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On the Combinatorics of Inflexion Arches of Saddle Spheres

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Abstract. Each saddle sphere $\Gamma \subset S^3$ is known to generate a spanning arrangement of at least four non-crossing oriented great semicircles on S^2. Each semicircle arises as the projection of an inflexion arch of the surface Γ. In the paper we prove the converse: each spanning arrangement of non-crossing oriented great semicircles is generated by some smooth saddle sphere.
In particular, this means the diversity of saddle spheres on S^3. Recall that each C^2-smooth saddle sphere leads directly to a counterexample to the following A.D. Alexandrov's conjecture:
Let $K \subset \mathbb{R}^3$ be a smooth convex body. If, for a constant C, at every point of ∂K, we have $R_1 \leq C \leq R_2$, then K is a ball (R_1 and R_2 stand for the principal curvature radii of ∂K).
In the framework of the conjecture, the main result of the paper means that all counterexamples can be classified by non-crossing arrangements of oriented great semicircles.

Key Words: A.D. Alexandrov's conjecture, inflexion point, inflexion arch, saddle surface, hyperbolic virtual polytope.

MSC 2000: 53C45, 53A10

1. Introduction

By $S^k \subset \mathbb{R}^{k+1}$ ($k = 2, 3$) we mean the unit sphere centered at the origin O. By a *plane (or spherical plane)* on S^k we mean a plane in the sense of spherical geometry, i.e., the intersection of S^k with some (Euclidean) hyperplane $e \subset \mathbb{R}^{k+1}$ passing through the origin O.

OSC-arrangements and saddle spheres

A *great semicircle* on S^k is a geodesic arch joining two antipodal points. By an *OSC-arrangement* we mean a finite set of disjoint oriented great semicircles $\{sc_i\}$ on the sphere S^2.

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An oriented semicircle sc_i yields its extension, i.e., a great circle c_i and an open hemisphere $S^+(sc_i)$ bounded by c_i (see Fig. 1). The hemisphere is chosen consistent with the orientation of sc_i.

Figure 1: An oriented semicircle yields a circle and a hemisphere

An OSC-arrangement $\mathcal{A} = \{sc_i\}$ is called *spanning* if

$$\bigcup_i S^+(sc_i) = S^2.$$

Clearly, a spanning arrangement contains at least four semicircles.

Two OSC-arrangements $\mathcal{A}_0$ and $\mathcal{A}_1$ are called *isotopic* if there exists a continuous family of OSC-arrangements $\mathcal{A}_t$ $(0 \leq t \leq 1)$ joining $\mathcal{A}_0$ and $\mathcal{A}_1$. The isotopy class of an OSC-arrangement is not determined by the number of its elements.

Definition 1.1 [9] A closed 2-dimensional surface $\Gamma \subset S^3$ is called *saddle* if no plane intersects Γ locally at just one point.
A closed surface $\Gamma \subset S^3$ is called a *saddle sphere* if

- Γ is saddle;
- Γ admits a bijective projection onto some (spherical) plane $S^2 \subset S^3$ (from now on we assume that S^2 is fixed). By the projection we mean the central projection (in the sense of spherical geometry) with the pole of S^2 as center.

In particular, the last item means that a saddle sphere is homeomorphic to S^2.

Figure 10 depicts a saddle sphere (red) and the projection (blue arrows). However, the dimension is higher than is depicted.

Definition 1.2 [9] An *inflexion arch* of a smooth saddle sphere Γ is a great semicircle $S \subset S^3$ such that:

- $S \subset \Gamma$;
- for each (spherical) plane $e \subset S^3$ that intersects S transversely, the point $e \bigcap S$ is an inflexion point of the curve $e \bigcap \Gamma$.

Note that an inflexion arch carries a natural orientation (see Fig. 2).

Each smooth saddle sphere on S^3 generates a spanning OSC-arrangement:

Theorem 1.3 [9] *Let $\Gamma \subset S^3$ be a C^2-smooth saddle sphere. Assume that Γ is non-degenerate, i.e., it does not coincide with a (spherical) plane. Then*

- *Γ contains at least four disjoint inflexion arches;*
- *The projections of all inflexion arches onto S^2 taken with inherited orientations form a spanning OSC-arrangement.*

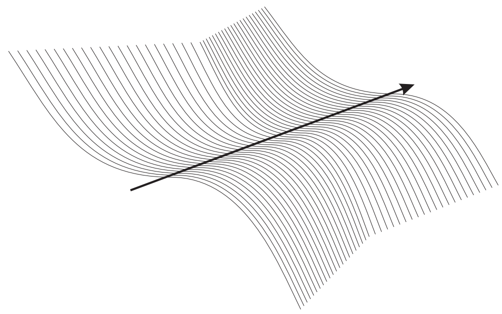

Figure 2: An inflexion arch

In the present paper we prove the converse:

Theorem 1.4 *Given a spanning arrangement $\mathcal{A} \subset S^2$ of disjoint oriented great semicircles, there exists a C^∞-smooth saddle sphere that generates (in the sense of Theorem 1.3) the arrangement $\mathcal{A}$ up to an isotopy.*

Smooth saddle spheres are in one-to-one correspondence with the counterexamples to A.D. Alexandrov's conjecture

Saddle spheres arose originally in a relationship to the following uniqueness conjecture, proven by A.D. Alexandrov in [1] for analytic surfaces:

Let $K \subset \mathbb{R}^3$ be a smooth convex body. If, for a constant C, at every point of ∂K, we have $R_1 \le C \le R_2$, then K is a ball. (R_1 and R_2 stand for the principal curvature radii of ∂K).

