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Generalized Arbelos in Aliquot Parts: Non-Intersecting Case

Hiroshi Okumura[1], **Masayuki Watanabe**[2]

[1]*Dept. of Life Science and Information, Maebashi Institute of Technology*
[2]*Dept. of Integrated Design Engineering, Maebashi Institute of Technology*
460-1 Kamisadori Maebashi, Gunma 371-0816, Japan
emails: okumura@maebashi-it.ac.jp, mwatanabe@maebashi-it.ac.jp

Abstract. We will extend the results on arbelos in [3] and [4] to the coaxial system of non-intersecting type. We get almost the same results except the ones about the embedded patterns that are limited in this case.

Key Words: Arbelos, Archimedean circles
MSC2007: 51M04

1. Introduction and preliminaries

Let α, β and γ be three circles such that the centers of these circles are collinear and α and β are inside and touching γ at different points. If α and β are tangent, the area bounded by these three circles is the usual arbelos. When α and β intersects at two points, we called the configuration of these three circles a *generalized arbelos of intersecting type* in [4]. In these two cases we studied an arbelos in n-aliquot parts, that is, an arbelos with $n-1$ members of the coaxial system generated by α and β such that the inscribed circles in the area divided by two of these circles and γ are congruent (see [3] and [4]). In this paper we study the case that the circles α and β are not intersecting. We call such configuration of α, β and γ a generalized arbelos of *non-intersecting type* (see Fig. 1). We do not avoid the case when α or β is a point. When α (resp. β) is a point, "a circle touches α (resp. β)" means "it passes through α (resp. β)".

Let Γ be a coaxial system of non-intersecting type with limiting points L and L', and E be a point on the line passing through the centers of circles in Γ. As in [4] we call the pair (Γ, E) a *coaxial system* with a *fixed point*. Throughout this paper except the final section, we take the line passing through the centers as the x-axis and the radical axis as the y-axis whenever a coaxial system with a fixed point (Γ, E) is given. Then the x-coordinate of L and L' are ℓ and $-\ell$ for some $\ell \in \mathbb{R}$. We choose L and L' such that ℓ is positive and denote by I the closed interval $\{x \in \mathbb{R} \mid -\ell \le x \le \ell\}$.

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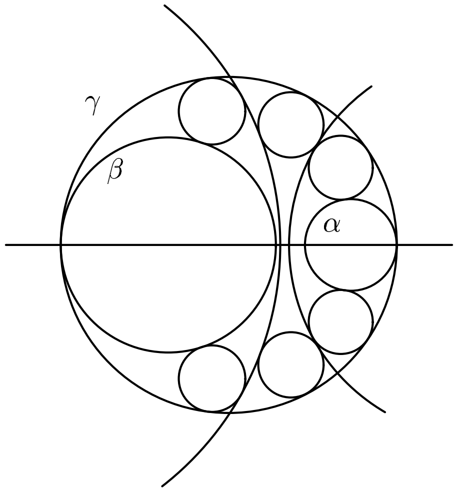

Figure 1: A generalized arbelos of non-intersecting type for $n = 3$

Let e be the x-coordinate of E. Any member $\alpha \in \Gamma$ meets the segment I in a single point. We denote the x-coordinate of this point by a and define the value $\mu(\alpha)$ as

$$\mu(\alpha) = \begin{cases} \dfrac{1}{a-e} & \text{if } \ |e| = \ell\,, \\[2ex] \dfrac{a-e+\sqrt{e^2-\ell^2}}{a-e-\sqrt{e^2-\ell^2}} & \text{if } \ |e| \neq \ell\,. \end{cases}$$

Let f denote the absolute value $|\sqrt{e^2-\ell^2}|$. When $|e| < \ell$, we consider that $\sqrt{e^2-\ell^2} = fi$ where i is the imaginary unit. Note that the value $\mu(\alpha)$ depends on the choice of the point E, but not on the choice of the coordinate system.

Since the radical axis is the y-axis, a member $\alpha \in \Gamma$ is the radical axis if and only if

$$\mu(\alpha) = \begin{cases} -\dfrac{1}{e} & \text{if } \ |e| = \ell\,, \\[2ex] \dfrac{e-\sqrt{e^2-\ell^2}}{e+\sqrt{e^2-\ell^2}} & \text{if } |e| \neq \ell\,. \end{cases}$$

In the case $|e| > \ell$, the value $\mu(\alpha)$ has the geometric meaning as follows:

Let ε be the circle with center E and radius f. Note that the circle ε is the member of Γ. It intersects the x-axis at two points which we denote by E^+ and E^-. We assume that the x-coordinate of E^+ is larger than that of E^-. Then $E^- \in I$ and $E^+ \notin I$ if $e > 0$, $E^- \notin I$ and $E^+ \in I$ if $e < 0$ and we have $\mu(\alpha) = \dfrac{AE^-}{AE^+}$, where A is the intersection point of α and the x-axis in I and we consider AE^+ (resp. AE^-) positive if A is left to E^+ (resp. E^-), and negative if A is right to E^+ (resp. E^-). If $\alpha = \varepsilon$ and $e > 0$, we have $A = E^-$ and then $\mu(\alpha) = 0$. If $\alpha = \varepsilon$ and $e < 0$, we have $A = E^+$ and we regard $\mu(\alpha) = \infty$. If $\alpha \neq \varepsilon$, the value $\mu(\alpha)$ is positive if and only if the member $\alpha \in \Gamma$ is outside the circle ε (see Fig. 2).

If $|e| < \ell$, $\mu(\alpha)$ is a complex number with $|\mu(\alpha)| = 1$. Since $f > 0$, we can assume that $0 < \arg(a-e+fi) < \pi$. Let $\omega(\alpha) = \arg(a-e+fi)$. Then $\mu(\alpha) = \exp(2\omega(\alpha)i)$ and $0 < \omega(\alpha) < \pi$.

In the following sections, we use the above notations.

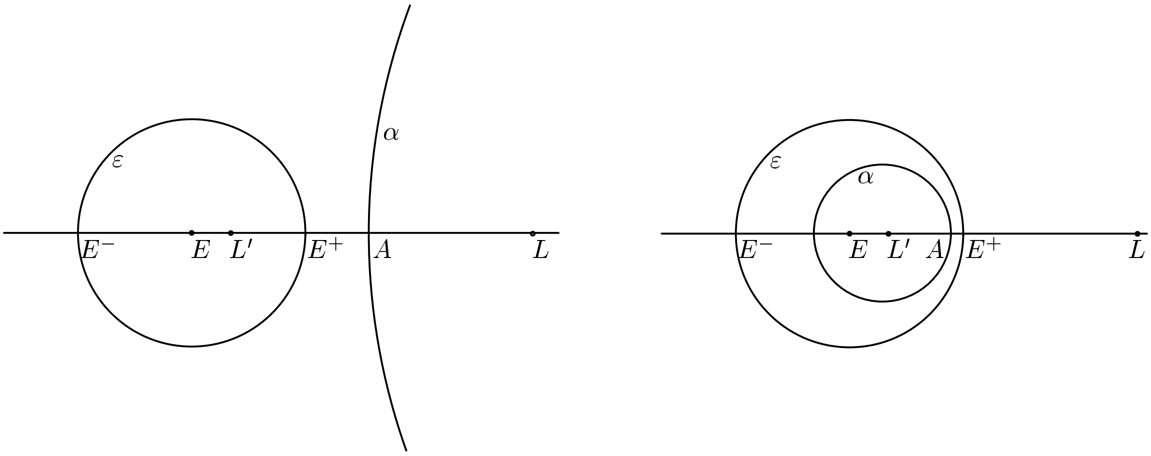

Figure 2: (a) $\mu(\alpha) > 0$ (b) $\mu(\alpha) < 0$

2. Incircles

Let (Γ, E) be a coaxial system with a fixed point and let γ be a circle with the center E. Let g be the radius of γ and assume that $g \geq |e| + \ell$. Then L and L' are either inside or on the circle γ respectively.

Let α and β be members in Γ intersecting or touching γ. They can degenerate when $g = |e| + \ell$. There exists a circle which is inside γ and touching α, β and γ at different points. We call such a circle an incircle of α and β in γ. If neither α nor β is a point, there are two incircles which are symmetric with respect to the x-axis and have the same radii. If α or β is a point, there is one incircle whose center is on the x-axis. If both α and β are points, the incircle coincides with γ.

Lemma 1 *Let a (resp. b) be the x-coordinate of the intersection point of α ($reps.$ β) and the x-axis in I. Let s and t be the x-coordinates of intersection points of γ and the x-axis with $t < s$. If $b < a$, the radius of the incircle of α and β in γ is*

$$\frac{(\ell^2 - st)(a-b)}{2(\ell^2 + ab - at - bs)}.$$

Proof: If α is a point, then $s = a = \ell$. If β is a point, then $t = b = -\ell$. In these cases the radius of the incircle is $(a-b)/2$, so the result follows.

