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A Stronger Form of the Steiner-Lehmus Theorem

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Abstract. We give a purely synthetic proof of a more general version of the Steiner-Lehmus theorem.

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1. Introduction

The Steiner-Lehmus theorem states that if the internal angle-bisectors of two angles of a triangle are congruent, then the triangle is isosceles. Despite its apparent simplicity, the problem has proved more than challenging ever since 1840. For a complete historical overview, see [2] and also [3] and [1]. In this paper, we give a short and purely synthetic proof of a more general statement.

2. The Main Theorem

We start with a simple lemma that will be used to prove the main theorem:

Lemma 1 *In the triangle ABC, let the two cevians BB' and CC' intersect at P. Then $BB' = CC'$ implies $PB' < PC$ and $PC' < PB$.*

Proof: Suppose that $PB' \geq PC$. Since $BB' = CC'$, it follows that $PC' \geq PB$. Therefore

$$\begin{aligned} \angle B'CP &\geq \angle CB'P, && \text{because } PB' \geq PC \\ &> \angle ABB', && \text{by the exterior angle theorem} \\ &\geq \angle PC'B, && \text{because } PC' \geq PB \\ &> \angle B'CP, && \text{by the exterior angle theorem.} \end{aligned}$$

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Thus we reach the contradiction $\angle B'CP > \angle B'CP$. Therefore $PB' < PC$. Similarly $PC' < PB$. □

The main result will be now split in two parts.

Theorem 1 *Let A' be the foot of the internal angle-bisector of the angle BAC of a given triangle ABC. Consider an arbitrary point P on the ray AA', different from A', and denote by B', C' the intersections of the lines BP, CP with the sidelines CA and AB, respectively. Then $BB' = CC'$ implies $AB = AC$.*

Proof: Erect a triangle $C'XC$ on the segment CC', that is congruent to the triangle BAB', and such that the points B and X do not lie on the same side of AC (see Fig. 1). We conclude that the angles $\angle C'AC$ and $\angle C'XC$ are equal, and thus the quadrilateral $C'AXC$ is cyclic, which means that $\angle CAX = \angle CC'X$. On the other hand, the angles $\angle CC'X$ and $\angle B'BA$ are equal, and therefore, $\angle CAX = \angle B'BA$.

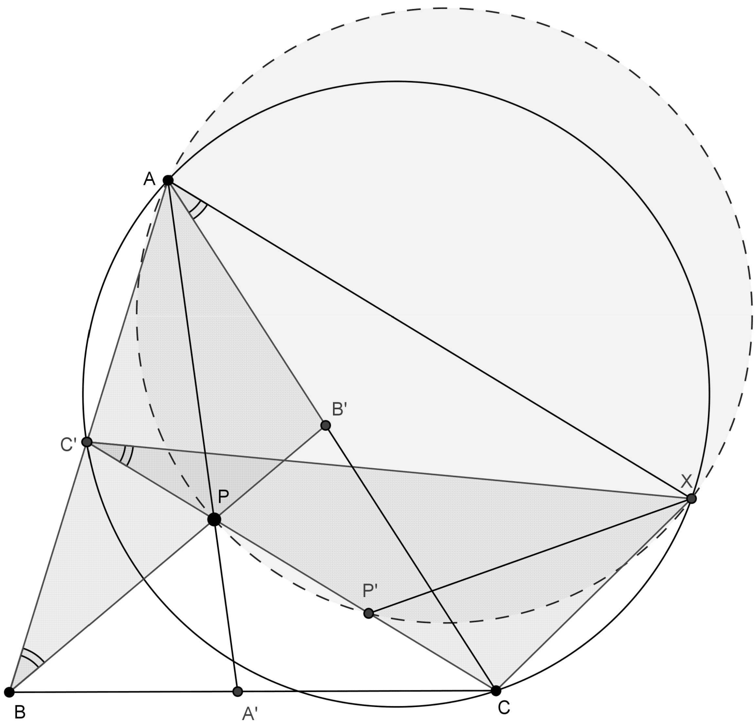

Figure 1: Proving Theorem 1

Let P' be the foot of the internal angle-bisector of the angle $C'XC$ in triangle $C'XC$. Since triangles $C'XC$ and BAB' are congruent, the previous Lemma 1 yields $CP' = B'P < CP$, which means that P' lies between C and C'. Moreover,

$$\angle CP'X = \angle B'PA = \angle BAP + \angle B'BA = \angle PAC + \angle CAX = \angle PAX.$$

From this we deduce that the quadrilateral $AXP'P$ is cyclic, and plus, since the segments AP and XP' are congruent, the quadrilateral $AXP'P$ is an isosceles trapezoid, and thus, we conclude that the lines AX and CC' are parallel. It now follows that $\angle CAX = \angle ACC'$, and hence, $\angle B'BA = \angle ACC'$. From this and the assumption $BB' = CC'$ we conclude that the triangles ABB' and ACC' are congruent, and therefore $AB = AC$. □

Theorem 2 *Let A' be the foot of the internal angle-bisector of the angle BAC of a given triangle ABC. Consider a point P on the ray AA' beyond A', and denote by B', C' the intersections of the lines BP, CP, with the sidelines CA and AB, respectively. Then $BB' = CC'$ implies $AB = AC$.*

Proof: Let A'' be the intersection of AA' with $B'C'$. It follows from Theorem 1 (applied to the triangle $AC'B'$) that $AC' = AB'$. It also follows that A'' is the midpoint of $B'C'$. By Ceva's theorem, we obtain $AB/BC' = AC/CB'$ and therefore $BC \| C'B'$. Thus $AB = AC$, as desired. □

Combining Theorems 1 and 2, we can now state the stronger version of the Steiner-Lehmus theorem:

Theorem 3 (Main Theorem) *Let A' be the foot of the internal angle-bisector of the angle BAC of a given triangle ABC. Consider P an arbitrary point on the ray AA', different from A', and denote by B', C' the intersections of the lines BP, CP, with the sidelines CA, and AB, respectively. Then $BB' = CC'$ implies $AB = AC$.*

Obviously, when P coincides with the incenter I of the triangle ABC, the Main Theorem reduces to the Steiner-Lehmus theorem.

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