Quite surprisingly, there exist diverse C^∞-smooth counterexamples to the conjecture. One of the key points is a relationship of the conjecture to saddle surfaces (its early idea can be traced in [1]). The saddle objects which arose here are called *hyperbolic hèrissons* (see [5] and [7]). Attempting to study the counterexamples to the conjecture, the author of the paper has developed the theory of *hyperbolic virtual polytopes*. The latter are objects dual to piecewise linear saddle spheres which appear in Section 3. Although hyperbolic virtual polytopes are very closely related to the subject of the paper, we make the paper self-contained and do not use this technique here, referring the reader to [7], [8], and [10]. However, here is a brief translation of the relationship into the language of support functions and their graphs:

Let K be a counterexample to the conjecture. Denote by $h_K : \mathbb{R}^3 \to \mathbb{R}$ its support function. (Remind that h_K is defined by the formula $h_K(\mathbf{x}) = \max_{\mathbf{y} \in K}(\mathbf{x}, \mathbf{y})$, where $(\mathbf{x}, \mathbf{y})$ stands for the scalar product. The support function is positively homogeneous.) Further, denote by h_C the support function of the ball of radius C and consider the difference $h = h_K - h_C$. Its graph γ is a conical surface in $\mathbb{R}^4$ with the apex at the origin O. It is crucial for us that the intersection $\gamma \cap S^3$ is a saddle sphere Γ (see Fig. 10).

Vice versa, given a C^2-smooth saddle sphere $\Gamma \subset S^3$, it spans a cone in $\mathbb{R}^4$ which can be interpreted as the graph of some positively homogeneous function h. For a sufficiently large C, the sum $h + h_C$ is a convex function. Then it is the support function of some convex body K which is a counterexample to the conjecture.

Structure of the paper and brief description of the proof

The main result of the paper is Theorem 1.4. Before we prove it, we first study OSC-arrangements. The main result we need (Theorem 2.9) asserts that any spanning OSC-

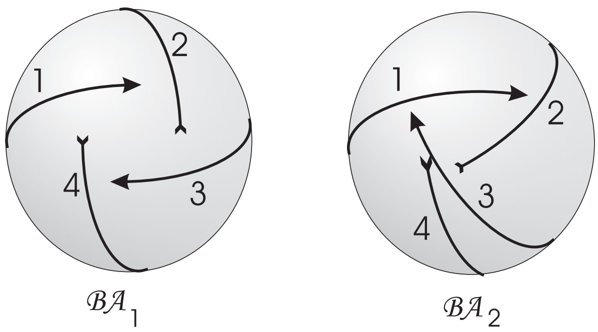

Figure 3: Two non-isotopic arrangements

arrangement can be constructed inductively via semicircle splittings starting from one of the basic arrangements of four semicircles depicted in Fig. 3. Next, we discretize the problem. We focus on *piecewise linear saddle spheres* (PLS-spheres). For PLS-spheres, the role of inflexion arches is played by *inflexion faces*.

In Section 4 we prove Theorem 1.4 for PLS-spheres by induction. We already know from [3], [8], and [9] that the discrete version of Theorem 1.4 is valid for arrangements of four semicircles. This plays the role of the induction base.

Thus we get in a position to show that semicircle splitting can be extended to inflexion arch splitting. For the sake of the induction transmission, we develop a deformation technique for a saddle sphere which can add any prescribed inflexion arch. Theorem 1.4 for C^∞-spheres follows from its discrete version immediately, since [7] provides a smoothing technique.

It should be mentioned that the interplay between piecewise linear saddle spheres and smooth saddle spheres is not well understood yet. The only proven result [7] asserts the existence of a C^∞-smooth saddle approximation for a very restricted class of piecewise linear saddle spheres. However, all the piecewise linear saddle spheres constructed in the paper fit this class: they have trivalent vertex-edge graphs and short edges (shorter than π).

A convention about spherical drawings

In the paper, we depict spherical objects (embedded graphs, arrangements, tilings). This can be done in three ways.

- We sometimes depict the sphere with an object as it is (as in Fig. 5 and Fig. 1).
- Alternatively, we sometimes depict the projection of a hemisphere from the origin O onto some plane $e \subset \mathbb{R}^3$ (as is done in Fig. 4 and Fig. 6).
- If a drawing does not fit a hemisphere, it makes sense to depict it schematically (as in Fig. 7 and Fig. 23).

2. Combinatorics of spanning arrangements

A *SC-arrangement* is a finite set of disjoint great semicircles on the sphere S^2. Each OSC-arrangement $\mathcal{A}$ generates a SC-arrangement $\overline{\mathcal{A}}$ by forgetting the orientations.

Lemma 2.1 [8]

1. *An OSC-arrangement is spanning if and only if it contains at least one of the arrangements $\mathcal{BA}_1$ and $\mathcal{BA}_2$ presented in Fig. 3 (up to an isotopy and a symmetry).*
   By this reason, the arrangements $\mathcal{BA}_1$ and $\mathcal{BA}_2$ are called the basic arrangements.