Suppose neither α nor β is a point. Then $t < b < a < s$, and if α (resp. β) is a circle the x-coordinate of the intersection point of α (resp. β) and the x-axis different from $(a, 0)$ (resp. $(b, 0)$) is ℓ^2/a (resp. ℓ^2/b). The inversion in the virtual circle with center $(b, 0)$ and radius $\sqrt{b^2 - \ell^2} = \sqrt{\ell^2 - b^2} \cdot i$ maps α and γ to circles $\overline{\alpha}$ and $\overline{\gamma}$ with centers on the x-axis and maps β to the y-axis which we denote by $\overline{\beta}$. Let m, n be the x-coordinates of the intersection points of $\overline{\alpha}$ and the x-axis with $m < n$, and p, q be the x-coordinates of the intersection points of $\overline{\gamma}$ and the x-axis with $p < q$. Then we have

$$m = \frac{ab - \ell^2}{a - b}, \quad n = \frac{\ell^2(a-b)}{ab - \ell^2}, \quad p = \frac{bs - \ell^2}{s - b}, \quad q = \frac{bt - \ell^2}{t - b}. \tag{1}$$

Incircles are mapped to congruent circles tangent to the circles $\overline{\alpha}$ and $\overline{\gamma}$ externally and tangent to the line $\overline{\beta}$ from the left. We denote one of such circles by $\overline{\mathcal{C}}$. We know immediately that $m < p \leq n < 0 \leq q$ and $p < b < q$, so the center of the inversion $(b, 0)$ is outside $\overline{\mathcal{C}}$ (Fig. 3).

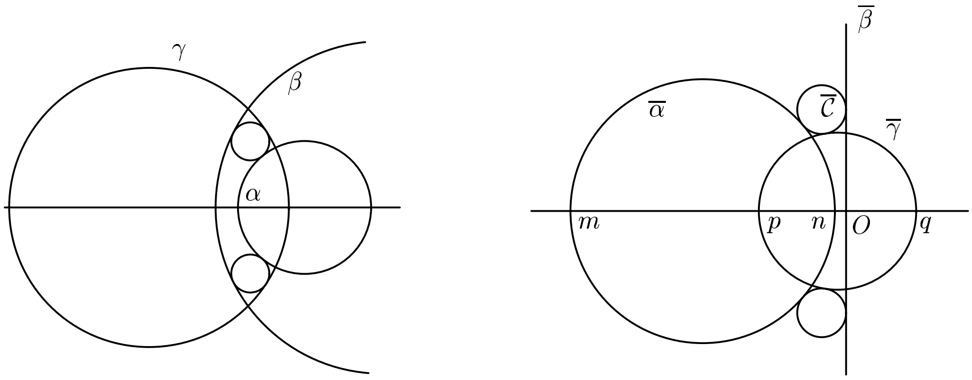

Figure 3: A configuration of α, β, γ and its inverted image

Let (x, y) be the center of $\overline{\mathcal{C}}$. Then x is a negative number and the radius of $\overline{\mathcal{C}}$ is $-x$. Since the circle $\overline{\mathcal{C}}$ is tangent to $\overline{\alpha}$ and $\overline{\gamma}$ externally, we have

$$\left(\frac{m+n}{2} - x\right)^2 + y^2 = \left(\frac{n-m}{2} - x\right)^2 \quad \text{and} \quad \left(\frac{p+q}{2} - x\right)^2 + y^2 = \left(\frac{q-p}{2} - x\right)^2,$$

and then we have

$$x = \frac{mn - pq}{2(m-p)} \quad \text{and} \quad y^2 = \frac{mp(n-q)}{m-p}. \tag{2}$$

Since the incircle $\mathcal{C}$ of α and β in γ is the inverted image of $\overline{\mathcal{C}}$ and the center of the inversion is outside $\overline{\mathcal{C}}$, the radius of $\mathcal{C}$ is

$$\frac{(b^2 - \ell^2)x}{|(x-b)^2 + y^2 - x^2|} = \frac{(b^2 - \ell^2)x}{b^2 - 2bx + y^2}.$$

Then the relations (1) and (2) imply the above formula for the radius of the incircle. □

Remark: We use this Lemma under the assumption $g \geq |e| + \ell$. But the result holds without this assumption if α and β intersect or touch γ. In the case that β is a circle and the point $(\ell^2/b, 0)$ is inside γ, the configuration of the inverted image is different from the one in Fig. 3 but the equations with respect to x and y are the same as in the proof.

Let $\{\alpha, \beta, \gamma\}$ be a generalized arbelos of non-intersecting type. Let E be the center of the circle γ and let (Γ, E) denote the coaxial system with a fixed point, where Γ is the coaxial system generated by the circles α and β. Note that the radius of γ is equal to or larger than $|e| + \ell$ since the limiting points L and L' are inside or on the circle γ. Let n be a natural number. We say that the configuration of figures $\{\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta, \gamma\}$ is a generalized arbelos of non-intersecting type in n-aliquot parts if α_j is a member of the coaxial system Γ which intersects the circle γ for each $j = 1, 2, \cdots, n-1$ and the incircles of α_{j-1} and α_j in γ are all congruent for $j = 1, 2, \cdots, n$. In this paper we shorten this long name to the *generalized arbelos in n-aliquot parts*. Each congruent incircle from this definition is called the *Archimedean circle* in n-aliquot parts.

Let g denote the radius of the circle γ. Then γ intersects the x-axis in the points $(e+g, 0)$ and $(e-g, 0)$. We choose the circles α and β such that they touch the circle γ at $(e+g, 0)$

and $(e-g,0)$, respectively. Then if $|e|\neq\ell$, we have

$$\mu(\alpha)=\frac{\dfrac{\ell^2}{e+g}-e+\sqrt{e^2-\ell^2}}{\dfrac{\ell^2}{e+g}-e-\sqrt{e^2-\ell^2}}=\frac{\left(g-\sqrt{e^2-\ell^2}\right)\left(e-\sqrt{e^2-\ell^2}\right)}{\left(g+\sqrt{e^2-\ell^2}\right)\left(e+\sqrt{e^2-\ell^2}\right)}\quad\text{and}$$

$$\mu(\beta)=\frac{\left(\dfrac{\ell^2}{e-g}-e+\sqrt{e^2-\ell^2}\right)}{\left(\dfrac{\ell^2}{e-g}-e-\sqrt{e^2-\ell^2}\right)}=\frac{\left(g+\sqrt{e^2-\ell^2}\right)\left(e-\sqrt{e^2-\ell^2}\right)}{\left(g-\sqrt{e^2-\ell^2}\right)\left(e+\sqrt{e^2-\ell^2}\right)}.$$

If $|e|=\ell$, we have

$$\mu(\alpha)=\frac{1}{\dfrac{\ell^2}{e+g}-e}=-\frac{e+g}{eg}\quad\text{and}\quad\mu(\beta)=\frac{1}{\dfrac{\ell^2}{e-g}-e}=\frac{e-g}{eg}.$$

Theorem 1 *Let $\alpha_1,\alpha_2,\cdots,\alpha_{n-1}$ be members in Γ intersecting γ. Then $\{\alpha=\alpha_0,\ \alpha_1,\ \cdots,\alpha_n=\beta,\ \gamma\}$ is a generalized arbelos in n-aliquot parts if and only if*

$$\mu(\alpha_0),\mu(\alpha_1),\cdots,\mu(\alpha_n)\ \textit{is}\ \begin{cases}\textit{an arithmetic sequence} & \textit{if } |e|=\ell\,,\\ \textit{a geometric sequence} & \textit{if } |e|\neq\ell\,.\end{cases}$$

Proof: Let a_j be the x-coordinate of the intersection point of α_j and the x-axis in I. By Lemma 1 the radius of the incircle of α_{j-1} and α_j in γ is

$$\frac{\left(\ell^2-(e+g)(e-g)\right)(a_{j-1}-a_j)}{2\left(\ell^2+a_{j-1}a_j-a_{j-1}(e-g)-a_j(e+g)\right)}=\frac{(\ell^2-e^2+g^2)(a_{j-1}-a_j)}{2\left(\ell^2-e^2+(a_{j-1}-e)(a_j-e)+g(a_{j-1}-a_j)\right)}.$$

First assume $|e|=\ell$. Then the radius of the incircle is

$$\frac{g^2(a_{j-1}-a_j)}{2\left((a_{j-1}-e)(a_j-e)+g(a_{j-1}-a_j)\right)}=\frac{g^2\left(\dfrac{1}{a_j-e}-\dfrac{1}{a_{j-1}-e}\right)}{2\left(1+g\left(\dfrac{1}{a_j-e}-\dfrac{1}{a_{j-1}-e}\right)\right)}$$
$$=\frac{g^2\left(\mu(\alpha_j)-\mu(\alpha_{j-1})\right)}{2\left(1+g\left(\mu(\alpha_j)-\mu(\alpha_{j-1})\right)\right)}.$$

So the incircles are all congruent if and only if the values $\mu(\alpha_j)-\mu(\alpha_{j-1})$ are constant for all $j=1,2,\cdots,n$, since the map $x\mapsto\dfrac{g^2x}{2(1+gx)}$ is injective.

