Journal for Geometry and Graphics
Volume 13 (2009), No. 2, 131–144.

Quadrics of Revolution on Given Points

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Abstract. In general, 4 points define a 3 parameter set of axisymmetric quadrics while 5 and 6 given points reduce these 3 degrees of freedom to 2 and 1, respectively. Similarly, 7 supporting points confine members of the set to a finite number. By imposing 2 constraints on the quadric coefficient matrix the 5 points are sufficient to find the axis direction of up to 6 right cylinders. Imposing only 1 constraint allows 6 points to support up to 12 right cones. Without either constraint, that implies a singular coefficient matrix or singular conic submatrix, up to 4 quadrics of revolution, possibly of mixed species, can contain 7 points. Formal arguments and proofs are presented to substantiate these observations. Algorithms are developed and applied to exhibit cases with 6 right cylinders, 12 right cones and 4 quadrics of revolution, at least 3 of which are of different type. Spheres, being uniquely defined on 4 points, are specifically excluded from consideration. The cases of 12 cones and 4 quadrics of revolution are believed to be original revelations. Methods to fit quadrics of revolution to more than 7 points are suggested.

Key Words: quadric of revolution, right cone, cone of revolution, right cylinder, cylinder of revolution, special quadric, surface specification, finite given point set, repeated eigenvalues

MSC 2000: 51N20, 51N35

1. Introduction

Real time precision camera aided inspection in industrial production, [3, 9, 10, 11, 18], by fitting pixels in digital camera images to conic curves is a process that continues to evolve as users gain confidence in achieving economic benefit. The next challenge is to reliably and accurately fit quadric surfaces via a similar technique. Previous work in this direction seems to have been confined to fitting cylinders of revolution. Results to date have established

ISSN 1433-8157/\$ 2.50

that, in general, up to six such cylinders may be disposed on five arbitrarily selected points. Some relevant references are listed in sequence of their appearance: [13, 14, 2, 1, 5, 19], from earliest to most recent. Remarkably, there is little backward cross citation, with the exception of Schaal, Strobel (see below) and Röschel [12] who were aware of early contributions, of Laguerre [4], Narasinga [7] and Narasinga & Srinivasachari [8], but who were not trying to find a finite number of quadric surfaces on minimal point set specifications. Furthermore there is no evidence of a unified approach to this sort of problem. The quadric of revolution appears to be a good starting point in this regard. Herein the original approach introduced to define four on seven given points will be extended to include the cylinder problem, to show how it fits into the general analytical procedure, and to expose for the first time how twelve cones of revolution may be placed on six points.

The purpose here is to clarify the way attributes of quadrics of revolution can be used to set up systems of equations whereby these surfaces can be determined on fewer than the nine given points required to uniquely define a general quadric. In the case of cylinders of revolution it has been known for some time, [2, 1, 5], and more recently demonstrated, [19], that in general as many as six such cylinders may be located on five given points. Furthermore it is common knowledge that four arbitrarily located points will support a sphere. By examining the structure of the conic section of a quadric of revolution, where it intersects the absolute plane, one sees, via cunning reparametrization, that the equivalent of two conditions or constraint equations are implied by the specification that a pair of identical eigenvalues be imposed on a 3×3 symmetric matrix that represents the coefficients of this conic section of a quadric of revolution. Essentially the six unknown coefficients in the matrix are reduced to four by the parametrization. So a discrete number of axisymmetric quadrics will in general contain seven given points. Imposing the condition that the overall 4×4 coefficient matrix determinant be zero supplies one additional constraint allowing a number of cones of revolution to be defined on six points. Adding that the conic section coefficient matrix determinant vanishes as well obtains the cylinders of revolution on five points, but via a unified method that begins with the general quadric of revolution, rather than with a special vector construction [19], used to find the cylinders' axial directions, that can be traced to early work by Schaal, [13, 14], and Strobel, [15, 16, 17].

2. Quadrics of revolution

A quadric $\mathcal{Q}$ in 3-space always has an equation of the form

$$\mathbf{x}^\top \mathbf{A}\, \mathbf{x} + 2\mathbf{c}^\top \mathbf{x} + c_0 \quad = \quad 0 \tag{1}$$

where $\mathbf{A}$ is a real symmetric 3×3 matrix, $\mathbf{c} = [c_1, c_2, c_3]^\top \in \mathbb{R}^3$, $c_0 \in \mathbb{R}$ and $\mathbf{x} = [x, y, z]^\top$ denotes the Cartesian position vector of any point on the surface. Eq. 1 can be written compactly as follows:

$$[1, \mathbf{x}^\top]\, \mathbf{Q} \begin{bmatrix} 1 \\ \mathbf{x} \end{bmatrix} \quad = \quad 0 \tag{2}$$

where

$$\mathbf{Q} \quad = \quad \begin{bmatrix} c_0 & \mathbf{c}^\top \\ \mathbf{c} & \mathbf{A} \end{bmatrix}. \tag{3}$$