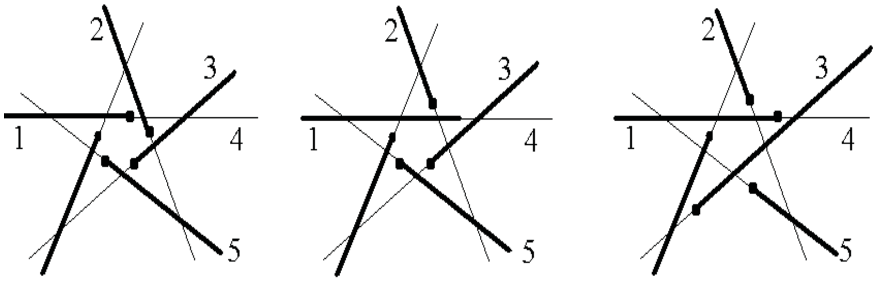

Figure 4: Arrangements of five semicircles

2. *Each SC-arrangement of four semicircles equals (up to an isotopy and a symmetry) one of the arrangements* $\overline{\mathcal{BA}_1}$ *and* $\overline{\mathcal{BA}_2}$*. The arrangements* $\overline{\mathcal{BA}_1}$ *and* $\overline{\mathcal{BA}_2}$ *are non-isotopic.*

Consider either a SC- or an OSC-arrangement $\mathcal{A} = \{sc_i\}$. Denote by $T(\mathcal{A})$ the tiling of the sphere S^2 generated by all the circles c_i spanned by sc_i. A *crossing point* of $\mathcal{A}$ is the intersection point of some two circles c_i and c_j.

Lemma 2.2 *For a SC-arrangement* $\mathcal{A} = \{sc_i\}$*, we have*

1. *Each crossing point belongs to exactly one of the semicircles* sc_i*.*
2. *The circles* c_i *lie in generic position, i.e., no three of them pass through one point.*
3. *If a SC-arrangement* $\mathcal{A}$ *contains more than four semicircles, then there exists a tile of* $T(\mathcal{A})$ *with more than four vertices.*

Proof: 1. The point $A = c_1 \bigcap c_2$ cannot belong to both of the semicircles c_1 and c_2 since they are non-crossing. If it belongs to none of them, then the antipodal point $-A$ belongs to both of c_1 and c_2. A contradiction.
2. This follows directly from 1.
3. It is easy to observe that just 5 circles give a pentagonal tile (see Fig. 5). Now start adding the other circles one by one. A new circle either hits the pentagon or not. If it does not, the pentagon (or more than pentagon) remains as it is. If not, the new circle splits the pentagon into two tiles. One of them necessarily has more than four vertices. □

Lemma 2.3 *Each SC-arrangement* $\mathcal{A}$ *of five semicircles is isotopic either to one of the arrangements presented in Fig. 4, or to the mirror image of one of them.*

Proof: We can assume that the great circles lie as is depicted in Fig. 5. Case analysis shows that the only possible positions of the semicircles are those in Fig. 4. The proving technique

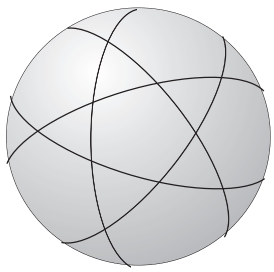

Figure 5: Five circles necessarily give a pentagon

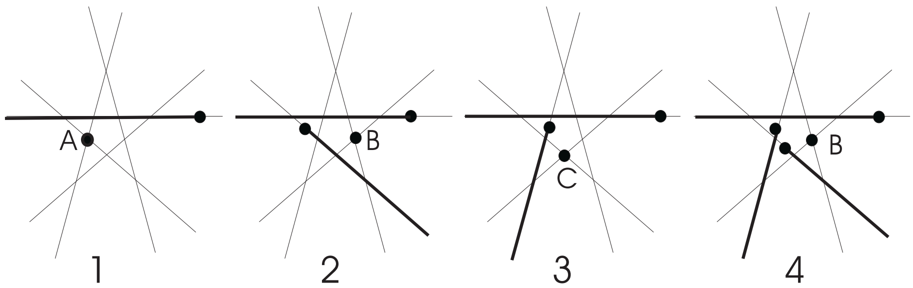

Figure 6: This way one gets stuck

comes from Lemma 2.2. We demonstrate through one particular example how this works. Namely, we prove that no semicircle lies in position 1 (see Fig. 6). The other cases are treated similarly.

Suppose the contrary: there is an arrangement of five semicircles containing a semicircle lying in position 1 (see Fig. 6). Consider the crossing point A. It belongs to one of the semicircles, therefore we have either the case 2 or the case 3 (see Fig. 6). The case 2 is impossible since the crossing point B cannot belong to a semicircle which does not intersect the fixed semicircles. For the position 3, the semicircle containing the point C is determined uniquely as depicted in Fig. 6, 4 . Finally, we get the crossing point B locked: each semicircle containing B intersects one of the three already fixed ones. A contradiction. $\square$

Definition 2.4 Let $\mathcal{A}$ be an OSC-arrangement. Two oriented semicircles $sc_1, sc_2 \in \mathcal{A}$ are called *twins* if there exists a continuous motion of sc_1 which brings sc_1 to sc_2. During the motion, sc_1 must not cross the other semicircles except for sc_2. Here we do not take the orientation into account, it may be either preserved or not.