Assume $|e|\neq\ell$. Since

$$\frac{(\ell^2-e^2+g^2)\left(1-\dfrac{\mu(\alpha_j)}{\mu(\alpha_{j-1})}\right)}{2\left(\left(g-\sqrt{e^2-\ell^2}\right)-\left(g+\sqrt{e^2-\ell^2}\right)\dfrac{\mu(\alpha_j)}{\mu(\alpha_{j-1})}\right)}$$
$$=\frac{(\ell^2-e^2+g^2)\left(\dfrac{a_{j-1}-e+\sqrt{e^2-\ell^2}}{a_{j-1}-e-\sqrt{e^2-\ell^2}}-\dfrac{a_j-e+\sqrt{e^2-\ell^2}}{a_j-e-\sqrt{e^2-\ell^2}}\right)}{2\left(\left(g-\sqrt{e^2-\ell^2}\right)\dfrac{a_{j-1}-e+\sqrt{e^2-\ell^2}}{a_{j-1}-e-\sqrt{e^2-\ell^2}}-\left(g+\sqrt{e^2-\ell^2}\right)\dfrac{a_j-e+\sqrt{e^2-\ell^2}}{a_j-e-\sqrt{e^2-\ell^2}}\right)}$$
$$=\frac{(\ell^2-e^2+g^2)(a_{j-1}-a_j)}{2\left(\ell^2-e^2+(a_{j-1}-e)(a_j-e)+g(a_{j-1}-a_j)\right)}$$

and the map

$$x \mapsto \frac{(\ell^2 - e^2 + g^2)(1-x)}{2\left(\left(g - \sqrt{e^2-\ell^2}\right) - \left(g + \sqrt{e^2-\ell^2}\right)x\right)}$$

is injective, the incircles are all congruent if and only if the values $\frac{\mu(\alpha_j)}{\mu(\alpha_{j-1})}$ are constant for all $j = 1, 2, \cdots, n$. Then the result follows. □

Corollary 1 *If n, m and l are positive integers with $n = ml$ and $\{\alpha = \alpha_0,\ \alpha_1, \cdots \alpha_n = \beta,\ \gamma\}$ is a generalized arbelos in n-aliquot parts. Then $\{\alpha = \alpha_0,\ \alpha_l,\ \alpha_{2l}, \cdots \alpha_{ml} = \beta,\ \gamma\}$ is a generalized arbelos in m-aliquot parts.*

The further arguments differ a little whether $|e| = \ell$, $|e| > \ell$ or $|e| < \ell$, that is, if the point E is one of the limiting points, is outside the segment I or is in its interior. In the following sections we consider these three cases separately.

In any case, let $2u = g + \sqrt{e^2 - \ell^2}$ and $2v = g - \sqrt{e^2 - \ell^2}$. Note that $u = v = \frac{1}{2}g$ when $|e| = \ell$ and that v is the conjugate of the complex number u when $|e| < \ell$.

3. The case $|e| = \ell$

Theorem 2 *Let $\{\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta, \gamma\}$ be a generalized arbelos in n-aliquot parts. Then the radius of the Archimedean circle in n-aliquot parts is $\frac{2u}{n+2}$.*

Proof: Since $\mu(\alpha) = -\frac{e+g}{eg}$ and $\mu(\beta) = \frac{e-g}{eg}$, the common difference of the arithmetic sequence $\mu(\alpha_0), \mu(\alpha_1), \cdots, \mu(\alpha_n)$ is $\frac{2}{ng}$. By the proof of Theorem 1 the radius is

$$\frac{\frac{2}{ng}g^2}{2\left(1 + \frac{2}{ng}g\right)} = \frac{g}{n+2} = \frac{2u}{n+2}.$$ □

Remark: The above result also holds when $\ell = 0$ (see [3]).

Corollary 2 *The member α_j in the generalized arbelos in n-aliquot parts is the radical axis of Γ if and only if $2j = n$.*

Proof: Since $\mu(\alpha_j) = \mu(\alpha) + \frac{2j}{ng} = -\frac{e+g}{eg} + \frac{2j}{ng}$, α_j is the radical axis of Γ if and only if $-\frac{e+g}{eg} + \frac{2j}{ng} = -\frac{1}{e}$. This is equivalent to $n = 2j$. □

Theorem 3 *Let $\{\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta, \gamma\}$ be a generalized arbelos in n-aliquot parts. Then there exists a circle γ' concentric to γ and tangent to all Archimedean circles in n-aliquot parts externally.*
The radius g' of γ' satisfies $(n+2)g' = ng$.
There exist two members α' and β' of the coaxial system Γ such that $\{\alpha', \alpha_0, \cdots, \alpha_n, \beta', \gamma'\}$ is a generalized arbelos in $(n+2)$-aliquot parts if and only if $g' \geq 2\ell$.

Remark: The circles α' and β' can be degenerate.

Proof: Let r be the radius of Archimedean circles in n-aliquot parts in $\{\alpha_0, \alpha_1, \cdots, \alpha_n, \gamma\}$. Since we have $g - 2r = \frac{ng}{n+2} > 0$, no Archimedean circles in n-aliquot parts contain the center of γ and we can draw a circle γ' with center E and radius $g' = \frac{ng}{n+2}$ which touches all Archimedean circles in n-aliquot parts externally.

If $g' \geq 2\ell$, the points L and L' are inside or on the circle γ' and there exist two circles α' and β' in the coaxial system Γ that touch from inside the circle γ'. We choose α' and β' such that α' touches γ' at $(e+g', 0)$ and β' touches γ' at $(e-g', 0)$. Since $\frac{2}{ng} = \frac{2}{(n+2)g'}$, we have

$$\mu(\alpha') = \frac{1}{\dfrac{\ell^2}{e+g'} - e} = -\frac{e+g}{eg} - \frac{2}{ng} = \mu(\alpha_0) - \frac{2}{(n+2)g'},$$

$$\mu(\beta') = \frac{1}{\dfrac{\ell^2}{e-g'} - e} = \frac{e-g}{eg} + \frac{2}{ng} = \mu(\alpha_n) + \frac{2}{(n+2)g'},$$

so that the sequence $\mu(\alpha'), \mu(\alpha_0), \mu(\alpha_1), \cdots, \mu(\alpha_n), \mu(\beta')$ is arithmetic and $\{\alpha', \alpha_0, \cdots, \alpha_n, \beta', \gamma'\}$ is a generalized arbelos in $(n+2)$-aliquot parts by Theorem 1.

If $g' < 2\ell$, one of the limiting points is outside γ' and there is only one circle in the coaxial system Γ that touches from inside the circle γ'. □

Let $\{\alpha, \beta, \gamma\}$ be a generalized arbelos of non-intersecting type, let α_j $(j \in \mathbb{Z})$ be a member of the coaxial system Γ generated by the circles α and β and let γ_j $(j \in \mathbb{Z})$ be a circle congruent to the circle γ. We call the configuration of figures

$$\{\cdots, \alpha_{-n}, \alpha_{-(n-1)}, \cdots, \alpha_{-1} = \alpha, \beta = \alpha_1, \cdots, \alpha_n, \cdots, \gamma_1 = \gamma, \gamma_3, \cdots, \gamma_{2n-1}, \cdots\}$$

the embedded pattern of generalized arbelos of *odd type* if the configuration $\{\alpha_{-n}, \alpha_{-(n-1)}, \cdots, \alpha_{-1}, \alpha_1, \cdots, \alpha_n, \gamma_{2n-1}\}$ is a generalized arbelos in $(2n-1)$-aliquot parts for any positive integer n. Also we call the configuration of figures

$$\{\cdots, \alpha_{-n}, \alpha_{-(n-1)}, \cdots, \alpha_{-1} = \alpha, \alpha_0, \beta = \alpha_1, \cdots, \alpha_n, \cdots, \gamma_2 = \gamma, \gamma_4, \cdots, \gamma_{2n}, \cdots\}$$

the embedded pattern of generalized arbelos of *even type* if the configuration $\{\alpha_{-n}, \alpha_{-(n-1)}, \cdots, \alpha_{-1}, \alpha_0, \alpha_1, \cdots, \alpha_n, \gamma_{2n}\}$ is a generalized arbelos in $2n$-aliquot parts for any positive integer n.

