every line meeting $\mathcal{L}^\infty$ in $X_{1380}^{-1}$ is a successful reflector of $DEF$. One member of this family of parallel lines is the Steiner minor axis.

### 4. Pivots

Suppose that for a given triangle $DEF$ there is a point $P$ such that every line through $P$ is a successful reflector. Then $P$ is called a pivot for $DEF$. If $U$ is a line through such a pivot and $D'E'F'$ is the reflection of $DEF$ in $U$, then the perspector of $D'E'F'$ and $ABC$ can be regarded as a function of the line $U$. As $U$ rotates about $P$, the perspector traces the pivotal curve of $P$. We shall see that several well known triangles have pivots. (Indeed, by Theorem 4, we may view $X_{1379}^{-1}$ as a pivot for the 1st Brocard triangle, with the Steiner minor axis as pivotal curve. Likewise, $X_{1380}^{-1}$ is a pivot for the 1st Brocard triangle, with the Steiner major axis as pivotal curve.)

**Theorem 5** The incenter is a pivot for the intouch triangle. The pivotal curve is the isogonal conjugate of the incircle.

**Proof.** The intouch triangle is defined by its vertices $D, E, F$ given by the following homogeneous trilinear coordinates (or simply trilinears):

\[
D = 0 : \sec^2(B/2) : \sec^2(C/2) \\
E = \sec^2(A/2) : 0 : \sec^2(C/2) \\
F = \sec^2(A/2) : \sec^2(B/2) : 0.
\]
(Geometrically, $D$ is the orthogonal projection of the incenter on sideline $BC$, and likewise for $E$ and $F$.) Suppose that $U$ is an arbitrary line through the incenter $I$. It can be easily checked by computer that the reflection $D'E'F'$ about $U$ is perspective to $DEF$ and that the perspector is given by trilinears

$$\frac{\csc^2(A/2)}{(bz - cy)^2} : \frac{\csc^2(B/2)}{(cx - az)^2} : \frac{\csc^2(C/2)}{(ay - bx)^2},$$

(10)

where $X$ is any point on $U$ except $I$. The pivotal curve of $I$ is a quartic curve whose isogonal conjugate, remarkably, is the incircle. Specifically, the point

$$(bz - cy)^2 \sin^2(A/2) : (cx - az)^2 \sin^2(B/2) : (ay - bx)^2 \sin^2(C/2)$$

obtained from reciprocals of trilinears in (10) lies on the incircle, given by the equation

$$a_1^4 \alpha^4 + b_1^4 \beta^4 + c_1^4 \gamma^4 - 2b_1^2 c_1^2 \beta \gamma - 2c_1^2 a_1^2 \gamma \alpha - 2ab_1^2 b_1^2 \alpha \beta = 0,$$

where

$$a_1 = \cos(A/2), \quad b_1 = \cos(B/2), \quad c_1 = \cos(C/2).$$

It frequently happens in geometry that a family of lines forms an envelope called a deltoid (as at [6], an interactive Java applet). For example, the Steiner deltoid [11] is the envelope of the Simson lines of a triangle. As a supplement to Theorem 5, we note that if $Q$ is the isogonal conjugate of the perspector and $X$ goes around the incircle, then the envelope of the lines $XQ$ is a deltoid. This intouch deltoid, as shown in Fig. 3, is homothetic to the Steiner
The homothetic center is the Feuerbach point, labeled $Fe$, and the homothetic ratio is $2r/R$, where $r$ and $R$ are the radii of the incircle and the circumcircle, respectively.

The method of proof of Theorem 5 enables the discovery and confirmation of other pivots, as in the following examples. Barycentrics for perspectors are omitted because of their lengths.

**Example 1.** Let $DEF$ be the tangential triangle [7]. The circumcenter $O$ is a pivot, and the pivotal curve is the isogonal conjugate of the circle with center $H$ and radius $2R$, where $H$ denotes the orthocenter and $R$ the circumradius of $ABC$. (The isogonal conjugate of a circle is a curve of degree at most 4.)

![Figure 4: Euler triangle and pivotal curve](image)

**Example 2.** Let $DEF$ be the Euler triangle [9]. The orthocenter $H$ is a pivot, and the pivotal curve is the isogonal conjugate of the circle with center $O$ and radius $R/2$, as shown in Fig. 4. In this figure, as $X$ moves around any circle with center $H$, the line $HX$ is a successful reflector. The perspector $P$ of triangles $ABC$ and $D'E'F'$ traces the pivotal curve.