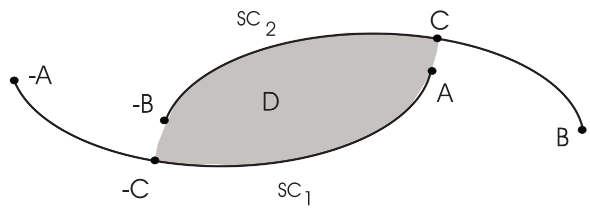

Figure 7: Twins

Lemma 2.5 (Twins Detection Lemma) *Let sc_1 and sc_2 be two semicircles of an OSC-arrangement $\mathcal{A}$. Denote by $\pm A$ and $\pm B$ their endpoints. Denote also $\pm C = c_1 \bigcap c_2$.*

1. *If the segments (A, C) and (C, B) are crossing points free (that is, they intersect none of the circles c_i for $i > 2$), then sc_1 and sc_2 are twins in $\mathcal{A}$ (see Fig. 7).*
2. *Conversely, if these semicircles are twins in $\mathcal{A}$, then for an appropriate choice of $\pm A$ and $\pm B$, the segments (A, C) and (C, B) are crossing points free.*

Proof: 1. Consider the open domain D formed by the circles c_1 and c_2 (it is marked grey in Fig. 7). We claim that there are no crossing points inside the domain. Indeed, the segments (A, C) and $(-C, -B)$ are crossing points free. If there is a crossing point $c_i \bigcap c_j$ inside D, then it belongs either to sc_i or to sc_j. This means that one of these semicircles intersects the

boundary of D, which is impossible.
Besides, the triangles $\triangle ABC$ and $\triangle(-A)(-B)(-C)$ intersect none of c_i (except for sc_1 and sc_2). Therefore the motion depicted in Fig. 11 is collision free. (The semicircles sc_1 and sc_2 are marked red. The black semicircles show the intermediate positions of sc_1.)
2. For a fixed semicircle sc, the set of crossing points lying on c has a natural cyclic ordering. Lemma 2.2 implies that this ordering is maintained during any non-colliding motion of the semicircles. □

Lemma 2.6 *Let $\mathcal{A}$ be a SC-arrangement. Assume that a tile t of $T(\mathcal{A})$ has at least 5 vertices. Let t be formed by the circles $c_1, \dots, c_k$. Denote by e_i the edges of t. For each of the semicircles $sc_1, \dots, sc_k$, only three cases are possible (see Fig. 8):*

1. *The segment e_i contains an endpoint of sc_i. We call this end* a normal end.
2. *The segment e_i does not intersect sc_i. Then the endpoint of sc_i lies in the next to e_i segment of c_i. We call this end* a short end.
3. *The segment e_i is contained in sc_i. Then the endpoint of sc_i lies in the next segment. We call this end* a long end.

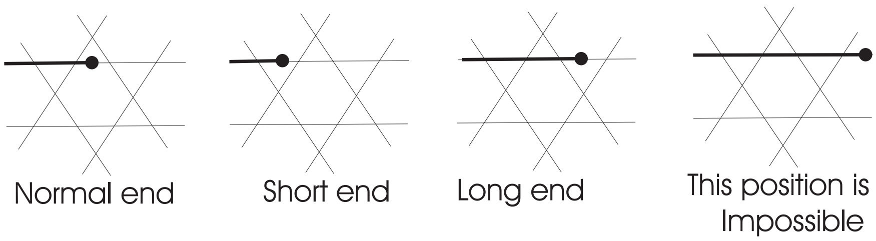

Figure 8: Possible types of ends

Proof: This follows directly from Lemma 2.3. □

Lemma 2.7 *Each SC-arrangement of more than three semicircles contains at least two disjoint pairs of twins.*

Proof: Let $\mathcal{A} = \{sc_i\}_{i=1}^n$. We will prove the theorem by induction. If the number of semicircles is four, we are done. Indeed, for the configuration $\overline{\mathcal{BA}_1}$, the pairs $(1,2), (2,3), (3,4)$ and $(4,1)$ are twins, whereas for the configuration $\overline{\mathcal{BA}_2}$, the pairs $(1,2)$ and $(3,4)$ are twins.

If the number of semicircles is greater than 4, consider a tile t of $T(\mathcal{A})$ which has at least 5 vertices. It always exists by Lemma 2.2. Let t be formed by $\mathcal{T} = \{c_1, \dots, c_k\}$. We assume that the circles are ordered consistent to the natural ordering of the edges of t. One should keep in mind that none of the circles c_i intersects the interior of t.
1. If among $c_1, \dots, c_k$ there are two consecutive normal ends, then they are twins by the Twins Detection Lemma.
2. Suppose that $sc_i \in \mathcal{T}$ has a long end. Then the consecutive semicircle sc_{i+1} has a short end (see Fig. 12). Consider the subarrangement $\mathcal{A}' \subset \mathcal{A}$ of all semicircles with slopes lying between the slopes of sc_i and sc_{i+1}, including the semicircles sc_i and sc_{i+1}. We mark the subarrangement red. We say that these semicircles *lie between sc_i and sc_{i+1}*. Case analysis based on Lemma 2.2 proves that their endpoints lie in the domain which is marked grey in Fig. 12.
Thus, each semicircle from $\mathcal{A} \setminus \mathcal{T}$ lies between some semicircle from $\mathcal{T}$ with a long end and

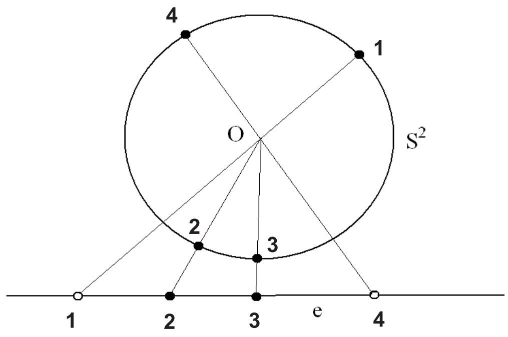

Figure 9: Points on the sphere yield black and white points in the plane

its consecutive semicircle from $\mathcal{T}$ with a short end.

If $\mathcal{A}'$ contains just two semicircles sc_i and sc_{i+1}, then they are twins. If not, then by inductive assumption, the arrangement $\mathcal{A}'$ has a twin which differs from the pair $(sc_i,\ sc_{i+1})$. This twin remains a twin in the whole arrangement $\mathcal{A}$.