As in [3] and [4], we can make two types of embedded patterns of arbelos. However, in the present situation we do not get infinite families of circles.

Theorem 4 *There exists an embedded pattern of odd type*

$$\{\alpha_{-n}, \alpha_{-(n-1)}, \cdots, \alpha_{-1} = \alpha, \alpha_1 = \beta, \cdots, \alpha_n, \gamma_1 = \gamma, \gamma_3, \cdots, \gamma_{2n-1}\}$$

if and only if $2n - 1 \leq \frac{g}{2\ell}$. *There exists an embedded pattern of even type*

$$\{\beta_{-n}, \beta_{-(n-1)}, \cdots, \beta_{-1} = \alpha,\ \beta_0,\ \beta_1 = \beta, \cdots, \beta_n, \gamma_2 = \gamma, \gamma_4, \cdots, \gamma_{2n}\}$$

if and only if $2n \leq \frac{g}{\ell}$.

Proof: By Theorem 3 the circle γ_n exists provided there exists a generalized arbelos in $(n-2)$-aliquot parts. Moreover, there exists a generalized arbelos in n-aliquot parts with γ_n as an outer circle if and only if its radius g_n satisfies $g_n \geq 2\ell$. Also we have

$$ng_n = (n-2)g_{n-2} = \cdots = \begin{cases} g_1 = g & \text{if } n \text{ is odd,} \\ 2g_2 = 2g & \text{if } n \text{ is even.} \end{cases}$$

Then the result follows. □

Corollary 3 *The following relations hold* $\gamma_{2(2n-1)} = \gamma_{2n-1}$, $\alpha_{-n} = \beta_{-(2n-1)}$, $\alpha_n = \beta_{(2n-1)}$.

Proof: By the above relations we have $g_{2(2n-1)} = g_{2n-1} = \dfrac{g}{2n-1}$. The second and the third assertions follow from the first assertion since $\alpha_{\pm n}$ (resp. $\beta_{\pm(2n-1)}$) is the unique member in Γ passing through $(\mp g_{(2n-1)}, 0)$ (resp. $(\mp g_{2(2n-1)}, 0)$). □

Figure 4 shows examples of both types with $g = 20$, $e = -4$ and $\ell = 4$. The part (a) is of the odd type and the part (b) is of the even type. Observe that in (a) there exists the circle γ_3 which is the same as the circle γ_6 in (b). But, there does not exist the circle α_{-2} in (a) nor there exist the circle β_{-3} in (b) since $\dfrac{g}{\ell} = 5$. So in this case the embedded pattern of the odd type is $\{\alpha = \alpha_{-1}, \beta = \alpha_1, \gamma = \gamma_1\}$ and the embedded pattern of the even type is $\{\beta_{-2}, \alpha = \beta_{-1}, \beta_0, \beta_1 = \beta, \beta_2, \gamma = \gamma_2, \gamma_4\}$.

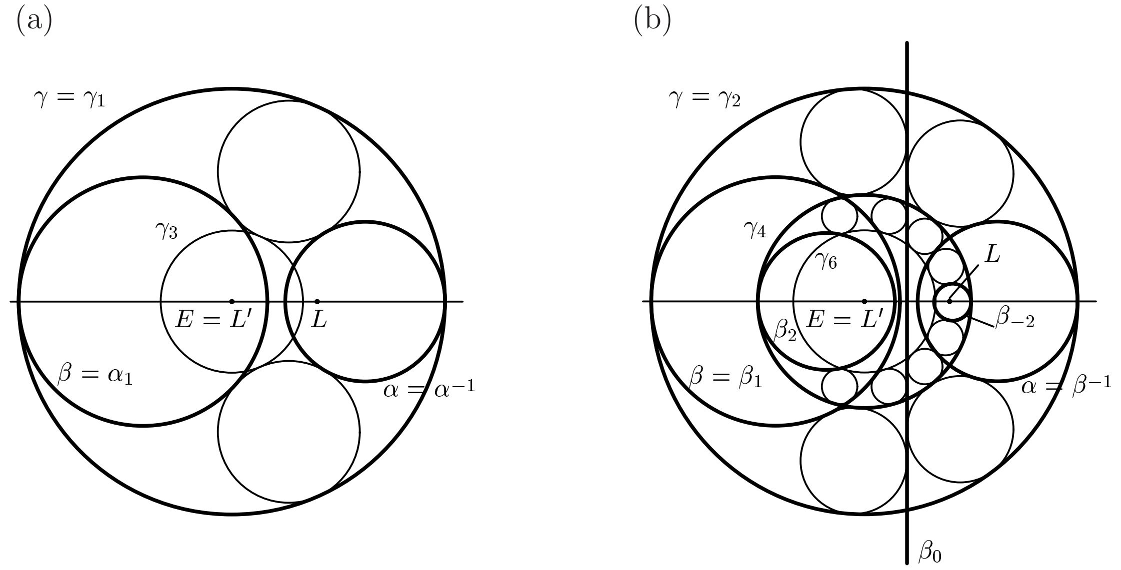

Figure 4: The embedded patterns of both types with $g = 20$, $e = -4$, $\ell = 4$

4. The case $|e| > \ell$

In this case, we have $f = \sqrt{e^2 - \ell^2}$, $2u = g + f$ and $2v = g - f$.

Theorem 5 *Let* $\{\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta, \gamma\}$ *be a generalized arbelos in* n*-aliquot parts. Then the radius of the Archimedean circle in* n*-aliquot parts is*

$$\frac{uv\left(u^{\frac{2}{n}} - v^{\frac{2}{n}}\right)}{u^{\frac{n+2}{n}} - v^{\frac{n+2}{n}}}.$$

Proof: The common ratio of the geometric sequence $\mu(\alpha_0), \mu(\alpha_1), \cdots, \mu(\alpha_n)$ is $\left(\frac{u}{v}\right)^{\frac{2}{n}}$ since $\mu(\alpha) = \frac{v\,(e-f)}{u\,(e+f)}$ and $\mu(\beta) = \frac{u\,(e-f)}{v\,(e+f)}$. By the proof of Theorem 1 the radius is

$$\frac{(g^2-f^2)\left(1-\left(\frac{u}{v}\right)^{\frac{2}{n}}\right)}{2\left((g-f)-(g+f)\left(\frac{u}{v}\right)^{\frac{2}{n}}\right)} = \frac{2u\cdot 2v\left(1-\left(\frac{u}{v}\right)^{\frac{2}{n}}\right)}{2\left(2v-2u\left(\frac{u}{v}\right)^{\frac{2}{n}}\right)} = \frac{uv\left(u^{\frac{2}{n}}-v^{\frac{2}{n}}\right)}{u^{\frac{n+2}{n}}-v^{\frac{n+2}{n}}}. \qquad \square$$

Remark: The above result also holds when $\ell = 0$ ([3]). In this case the radii of the circles α and β are u and v, or v and u.

Corollary 4 *The member α_j in the generalized arbelos is the radical axis of the coaxial system Γ if and only if $2j = n$.*

Proof: Since

$$\mu(\alpha_j) = \mu(\alpha)\cdot\left(\frac{u}{v}\right)^{\frac{2j}{n}} = \left(\frac{u}{v}\right)^{\frac{2j}{n}-1}\cdot\frac{e-\sqrt{e^2-\ell^2}}{e+\sqrt{e^2-\ell^2}},$$

the member $\alpha_j \in \Gamma$ is the radical axis if and only if $\left(\frac{u}{v}\right)^{\frac{2j}{n}-1} = 1$. The result follows from this. $\square$

Theorem 6 *Let $\{\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta, \gamma\}$ be a generalized arbelos in n-aliquot parts. Then there exists a circle γ' concentric to γ and tangent to all Archimedean circles in n-aliquot parts externally.*

The radius g' of γ' satisfies $\left(\frac{g'+f}{g'-f}\right)^{\frac{1}{n+2}} = \left(\frac{g+f}{g-f}\right)^{\frac{1}{n}}$.

There exist two members α' and β' of the coaxial system Γ such that $\{\alpha', \alpha_0, \cdots, \alpha_n, \beta', \gamma'\}$ is a generalized arbelos in $(n+2)$-aliquot parts if and only if $g' \geq |e| + \ell$.

