A Stronger Form of the Steiner-Lehmus Theorem

Virgil Nicula¹, Cosmin Pohoată²

¹10 Armenis Street, Bucharest 032483, Romania
email: levinicula@yahoo.com

²13 Pridvorului Street, Bucharest 41202, Romania
email: pohoata_cosmin2000@yahoo.com

Abstract. We give a purely synthetic proof of a more general version of the Steiner-Lehmus theorem.

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1. Introduction

The Steiner-Lehmus theorem states that if the internal angle-bisectors of two angles of a triangle are congruent, then the triangle is isosceles. Despite its apparent simplicity, the problem has proved more than challenging ever since 1840. For a complete historical overview, see [2] and also [3] and [1]. In this paper, we give a short and purely synthetic proof of a more general statement.

2. The Main Theorem

We start with a simple lemma that will be used to prove the main theorem:

Lemma 1 In the triangle $ABC$, let the two cevians $BB'$ and $CC'$ intersect at $P$. Then $BB' = CC'$ implies $PB' < PC$ and $PC' < PB$.

Proof: Suppose that $PB' \geq PC$. Since $BB' = CC'$, it follows that $PC' \geq PB$. Therefore

\[ \angle B'CP \geq \angle CB'P, \quad \text{because} \quad PB' \geq PC \]

\[ > \angle ABB', \quad \text{by the exterior angle theorem} \]

\[ \geq \angle PC'B, \quad \text{because} \quad PC' \geq PB \]

\[ > \angle B'CP, \quad \text{by the exterior angle theorem}. \]
Thus we reach the contradiction $\angle B'CP > \angle B''CP$. Therefore $PB' < PC$. Similarly $PC' < PB$. 

The main result will be now split in two parts.

**Theorem 1** Let $A'$ be the foot of the internal angle-bisector of the angle $BAC$ of a given triangle $ABC$. Consider an arbitrary point $P$ on the ray $AA'$, different from $A'$, and denote by $B'$, $C'$ the intersections of the lines $BP$, $CP$ with the sidelines $CA$ and $AB$, respectively. Then $BB' = CC'$ implies $AB = AC$.

**Proof:** Erect a triangle $C'XC$ on the segment $CC'$, that is congruent to the triangle $BAB'$, and such that the points $B$ and $X$ do not lie on the same side of $AC$ (see Fig. 1). We conclude that the angles $\angle C'AC$ and $\angle C'XC$ are equal, and thus the quadrilateral $C'AXC$ is cyclic, which means that $\angle CAX = \angle C'CX$. On the other hand, the angles $\angle CC'X$ and $\angle B'BA$ are equal, and therefore, $\angle CAX = \angle B'BA$.

Let $P'$ be the foot of the internal angle-bisector of the angle $C'XC$ in triangle $C'XC$. Since triangles $C'XC$ and $BAB'$ are congruent, the previous Lemma 1 yields $CP' = B'P < CP$, which means that $P'$ lies between $C$ and $C'$. Moreover,

$$\angle C'PX = \angle B'PA = \angle BAP + \angle B'BA = \angle PAC + \angle CAX = \angle PAX.$$ 

From this we deduce that the quadrilateral $AXP'P$ is cyclic, and plus, since the segments $AP$ and $XP'$ are congruent, the quadrilateral $AXP'P$ is an isosceles trapezoid, and thus, we conclude that the lines $AX$ and $CC'$ are parallel. It now follows that $\angle CAX = \angle ACC'$, and hence, $\angle B'BA = \angle ACC'$. From this and the assumption $BB' = CC'$ we conclude that the triangles $ABB'$ and $ACC'$ are congruent, and therefore $AB = AC$. 

\[\square\]
Theorem 2 Let $A'$ be the foot of the internal angle-bisector of the angle $BAC$ of a given triangle $ABC$. Consider a point $P$ on the ray $AA'$ beyond $A'$, and denote by $B'$, $C'$ the intersections of the lines $BP$, $CP$, with the sidelines $CA$ and $AB$, respectively. Then $BB' = CC'$ implies $AB = AC$.

Proof: Let $A''$ be the intersection of $AA'$ with $B'C'$. It follows from Theorem 1 (applied to the triangle $AC'B'$) that $AC' = AB'$. It also follows that $A''$ is the midpoint of $B'C'$. By Ceva’s theorem, we obtain $AB/BC' = AC/CB'$ and therefore $BC || C'B'$. Thus $AB = AC$, as desired.

Combining Theorems 1 and 2, we can now state the stronger version of the Steiner-Lehmus theorem:

Theorem 3 (Main Theorem) Let $A'$ be the foot of the internal angle-bisector of the angle $BAC$ of a given triangle $ABC$. Consider $P$ an arbitrary point on the ray $AA'$, different from $A'$, and denote by $B'$, $C'$ the intersections of the lines $BP$, $CP$, with the sidelines $CA$, and $AB$, respectively. Then $BB' = CC'$ implies $AB = AC$.

Obviously, when $P$ coincides with the incenter $I$ of the triangle $ABC$, the Main Theorem reduces to the Steiner-Lehmus theorem.

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References


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