Let us summarize the above. On the one hand, a pair of consecutive normal ends is a twin. On the other hand, a pair of type (a long end, the consecutive short end) yields a twin as well. Since the arrangement $\mathcal{T}$ has at least two pairs of these types, they give two disjoint pairs of twins. □

Lemma 2.8 *Let the pairs (sc_1, sc_2) and (sc_3, sc_4) be twins in a spanning OSC-arrangement $\mathcal{A}$. Assume that $\mathcal{A}$ contains more than four semicircles. Then for some $i = 1, 2, 3,$ or 4, the arrangement $\mathcal{A} \setminus \{sc_i\}$ is spanning.*

Proof: Assume without loss of generality that $\mathcal{A}$ consists of five semicircles. Make first some reformulations. Denote by $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5} \in S^2$ the poles of the hemispheres $S^+(sc_1), \ldots, S^+(sc_5)$. Choose a plane $e \subset \mathbb{R}^3$ not passing through the origin O. Each point $\mathbf{i}$ gives a point $e \bigcap (O, \mathbf{i})$ where $(O, \mathbf{i})$ stands for the line passing through O and $\mathbf{i}$. We denote the generated points with the same numbers and mark each of the points either black or white, according to the directions of the rays (see Fig. 9).

We say that a collection of black and white points in the plane is a $B \bigcap W$*-collection* if the convex hull of the white points has a non-empty intersection with the convex hull of the black points.

The three conditions are equivalent:

- An arrangement $sc_1, \ldots, sc_k$ is spanning.
- The radius-vectors of $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}$ span positively $\mathbb{R}^3$.
- The generated collection of black and white points is a $B \bigcap W$-collection.

We may assume that the points generated by $sc_1, \ldots, sc_4$ lie in the convex position. Since we have a $B \bigcap W$-collection, a point can be deleted such that the rest is a $B \bigcap W$-collection. □

Theorem 2.9 *Each spanning OSC-arrangement $\mathcal{A}$ can be obtained inductively in some steps from a 4-element spanning OSC-arrangement by adding a twin to an already existing semicircle.*

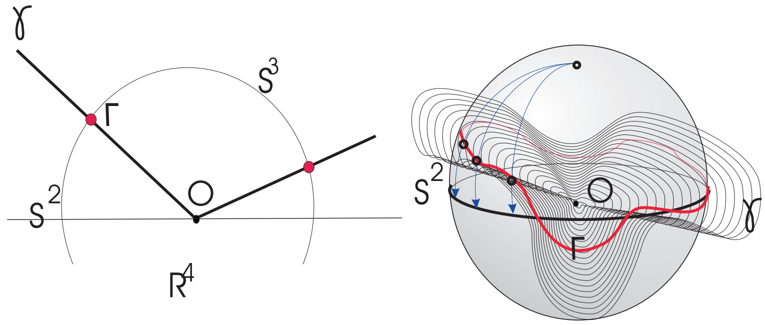

Figure 10: A saddle sphere and the spanned cone

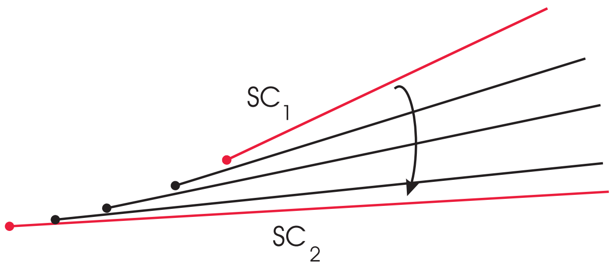

Figure 11: This motion brings one of the twins to another

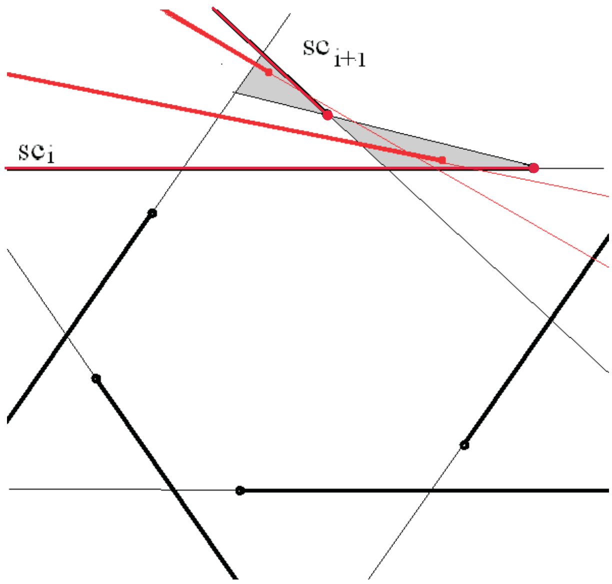

Figure 12: A hexagonal tile t, the subarrangement $\mathcal{T}$ (black) and the semicircles lying between sc_i and sc_{i+1} (red)

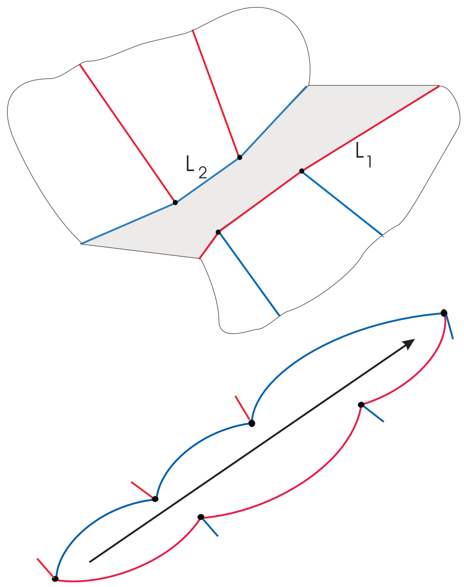

Figure 13: An inflexion face, its projection on S^2, and an oriented semicircle inside the projection. The projection is depicted schematically.