Proof: Let r be the radius of Archimedean circles in n-aliquot parts in $\{\alpha_0, \alpha_1, \cdots, \alpha_n, \gamma\}$. Since

$$g - 2r = \frac{(u-v)\left(u^{\frac{n+2}{n}}+v^{\frac{n+2}{n}}\right)}{u^{\frac{n+2}{n}}-v^{\frac{n+2}{n}}} > 0,$$

we can draw a circle γ' with center E and radius $g' = g - 2r$ which touches all Archimedean circles in n-aliquot parts externally. Since

$$g'+f = \frac{(u-v)\left(u^{\frac{n+2}{n}}+v^{\frac{n+2}{n}}\right)}{u^{\frac{n+2}{n}}-v^{\frac{n+2}{n}}} + (u-v) = \frac{2u^{\frac{n+2}{n}}(u-v)}{u^{\frac{n+2}{n}}-v^{\frac{n+2}{n}}} \quad \text{and}$$

$$g'-f = \frac{(u-v)\left(u^{\frac{n+2}{n}}+v^{\frac{n+2}{n}}\right)}{u^{\frac{n+2}{n}}-v^{\frac{n+2}{n}}} - (u-v) = \frac{2v^{\frac{n+2}{n}}(u-v)}{u^{\frac{n+2}{n}}-v^{\frac{n+2}{n}}},$$

we have

$$\frac{g'+f}{g'-f} = \left(\frac{u}{v}\right)^{\frac{n+2}{n}} = \left(\frac{g+f}{g-f}\right)^{\frac{n+2}{n}}.$$

If $g' \geq |e| + \ell$, there exist two circles α' and β' in the coaxial system Γ such that α' touches from inside the circle γ' at $(e+g', 0)$ and β' touches from inside it at $(e-g', 0)$. Then we have

$$\mu(\alpha') = \frac{(g'-f)(e-f)}{(g'+f)(e+f)} = \left(\frac{v}{u}\right)^{\frac{n+2}{n}} \frac{e-f}{e+f} = \mu(\alpha) \Big/ \left(\frac{u}{v}\right)^{\frac{2}{n}},$$
$$\mu(\beta') = \frac{(g'+f)(e-f)}{(g'-f)(e+f)} = \left(\frac{u}{v}\right)^{\frac{n+2}{n}} \frac{e-f}{e+f} = \mu(\beta) \left(\frac{u}{v}\right)^{\frac{2}{n}}.$$

So the sequence $\mu(\alpha'), \mu(\alpha_0), \mu(\alpha_1), \cdots, \mu(\alpha_n), \mu(\beta')$ is geometric and $\{\alpha', \alpha_0, \cdots, \alpha_n, \beta', \gamma'\}$ is a generalized arbelos in $(n+2)$-aliquot parts by Theorem 1.

If $g' < |e| + \ell$, then either L or L' is outside the circle γ' so that there is at most one circle of the coaxal system Γ that touches from inside the circle γ'. □

Theorem 7 *Let $\{\alpha, \beta, \gamma\}$ be a generalized arbelos of non-intersecting type. There exists an embedded pattern of odd type*

$$\{\alpha_{-n}, \alpha_{-(n-1)}, \cdots, \alpha_{-1} = \alpha, \alpha_1 = \beta, \cdots, \alpha_n, \gamma_1 = \gamma, \gamma_3, \cdots, \gamma_{2n-1}\}$$

if and only if

$$2n - 1 \leq \frac{\log(|e| + f) - \log(|e| - f)}{2(\log(g+f) - \log(g-f))}. \tag{3}$$

There exists an embedded pattern of even type

$$\{\beta_{-n}, \beta_{-(n-1)}, \cdots, \beta_{-1} = \alpha,\ \beta_0,\ \beta_1 = \beta, \cdots, \beta_n, \gamma_2 = \gamma, \gamma_4, \cdots, \gamma_{2n}\}$$

if and only if

$$2n \leq \frac{\log(|e| + f) - \log(|e| - f)}{\log(g+f) - \log(g-f)}. \tag{4}$$

Proof: By Theorem 6 the circle γ_n exists provided there exists a generalized arbelos in $(n-2)$-aliquot parts. Moreover, there exists a generalized arbelos in n-aliquot parts with γ_n as an outer circle if and only if its radius g_n satisfies $g_n \geq |e| + \ell$. Since

$$\left(\frac{g_n+f}{g_n-f}\right)^{\frac{1}{n}} = \left(\frac{g_{n-2}+f}{g_{n-2}-f}\right)^{\frac{1}{n-2}} = \cdots = \begin{cases} \dfrac{g_1+f}{g_1-f} = \dfrac{g+f}{g-f} & \text{if } n \text{ is odd,} \\ \left(\dfrac{g_2+f}{g_2-f}\right)^{\frac{1}{2}} = \left(\dfrac{g+f}{g-f}\right)^{\frac{1}{2}} & \text{if } n \text{ is even,} \end{cases}$$

we have

$$g_{2n-1} = \frac{f\left(\left(\frac{g+f}{g-f}\right)^{2n-1} + 1\right)}{\left(\frac{g+f}{g-f}\right)^{2n-1} - 1}, \qquad g_{2n} = \frac{f\left(\left(\frac{g+f}{g-f}\right)^{n} + 1\right)}{\left(\frac{g+f}{g-f}\right)^{n} - 1}.$$

Then the inequality $g_{2n-1} \geq |e| + \ell$ holds if and only if

$$2n - 1 \leq \frac{\log(|e| + \ell + f) - \log(|e| + \ell - f)}{\log(g+f) - \log(g-f)}, \tag{5}$$

and the inequality $g_{2n} \geq |e| + \ell$ holds if and only if

$$2n \leq \frac{2(\log(|e| + \ell + f) - \log(|e| + \ell - f))}{\log(g+f) - \log(g-f)}. \tag{6}$$

Since we have

$$\left(\frac{|e|+\ell+f}{|e|+\ell-f}\right)^2 = \frac{|e|+f}{|e|-f},$$

the inequality (5) is equivalent to (3) and the inequality (6) is equivalent to (4). □

In analogy with Corollary 3, we have the following

Corollary 5 *The following relations hold* $\gamma_{2(2n-1)} = \gamma_{2n-1},\ \alpha_{-n} = \beta_{-(2n-1)},\ \alpha_n = \beta_{(2n-1)}$.

Figure 5 shows examples of both types with $g = 21,\ e = -5,\ \ell = 4$. The part (a) is of the odd type and the part (b) is of the even type. Observe that in (a) there exists the circle γ_3 which is the same as the circle γ_6 in (b). But there does not exist the circle α_{-2} in (a) nor there exist the circle β_{-3} in (b) since

$$\frac{\log(|e|+f) - \log(|e|-f)}{2(\log(g+f) - \log(g-f))} < 2\cdot 2 - 1, \quad 2\cdot 2 \le \frac{\log(|e|+f) - \log(|e|-f)}{\log(g+f) - \log(g-f)} < 2\cdot 3.$$

So in this case the embedded pattern of the odd type is $\{\alpha = \alpha_{-1}, \beta = \alpha_1, \gamma = \gamma_1\}$ and the embedded pattern of the even type is $\{\beta_{-2}, \alpha = \beta_{-1}, \beta_0,\ \beta_1 = \beta, \beta_2, \gamma = \gamma_2, \gamma_4\}$.

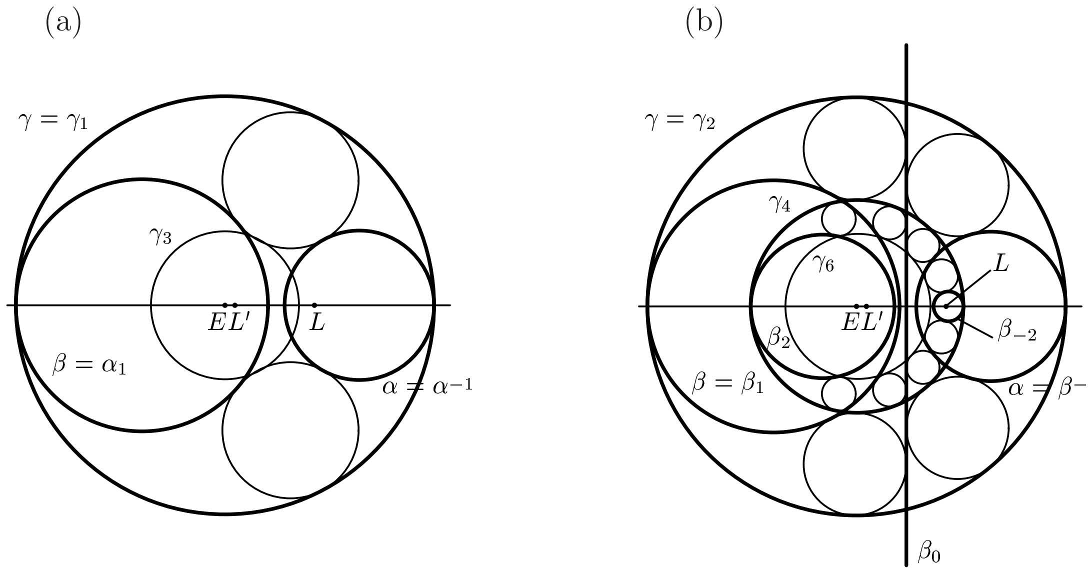

Figure 5: The embedded patterns of both types with $g = 21,\ e = -5,\ \ell = 4$

5. Case $|e| < \ell$

In this case, we have $f = \sqrt{\ell^2 - e^2}$, $2u = g + fi$, $2v = g - fi$,

$$\mu(\alpha) = \frac{(g-fi)(e-fi)}{(g+fi)(e+fi)} = \frac{v(e-fi)}{u(e+fi)} \quad \text{and} \quad \mu(\beta) = \frac{(g+fi)(e-fi)}{(g-fi)(e+fi)} = \frac{u(e-fi)}{v(e+fi)}.$$