**Example 3.** Let $DEF$ be the hexyl triangle [10]. The incenter $I$ is a pivot, and the pivotal curve is the isogonal conjugate of the circle $\odot$ with center $I$ and radius $2R$ (so that $\odot$ is the circumcircle of the hexyl triangle).

**Example 4.** Let $DEF$ be the Gossard triangle [5]. The Zeeman-Gossard point $X_{402}$ is a pivot, and the perspector lies on $\mathcal{L}^\infty$.

**Example 5.** Let $DEF$ be the Kosnita triangle ([1], [2]), whose vertices are the circumcenters of the triangles $OBC$, $OCA$, $OAB$. The circumcenter $O$ is a pivot, and the pivotal curve is the isogonal conjugate of the circle having center $H$ and radius $R$. This circle passes through the point $X_{265}$, so that the isogonal conjugate of this point, namely $X_{186}$, lies on the pivotal curve. In [4], $X_{182}$ is identified as the inverse-in-circumcircle of $H$. 
5. Successful reflectors through the incenter

**Theorem 6** The cevian triangle \(DEF\) of a point \(X\) has four successful reflectors through the incenter, \(I\). Three of them are the lines \(AI, BI, CI\).

**Proof:** Let \(X = x : y : z\) be a point not on a sideline \(BC, CA, AB\). The cevian triangle \(DEF\) of \(X\) has vertices

\[
D = AX \cap BC, \quad E = BX \cap CA, \quad F = CX \cap AB.
\]

The determinant \(u_2u_6u_7 - u_3u_4u_8\) in (5) factors as

\[
(a + b + c)^4 (ct - b) (b + at + bt) (a + c + ct) (h - kt) (b^2 - a^2t + b^2t + c^2t + c^2t^2),
\]

where the linear factor \(h - kt\) is too long to be shown here. The other three factors that are linear in \(t\) give equations of the lines \(AI, BI, CI\). \(\square\)

**Example 6.** In Theorem 6, we take \(X = G\), so that \(DEF\) is the medial triangle. In this case, the successful reflector other than the lines \(AI, BI, CI\) is the line

\[
u(a, b, c)\alpha + u(b, c, a)\beta + u(c, a, b)\gamma = 0,
\]

where

\[
u(a, b, c) = (b + c - a)(b^2 + c^2 - 4bc + ab + ac),
\]

and the perspector, \(u_4u_7 : u_4u_8 : u_6u_7\), is the triangle center \(h(a, b, c) : h(b, c, a) : h(c, a, b)\), where

\[
h(a, b, c) = \frac{3a - b - c}{(2a - b - c)(b^2 + c^2 - 4bc + ba + ca)}.
\]

**Example 7.** As another illustration of Theorem 6, let \(X\) be the Nagel point, given in trilinears (as are all coordinates in this example) by

\[
X_8 = bc(b + c - a) : ca(c + a - b) : ab(a + b - c).
\]

The cevian triangle of \(X_8\) is known as the extouch triangle — whereas the intouch triangle in Theorem 5 has as vertices the touchpoints of the incircle on the sidelines \(BC, CA, AB\), the vertices of the extouch triangle are touchpoints of the excircles on the sidelines. Let

\[
g(a, b, c) = b^3 + c^3 + 2a^3 - 3a^2b - 3a^2c - 4ab^2 - 4ac^2 - b^2c - bc^2 + 12abc.
\]

The successful reflector other than the lines \(AI, BI, CI\) is the line

\[
l(a, b, c)\alpha + l(b, c, a)\beta + l(c, a, b)\gamma = 0
\]

given by

\[
l(a, b, c) = (b + c - a)\ g(a, b, c),
\]

and the perspector is the point \(p(a, b, c) : p(b, c, a) : p(c, a, b)\) given by

\[
p(a, b, c) = \frac{a(b + c - a)(a^2 + b^2 + c^2 + 2ab + 2ac - 6bc)}{(2a - b - c)g(a, b, c)}.
\]

Figure 5 shows the extouch triangle \(DEF\), its reflection \(D'E'F'\) in the successful reflector \(IJ\) (which is the trilinear polar of the point \(Q\)), and the perspector \(P\) in which the lines \(AD', BE', CF'\) concur.
6. Concluding remark

Unlike traditional geometry, the methods employed in this article are “symbolic” in two ways that bear comment: first, they depend largely on the factoring of determinants that are functions of the symbols $a, b, c$, and second, they are highly computer-dependent on expressions that are in some cases of great length. Consequently, one might feel that, although the results have geometric meaning, the methods leave something to be desired. The authors agree that traditional geometric proofs would be of interest. On the other hand, the proofs given here cover more ground, since $a, b, c$ need not be sidelengths of a traditional triangle.

References


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