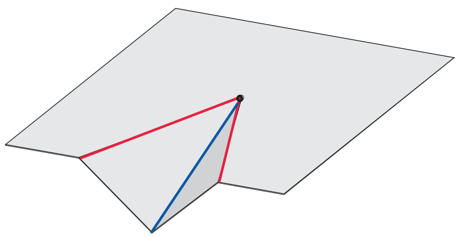

Figure 14: A simple PLS-sphere looks locally like this

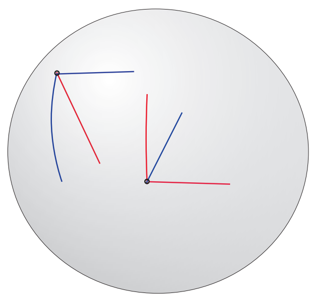

Figure 15: Two possible local colorings

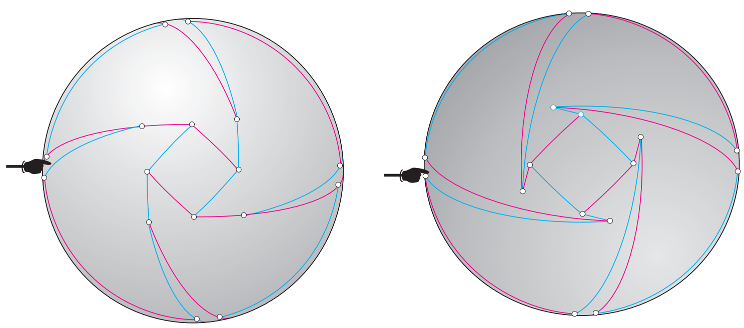

Figure 16: This fan yields the arrangement $\mathcal{BA}_1$

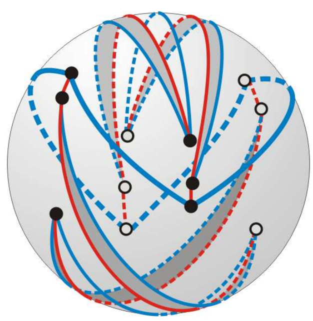

Figure 17: This fan yields the arrangement $\mathcal{BA}_2$

Proof: If the arrangement $\mathcal{A}$ contains just four semicircles, we are done. If it has more than four elements, then by Lemma 2.2, it contains at least two disjoint pairs of twins. Finally, by Lemma 2.8, we can remove one of the twins maintaining the spanning property of the arrangement. □

3. Piecewise linear saddle spheres

A *spherical polygon* is a subset of a (spherical) plane bounded by a piecewise geodesic line. A *piecewise linear saddle sphere in* S^3 *(a PLS-sphere)* is a saddle sphere (in the sense of Definition 1.1) patched of spherical polygons. A PLS-sphere is called *simple* if each its vertex is trivalent, i.e., has exactly 3 incident edges (see Fig. 14). A PLS-sphere has *short edges* if the projections of its edges on S^2 are shorter than π. This property is necessary for further smoothing technique developed in [7].

It is difficult to visualize a PLS-sphere. But since we are interested in the generated OSC-arrangement, we will depict the projections of all its edges onto S^2. Convex edges we will mark red, and concave edges we will mark blue. The projection yields a spherical tiling Σ_Γ which we call the *fan* of the surface. The tiles of the fan are projections of the faces of Γ. For example, Fig. 13 depicts a part of a PLS-sphere Γ and the corresponding part of its fan.

Definition 3.1 and Theorem 3.2 below are discrete versions of Definition 1.2 and Theorem 1.3. In particular, they explain that a PLS-sphere with short edges generates a spanning OSC-arrangement.

Definition 3.1 [9] A face f of a PLS-sphere Γ is called an *inflexion face* if the following conditions hold:

1. f is bounded by two piecewise geodesic convex lines (say, by L_1 and L_2) such that the convexity directions look like in Fig. 13;
2. the surface Γ is convex along the edges of L_1 and concave along the edges of L_2.

Theorem 3.2 [9]

1. *Each PLS-sphere with short edges has at least four inflexion faces.*
2. *Each inflexion face contains a great semicircle. It carries a natural orientation (see Fig. 13). If we fix one semicircle for each inflexion face, then the projections of all semicircles give a spanning OSC-arrangement.*

In other words, inflexion faces play the role of inflexion arches. The fan of a simple PLS-sphere encodes the complete information about its inflexion faces:

Lemma 3.3 [7] *For a simple PLS-sphere* Γ*, we have:*

1. *each vertex of* Σ_Γ *is incident to an angle larger than* π*;*
2. *at each vertex of* Σ_Γ*, only two types of local coloring are possible (see Fig. 15);*
3. *a tile of* Σ_Γ *corresponds to an inflexion face if and only is the edge color changes exactly twice when going around the boundary of the tile.*

Sketch of the *proof*. The surface Γ is saddle and simple. Therefore, at each of its vertex it looks like the one depicted in Fig. 14 (up to an affine transform). This implies (1) and (2).
The color changes twice when going around the boundary of a tile t if and only if t is bounded by two convex lines (that is, the red part of the boundary and the blue part of the boundary) such that their convexity directions look like in Fig. 13. In particular, this implies that the tile t contains a great semicircle. (3) is proven. □

Example 3.4

- Paper [8] presents a simple PLS-sphere Γ_1 which generates the basic arrangement $\mathcal{BA}_1$. Its fan is depicted in Fig. 16. The black arrow points at one of four tiles corresponding to inflexion faces.
  It should be mentioned that chronologically, the very first PLS-sphere (generating the same arrangement) was presented in [6].
- Papers [3] and [2] present a simple PLS-sphere Γ_2 which generates the basic arrangement $\mathcal{BA}_2$. Its fan is depicted in Fig. 17 (see also [10] for its 3D image). There are four tiles (marked grey) corresponding to inflexion faces.