Let $\eta = \arg(g+fi) = \arg(u)$ and $zeta = \arg(e+fi)$. We can assume that $0 < \eta \le \dfrac{\pi}{4}$ and $0 < \zeta < \pi$ since $g \ge \ell \ge f > 0$. Then we have $\omega(\alpha) = \pi - \eta - \zeta$ and $\omega(\beta) = \pi + \eta - \zeta$.

Theorem 8 *Let $\{\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta, \gamma\}$ be a generalized arbelos in n-aliquot parts. Then the radius of the Archimedean circle in n-aliquot parts is*

$$\frac{uv\left(\exp\left(\frac{2\eta}{n}i\right)\right) - \exp\left(-\frac{2\eta}{n}i\right)}{u\exp\left(\frac{2\eta}{n}i\right) - v\exp\left(-\frac{2\eta}{n}i\right)} .$$

Proof. Let a_j be the x-coordinate of the intersection point of α_j and the x-axis in I. We can assume that $a_0 > a_1 > \cdots > a_n$. Then the following chain of inequalities holds

$$0 < \pi - \eta - \zeta = \omega(\alpha_0) < \omega(\alpha_1) < \cdots < \omega(\alpha_n) = \pi + \eta - \zeta < \pi .$$

Since $\mu(\alpha_0) = \exp(2\omega(\alpha_0)i)$, $\mu(\alpha_1) = \exp(2\omega(\alpha_1)i)$, $\cdots$, $\mu(\alpha_n) = \exp(2\omega(\alpha_n)i)$ is a geometric sequence, the sequence $2\omega(\alpha_0), 2\omega(\alpha_1), \cdots, 2\omega(\alpha_n)$ is arithmetic with $\frac{4\eta}{n}$ as its common difference, so the common ratio of the geometric sequence is $\exp(\frac{4\eta}{n}i)$. By the proof of Theorem 1, the radius of the Archimedean circle in n-aliquot parts is

$$\begin{aligned}&\frac{(\ell^2 - e^2 + g^2)\left(1 - \exp\left(\frac{4\eta}{n}i\right)\right)}{2\left(\left(g - \sqrt{e^2 - \ell^2}\right) - \left(g + \sqrt{e^2 - \ell^2}\right)\exp\left(\frac{4\eta}{n}i\right)\right)}\\ &= \frac{(g^2 + f^2)\left(1 - \exp\left(\frac{4\eta}{n}i\right)\right)}{2\left((g - fi) - (g + fi)\exp\left(\frac{4\eta}{n}i\right)\right)} = \frac{uv\left(\exp\left(-\frac{2\eta}{n}i\right)\right) - \exp\left(\frac{2\eta}{n}i\right)}{v\exp\left(-\frac{2\eta}{n}i\right) - u\exp\left(\frac{2\eta}{n}i\right)} .\end{aligned}$$

□

Remark: If we denote the complex number $|u|^{\frac{q}{p}}\exp(\frac{q}{p}\eta i)$ by $u^{\frac{q}{p}}$ and $|v|^{\frac{q}{p}}\exp(-\frac{q}{p}\eta i)$ by $v^{\frac{q}{p}}$, the above expression for the radius is the same as the one in Theorem 5.

$$\frac{uv\left(|u|^{\frac{2}{n}}\exp\left(\frac{2}{n}\eta i\right) - |v|^{\frac{2}{n}}\exp\left(-\frac{2}{n}\eta i\right)\right)}{|u|^{\frac{n+2}{n}}\exp\left(\frac{n+2}{n}\eta i\right) - |v|^{\frac{n+2}{n}}\exp\left(-\frac{n+2}{n}\eta i\right)} = \frac{uv\left(u^{\frac{2}{n}} - v^{\frac{2}{n}}\right)}{u^{\frac{n+2}{n}} - v^{\frac{n+2}{n}}} .$$

Corollary 6 *The member α_j in the generalized arbelos is the radical axis of the coaxial system Γ if and only if $2j = n$.*

Proof: Since

$$\mu(\alpha_j) = \mu(\alpha)\cdot\exp\left(\frac{4j\eta}{n}i\right) = \exp\left(\left(\frac{4j\eta}{n} - 2\eta\right)i\right)\cdot\frac{e - fi}{e + fi},$$

the member $\alpha_j \in \Gamma$ is the radical axis if and only if $\exp\left(\left(\frac{4j\eta}{n} - 2\eta\right)i\right) = 1$. This is equivalent to $2j = n$ since $-\pi < \frac{4j\eta}{n} - 2\eta < \pi$. □

Theorem 9 *Let $\{\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta, \gamma\}$ be a generalized arbelos in n-aliquot parts. Then there exists a circle γ' concentric to γ and tangent to all Archimedean circles in n-aliquot parts externally if and only if either $n \geq 3$, $n = 2$ and $g > f$, or $n = 1$ and $g > \sqrt{3}f$. If such a circle γ' with the radius g' exists and η' with $0 < \eta' < \frac{\pi}{2}$ is the argument of $g' + fi$, then $\frac{\eta'}{n+2} = \frac{\eta}{n}$.*

Proof: Let r be the radius of Archimedean circles in n-aliquot parts in $\{\alpha_0, \alpha_1, \cdots, \alpha_n, \gamma\}$. The existence of γ' is equivalent to $g - 2r > 0$ and we have

$$\begin{aligned} g - 2r &= u + v - \frac{2uv\left(\exp\left(\frac{2\eta}{n}i\right)\right) - \exp\left(-\frac{2\eta}{n}i\right)}{u\exp\left(\frac{2\eta}{n}i\right) - v\exp\left(-\frac{2\eta}{n}i\right)} = \frac{(u - v)\left(u\exp\left(\frac{2}{n}\eta i\right) + v\exp\left(-\frac{2}{n}\eta i\right)\right)}{u\exp\left(\frac{2}{n}\eta i\right) - v\exp\left(-\frac{2}{n}\eta i\right)}\\ &= \frac{(u - v)\left(\exp\left(\frac{n+2}{n}\eta i\right) + \exp\left(-\frac{n+2}{n}\eta i\right)\right)}{\exp\left(\frac{n+2}{n}\eta i\right) - \exp\left(-\frac{n+2}{n}\eta i\right)} = \frac{f\cdot\cos\left(\frac{n+2}{n}\eta\right)}{\sin\left(\frac{n+2}{n}\eta\right)} .\end{aligned}$$

Note that $\sin\left(\frac{n+2}{n}\eta\right) > 0$ since $0 < \frac{n+2}{n}\eta \leq \frac{3\pi}{4}$, so the inequality $g - 2r > 0$ holds if and only if $\cos\left(\frac{n+2}{n}\eta\right) > 0$ and this is equivalent to $\frac{n+2}{n}\eta < \frac{\pi}{2}$. This holds for any $n \geq 3$, is equivalent to $\eta < \frac{\pi}{4}$ if $n = 2$ and is equivalent to $\eta < \frac{\pi}{6}$ if $n = 1$. The condition $\eta < \frac{\pi}{4}$ is equivalent to $g > f$ and the condition $\eta < \frac{\pi}{6}$ is equivalent to $g > \sqrt{3}f$.