Our next aim is to develop some deformation tricks for PLS-spheres. The general idea looks as follows. Assume that a simple PSL-sphere Γ is fixed. The *hull* of its face f is the (spherical) plane $hull(f)$ spanned by f. We deform a bit $hull(f)$, replacing it by some piecewise linear surface. The other hulls are maintained, but some vertices and edges get changed. New vertices and edges appear as the intersections of the new hulls. If one takes some special care of consistent patching (described below), this procedure yields another simple PLS-sphere.

$\mathcal{H}$-operation

Suppose two inner points of some red (respectively, blue) edges of Σ_Γ can be connected by a geodesic segment avoiding intersections with other edges. Let the new segment s belong to a tile t, which corresponds to a face f. We break somewhat the $hull(f)$ along the segment s to make it concave (respectively, convex) (see Fig. 18). Next, we replace the $hull(f)$ by the broken plane, maintaining the hulls of the other faces. Thus we get two new vertices and one new edge. The fan changes as is shown in Fig. 19.

We shall apply $\mathcal{H}$-operations in Section 4 for shortening the edges of hyperbolic fans, which is necessary for further smoothing.

$\mathcal{C}$-operation

Suppose that the described below $\mathcal{C}$-configuration of four geodesic segments $1, 2, 3$, and 4 can be placed on S^2 (see Fig. 20) such that:

- the endpoints of 1 and 4 lie on edges of Σ_Γ of the same color;
- intersections of the configuration with the edges of Σ_K are avoided (except for the endpoints of 1 and 4);
- segments 2 and 3 are great semicircles;
- each vertex of the configuration has an incident angle larger than π;
- segments 1 and 4 lie on one and the same great circle.

Suppose that the $\mathcal{C}$-configuration lies in a tile t of the fan Σ_Γ. Let f be a face of Γ corresponding to the tile t. We replace the $hull(f)$ by a piecewise linear surface consisting of three linear parts as is shown in Fig. 21. The hulls of other faces remain unchanged.

A $\mathcal{C}$-operation is useful because it can add a new inflexion face to a PLS-sphere. An $\mathcal{H}$-operation can alter inflexion faces, but it never changes the isotopy type of the induced OSC-arrangement.

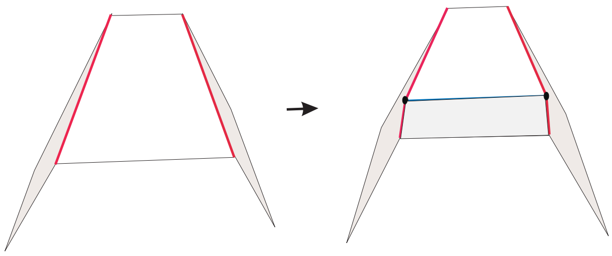

Figure 18: $\mathcal{H}$-operation breaks a face along a segment

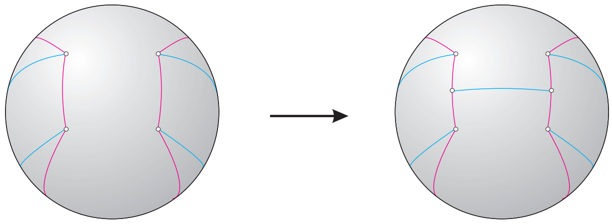

Figure 19: $\mathcal{H}$-operation alters the fan this way

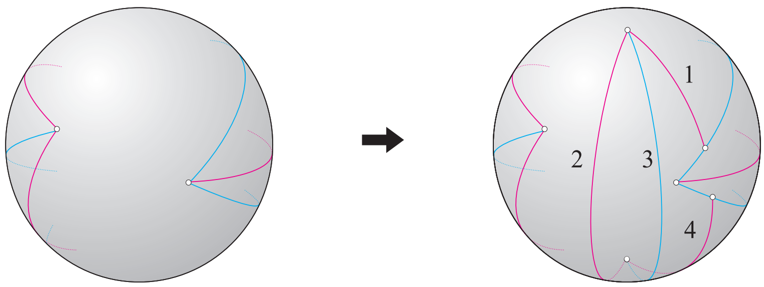

Figure 20: $\mathcal{C}$-operation

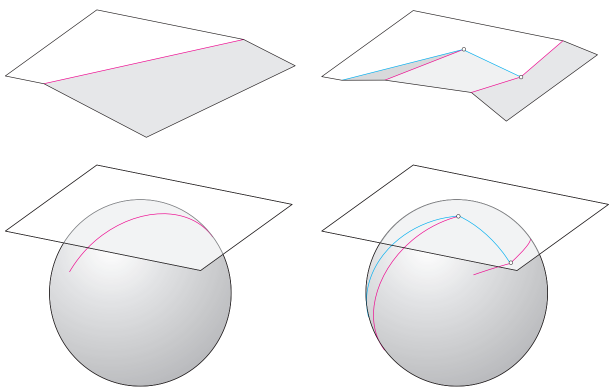

Figure 21: $\mathcal{H}$- and $\mathcal{C}$-operations alter the surface according to the drawing on the sphere