When the circle γ' exists, we have

$$g' + fi = g - 2r + fi = \frac{f\cos\left(\dfrac{n+2}{n}\eta\right)}{\sin\left(\dfrac{n+2}{n}\eta\right)} + fi = \frac{f}{\sin\left(\dfrac{n+2}{n}\eta\right)}\exp\left(\frac{n+2}{n}\eta\right).$$

Since $f/\sin\left(\frac{n+2}{n}\eta\right)$ is a positive real number and $0 < \frac{n+2}{n}\eta < \frac{\pi}{2}$, we have $\eta' = \frac{n+2}{n}\eta$. □

Remark: There exists a generalized arbelos in 2-aliquot parts with $g = f$. In this case, both circles α and β degenerate to points and the Archimedean circles in 2-aliquot parts pass through the center of the circle γ, so the circle γ does not exist.

Theorem 10 *Let $\{\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta, \gamma\}$ be a generalized arbelos in n-aliquot parts, and assume that the circle γ' in Theorem 9 exists. Then there exist two circles α' and β' of the coaxial system Γ such that $\{\alpha', \alpha_0, \cdots, \alpha_n, \beta', \gamma'\}$ is a generalized arbelos in $(n+2)$-aliquot parts if and only if $g' \geq |e| + \ell$.*

Proof: There exist two circles α and β in the coaxial system Γ which are inside and touching the circle γ' if and only if $g' \geq |e| + \ell$. If α' and β' are such circles with α touching γ' at $(e + g', 0)$ and β' touching γ' at $(e - g', 0)$, we have

$$\mu(\alpha') = \frac{(g' - fi)(e - fi)}{(g' + fi)(e + fi)} = \frac{\exp\left(-\dfrac{n+2}{n}\eta i\right)(e - fi)}{\exp\left(\dfrac{n+2}{n}\eta i\right)(e + fi)}$$

by Theorem 9. Hence,

$$\frac{\mu(\alpha)}{\mu(\alpha')} = \frac{v}{u}\frac{\exp\left(\dfrac{n+2}{n}\eta i\right)}{\exp\left(-\dfrac{n+2}{n}\eta i\right)} = \exp\left(\frac{4}{n}\eta i\right).$$

Similarly, $\frac{\mu(\beta')}{\mu(\beta)} = \exp\left(\frac{4}{n}\eta i\right)$. Then the sequence $\mu(\alpha'), \mu(\alpha), \mu(\alpha_1), \cdots, \mu(\alpha_{n-1}), \mu(\beta), \mu(\beta')$ is geometric and $\{\alpha', \alpha_0, \cdots, \alpha_n, \beta', \gamma'\}$ is a generalized arbelos in $(n+2)$-aliquot parts by Theorem 1. □

Theorem 11 *Let $\{\alpha,\ \beta,\ \gamma\}$ be a generalized arbelos of non-intersecting type. There exists an embedded pattern of odd type*

$$\{\alpha_{-n}, \alpha_{-(n-1)}, \cdots, \alpha_{-1} = \alpha, \alpha_1 = \beta, \cdots, \alpha_n, \gamma_1 = \gamma, \gamma_3, \cdots, \gamma_{2n-1}\}$$

if and only if

$$2n - 1 \leq \frac{\log(|e| + fi) - \log(|e| - fi)}{2(\log(g + fi) - \log(g - fi))}. \tag{7}$$

Similarly, there exists an embedded pattern of even type

$$\{\beta_{-n}, \beta_{-(n-1)}, \cdots, \beta_{-1} = \alpha,\ \beta_0,\ \beta_1 = \beta, \cdots, \beta_n, \gamma_2 = \gamma, \gamma_4, \cdots, \gamma_{2n}\}$$

if and only if

$$2n \le \frac{\log(|e|+fi)-\log(|e|-fi)}{\log(g+fi)-\log(g-fi)}\,, \tag{8}$$

where $\log z$ *denote the complex number* $\log|z|+\arg(z)i$ *with* $-\pi<\arg(z)\le\pi$.

Proof: Let $\xi=\arg(|e|+\ell+fi)$ with $0<\xi<\dfrac{\pi}{2}$. Since

$$\tan 2\xi = \frac{2f}{|e|+\ell} \Big/ \left(1-\left(\frac{f}{|e|+\ell}\right)^2\right) = f/|e|\,,$$

we have $2(\log(|e|+\ell+fi)-\log(|e|+\ell-fi)) = \log(|e|+fi)-\log(|e|-fi)$, and then

$$\frac{\log(|e|+fi)-\log(|e|-fi)}{2(\log(g+fi)-\log(g-fi))} = \frac{\log(|e|+\ell+fi)-\log(|e|+\ell-fi)}{\log(g+fi)-\log(g-fi)}\,. \tag{9}$$

So the inequalities (7) and (8) hold for $n=1$ since $g\ge|e|+\ell$ and

$$\frac{\log(|e|+\ell+fi)-\log(|e|+\ell-fi)}{\log(g+fi)-\log(g-fi)} \ge 1\,.$$

If $g\le\sqrt{3}f$, then $\tan^{-1}\frac{f}{g}\ge\frac{\pi}{6}$ and the inequality

$$\frac{\log(|e|+fi)-\log(|e|-fi)}{2(\log(g+fi)-\log(g-fi))} < \frac{3}{2}$$

holds, so the inequality (7) holds only for $n=1$.

The case $g=f$ occurs only when $e=0$ and we have

$$\frac{\log(|e|+fi)-\log(|e|-fi)}{\log(g+fi)-\log(g-fi)} = 2\,.$$

So, the inequality (8) holds only for $n=1$ if $g=f$.

On the other hand by Theorem 9, the embedded pattern of odd type is $\{\alpha=\alpha_{-1},\ \beta=\alpha_1,\ \gamma=\gamma_1\}$ if $g\le\sqrt{3}f$, and the embedded pattern of even type is $\{\alpha=\beta_{-1},\ \beta_0,\ \beta_1=\beta,\ \gamma=\gamma_2\}$ if $g=f$. In this case β_0 is the radical axis of the coaxial system Γ.

Let us assume that the circle γ_n with the radius g_n exists. By Theorems 9 and 10 and the above argument, there exists a generalized arbelos in n-aliquot parts with γ_n as an outer circle if and only if $g_n\ge|e|+\ell$. If we denote $\eta_n=\arg(g_n+fi)$ with $0<\eta_n<\frac{\pi}{2}$, the inequality $g_n\ge|e|+\ell$ is equivalent to $\eta_n\le\xi$. By Theorem 9 we have

$$\frac{\eta_n}{n} = \frac{\eta_{n-2}}{n-2} = \cdots = \begin{cases} \eta_1=\eta & \text{if } n \text{ is odd,} \\ \dfrac{\eta_2}{2}=\dfrac{\eta}{2} & \text{if } n \text{ is even,} \end{cases}$$

and then $\eta_{2n-1}=(2n-1)\eta$ and $\eta_{2n}=n\eta$.

So $\eta_{2n-1}\le\xi$ holds if and only if $2n-1\le\frac{\xi}{\eta}$ holds and this is equivalent to

$$2n-1 \le \frac{\log(|e|+\ell+fi)-\log(|e|+\ell-fi)}{\log(g+fi)-\log(g-fi)}\,. \tag{10}$$

Similarly, the inequality $\eta_{2n} \leq \xi$ is equivalent to

$$2n \leq \frac{2\left(\log\left(|e| + \ell + fi\right) - \log\left(|e| + \ell - fi\right)\right)}{\log\left(g + fi\right) - \log\left(g - fi\right)} \, . \tag{11}$$

Clearly, the inequality (10) is equivalent to the inequality (7) while the inequality (8) and (11) are equivalent because of the relation (9). □

Corollary 7 *The following relations hold:*

$$\gamma_{2(2n-1)} = \gamma_{2n-1}\,, \quad \alpha_{-n} = \beta_{-(2n-1)}\,, \quad \alpha_n = \beta_{(2n-1)}\,.$$

Figure 6 shows an example of the odd type with $g = 9$, $e = -1$, $\ell = 6$. The incircle of α and β in γ contains the center of γ so that we can not draw the circle γ' and the embedded pattern is $\{\alpha = \alpha_{-1},\ \beta = \alpha_1,\ \gamma = \gamma_1\}$. In this case $g^2 = 81 < 105 = 3f^2$.