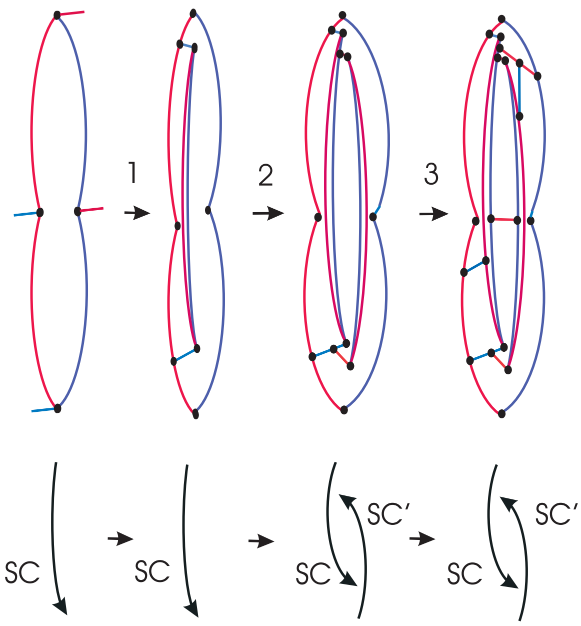

Figure 22: First type of splitting

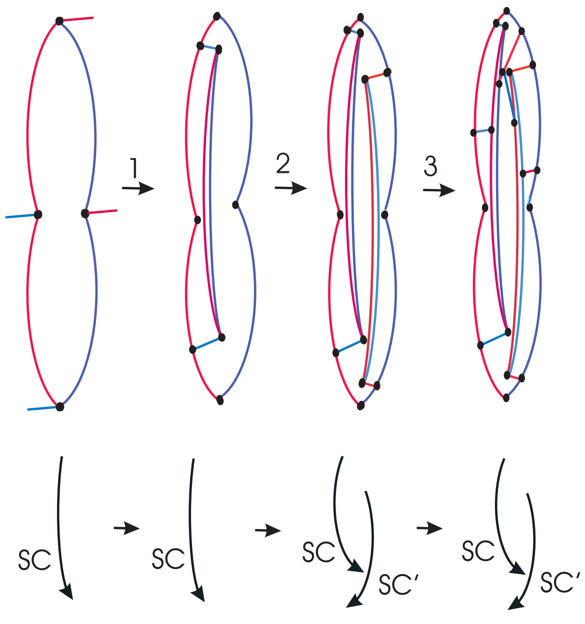

Figure 23: Second type of splitting

4. Proof of Theorem 1.4

We first prove the discrete version of the main theorem:

Theorem 4.1 *Given a spanning arrangement $\mathcal{A} \subset S^2$ of oriented great semicircles, there exists a simple PLS-sphere Γ with short edges which generates the arrangement $\mathcal{A}$ (up to an isotopy).*

Proof: We prove the theorem inductively by the number of semicircles in $\mathcal{A}$.

Base of induction is already proven due to Example 3.4. Namely, a minimal spanning OSC-arrangement consists of four semicircles. Up to isotopy and symmetry, it is either $\mathcal{BA}_1$ or $\mathcal{BA}_2$ (Lemma 2.1).
Note that the PLS-sphere generating $\mathcal{BA}_2$ has some edges of length π, i.e., not all its edges are short. However, the long edges can easily be shortened by a series of $\mathcal{H}$-operations, as is done in [2]. We present here the fan with long edges just because it is less complicated.

Induction transmission. Suppose that the number of semicircles is greater than four. Using Theorem 2.9, we delete a twin sc' of some semicircle $sc \in \mathcal{A}$ such that the rest $\mathcal{A}'$ is a spanning OSC-arrangement. By inductive assumption, $\mathcal{A}'$ is generated by some simple PLS-sphere Γ' with short edges. Using the $\mathcal{C}$- and $\mathcal{H}$-operations, we deform the surface Γ' in 3 steps. This deformation adds an extra inflexion face and splits the semicircle sc.

Step 1. Let the inflexion face f of Γ' correspond to the semicircle sc. Denote by t the projection of f onto S^2. We now add a $\mathcal{C}$-configuration inside t and apply the corresponding $\mathcal{C}$-operation to the surface Γ'. We get a new PLS-sphere, but the isotopy type of the generated OSC-arrangement is maintained (this can be checked using Lemma 3.3).

Step 2. Next, we add one more $\mathcal{C}$-configuration inside t and apply one more $\mathcal{C}$-operation. On this step one should take into account the position and the orientation of sc' with respect to sc. Therefore we have different (but similar) cases depicted in Fig. 22 and Fig. 23. This gives us one more inflexion face which yields sc'. Now we have a simple saddle sphere with the required OSC-arrangement, but some of its edges are of length π.

Step 3. It remains to shorten the long edges. This can be easily done by H-operations. This yields a required simple PLS-sphere Γ. $\square$

Theorem 1.4 for C^∞-smooth saddle spheres follows immediately due to the smoothing technique. Indeed, paper [7] asserts that each simple saddle sphere Γ with short edges can be approximated by a C^∞-smooth saddle sphere $\overline{\Gamma}$. The way of approximation (which is presented explicitly) yields that the inflexion faces of Γ and inflexion arches of $\overline{\Gamma}$ are in one-to-one correspondence.

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[10] A collection of 3D illustrations is available at
`http://club.pdmi.ras.ru/$\sim$panina/hyperbolicpolytopes.html`

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