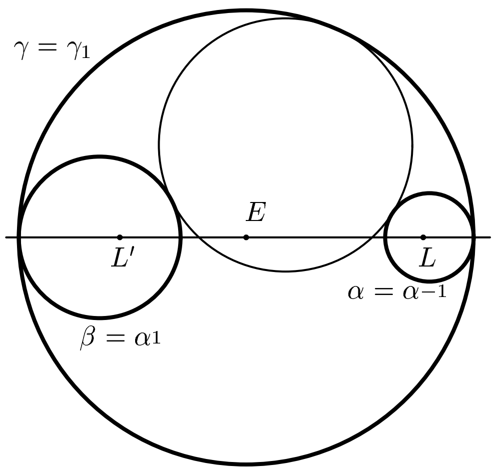

Figure 6: The embedded pattern of odd type with $g = 9$, $e = -1$, $\ell = 6$

Figure 7 shows examples of both types with $g = 21$, $e = -4$, $\ell = 5$. The part (a) is of the odd type and the part (b) is of the even type. In the part (a), we can draw the circle γ_3 since $g^2 - 3f^2 > 0$. But, we can not draw the circle α_{-2} since

$$\frac{\log\left(|e| + fi\right) - \log\left(|e| - fi\right)}{2\left(\log\left(g + fi\right) - \log\left(g - fi\right)\right)} = \frac{\tan^{-1}(3/4)}{2\tan^{-1}(3/21)} < 2 \cdot 2 - 1$$

and the circle γ_3 does not contain the point L. So the embedded pattern is just $\{\alpha = \alpha_{-1},\ \beta = \alpha_1,\ \gamma = \gamma_1\}$.

In the part (b), we have

$$2 \cdot 2 \leq \frac{\log\left(|e| + fi\right) - \log\left(|e| - fi\right)}{\log\left(g + fi\right) - \log\left(g - fi\right)} = \frac{\tan^{-1}(3/4)}{\tan^{-1}(3/21)} < 2 \cdot 3\,,$$

so the embedded pattern is $\{\beta_{-2},\ \alpha = \beta_{-1}, \beta_0,\ \beta_1 = \beta,\ \beta_2,\ \gamma = \gamma_2, \gamma_4\}$. The circle γ_3 in the part (a) is the same as the circle γ_6 in the part (b).

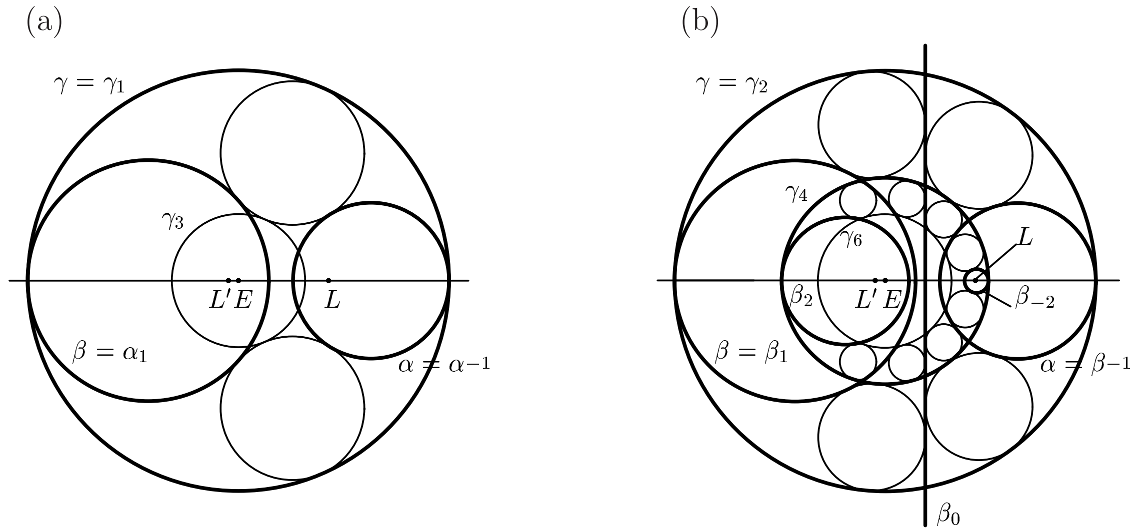

Figure 7: The embedded patterns of both types with $g = 21,\ e = -4,\ \ell = 5$

6. A new family of Archimedean circles

Let $\{\alpha, \beta, \gamma\}$ be an usual arbelos and ρ be the line passing through the centers of α and β. In this section we construct a family of circles having the same radii as the Archimedean twin circles in $\{\alpha, \beta, \gamma\}$ which we call *Archimedean circles* (see [1] and [2]).

First assume α and β are not congruent. Then there exists a circle ε concentric to γ and passing through the tangent point of α and β. Let L be a point on the line ρ between γ and ε, let L' be the inverted image of L by the circle ε and let Γ be the coaxial system having two points L and L' as its limiting points. Note that Γ is also the coaxial system generated by ε and the line λ perpendicular to ρ and passing through the midpoint of L and L'. So ε is always the member of Γ for any L. Since the points L and L' are in the circle γ there exist two circles α' and β' of the coaxial system Γ such that $\{\alpha', \beta', \gamma\}$ is a generalized arbelos of non-intersecting type. By Theorem 5 the radii of the incircles in the generalized arbelos in two aliquot parts $\{\alpha', \lambda, \beta', \gamma\}$ are determined only by the radii of ε and γ. When the point L goes to the tangent point of α and β, the point L' also goes to the same point, the circle α' goes to the circle α and the circle β' goes to the circle β. Then the radii of the Archimedean circles in two-aliquot parts in $\{\alpha', \beta', \gamma\}$ are the same as the radii of the Archimedean twin circles in $\{\alpha, \beta, \gamma\}$ by the Remark below Theorem 5. Then we get infinitely many Archimedean circles, according as the point L moves on the line ρ between γ and ε (see Fig. 8(a)).

Now assume α and β are congruent. Then the center of γ and the tangent point of α and β are coincide. We denote this point by E. Let L be any point on the line ρ and inside γ and let Γ be the coaxial system having two points E and L as its limiting points. There exist two circles α' and β' of the coaxial system Γ such that $\{\alpha', \beta', \gamma\}$ is a generalized arbelos. Since any Archimedean circle in two aliquot parts in the generalized arbelos $\{\alpha', \lambda, \beta', \gamma\}$ where λ is the radical axis of Γ, is an Archimedean circle in $\{\alpha, \beta, \gamma\}$ by Theorem 2 and the Remark below it, we get infinitely many Archimedean circles according as the point L moves on the line ρ in γ (see Fig. 8(b)).

Above arguments hold for any arbelos in n-aliquot parts. So we can construct a family of Archimedean circles in n-aliquot parts similarly.

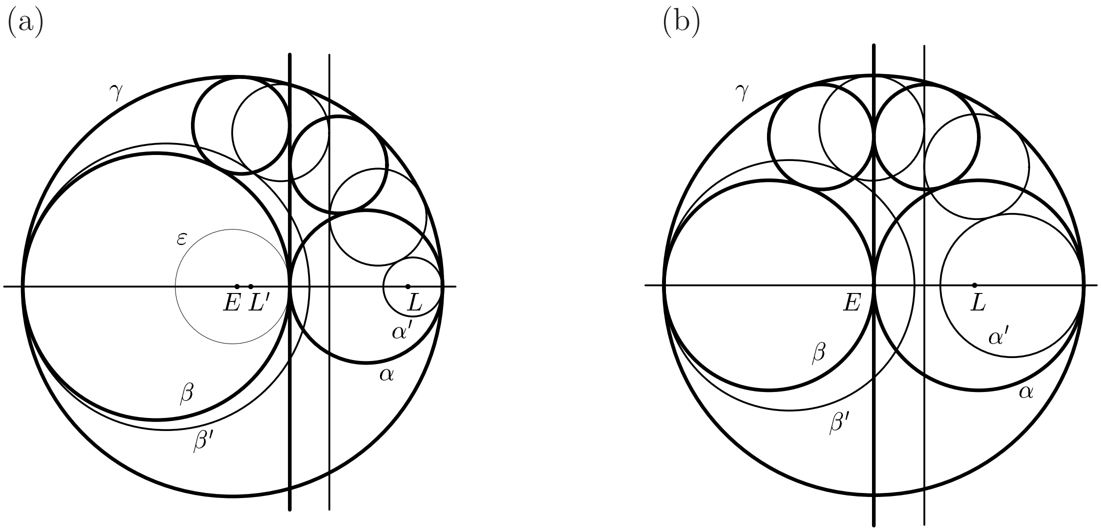

Figure 8: A family of Archimedean circles

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References

[1] C.W. Dodge, T. Schoch, P.Y. Woo, P. Yiu: *Those ubiquitous Archimedean circles*. Math. Mag. **72**, 202–213 (1999).

[2] H. Okumura, M. Watanabe: *The Archimedean Circles of Schoch and Woo*. Forum Geom. **4**, 27–34 (2004).

[3] H. Okumura, M. Watanabe: *The Arbelos in n-Aliquot Parts*. Forum Geom. **5**, 37–45 (2005).

[4] H. Okumura, M. Watanabe: *Generalized Arbelos in Aliquot Parts: Intersecting case*. J. Geometry Graphics **12**, 53–62 (2008).

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