The following well known facts, drawn from the theory of quadrics will be used:

$\mathcal{Q}$ is a quadric of revolution, but not a sphere[1], if and only if the symmetric matrix $\mathbf{A}$ has exactly two distinct eigenvalues λ_1 and λ_2 where λ_1 has multiplicity 1, λ_2 has multiplicity 2 and $\lambda_2 \neq 0$. The 1-dimensional[2] eigenspace belonging to λ_1 represents the (direction of the) axis of $\mathcal{Q}$ whereas the 2-dimensional eigenspace belonging to λ_2 represents the planes perpendicular to this axis — the planes which intersect $\mathcal{Q}$ on circles.

Proposition 1. *Let* $\mathbf{A}$ *be a symmetric* 3×3 *matrix with eigenvalues* $\lambda_1, \lambda_2 = \lambda_3$ *(two of them are identical); then* $\mathbf{A}$ *can be written as*

$$\mathbf{A} \;=\; (\lambda_1 - \lambda_2)\,\mathbf{a}\,\mathbf{a}^\top + \lambda_2\,\mathbf{I}_3 \tag{4}$$

where

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \|\mathbf{a}\| = 1, \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

Proof. Since $\mathbf{A}$ is symmetric there exists a matrix[3]

$$\mathbf{R} \;=\; [\mathbf{a}, \mathbf{b}, \mathbf{c}] \;=\; \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

with

$$\mathbf{A} \;=\; \mathbf{R} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \mathbf{R}^\top,$$

where $\mathbf{R}$ is orthogonal, i.e., $\mathbf{R}\,\mathbf{R}^\top = \mathbf{I}_3$. Hence,

$$\begin{aligned} \mathbf{A} \;&=\; [\mathbf{a}, \mathbf{b}, \mathbf{c}] \cdot \begin{bmatrix} \lambda_2 + \lambda_1 - \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}^\top \\ \mathbf{b}^\top \\ \mathbf{c}^\top \end{bmatrix} \\ &=\; [\mathbf{a}, \mathbf{b}, \mathbf{c}] \cdot \begin{bmatrix} \lambda_1 - \lambda_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{a}^\top \\ \mathbf{b}^\top \\ \mathbf{c}^\top \end{bmatrix} \;+\; \lambda_2\,\mathbf{R}\,\mathbf{R}^\top \\ &=\; (\lambda_1 - \lambda_2)\,\mathbf{a}\,\mathbf{a}^\top + \lambda_2\,\mathbf{I}_3. \end{aligned}$$

□

Proposition 2. *Let* $\mathcal{Q}$ *be a quadric of revolution different from a sphere or a pair of parallel planes, then* $\mathcal{Q}$ *has an equation of the form (2) where*

$$\mathbf{A} \;=\; \begin{bmatrix} a_1^2 + b & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 + b & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 + b \end{bmatrix} \tag{6}$$

and $b \neq 0$.

[1] Spheres are characterized by a single eigenvalue $\lambda \neq 0$ whose multiplicity is then 3.

[2] In case of symmetric matrices the algebraic multiplicity of an eigenvalue is the same as the geometric multiplicity, i.e., the dimension of its corresponding eigenspace.

[3] The first, second and third column of $\mathbf{R}$ contains an eigenvector of $\mathbf{A}$ belonging to λ_1, λ_2 and $\lambda_3 = \lambda_2$, respectively.

Proof. Because $\mathcal{Q}$ is a quadric of revolution matrix $\mathbf{A}$ has exactly two different eigenvalues λ_1 and λ_2 where λ_1 has multiplicity 1, λ_2 has multiplicity 2 and $\lambda_2 \neq 0$. According to Proposition 1 matrix $\mathbf{A}$ can be written in the form (4).

Since $\lambda_1 \neq \lambda_2$ one can divide the quadric equation by $\lambda_1 - \lambda_2$ which after having put

$$b = \frac{\lambda_2}{\lambda_1 - \lambda_2}$$

yields matrix $\mathbf{A}$ according to Eq. 6. □

Remark 1. The following items are emphasized:

a) Vector $\mathbf{a}$ is a direction vector of the quadric of revolution's axis. The condition that $\mathbf{a}$ is normalized is optional because the matrix $\mathbf{Q}$ can be multiplied by an arbitrary factor $\neq 0$.

b) If $b = 0$ parabolic cylinders or pairs of parallel planes are obtained instead of quadrics of revolution.[4] Their intersection with the plane at infinity is the absolute line of the plane $a_1x + a_2y + a_3z = 0$.

Due to Proposition 2 *every* quadric of revolution has an equation of the form (2) where the 3×3 submatrix $\mathbf{A}$ of $\mathbf{Q}$ can be parameterized according to Eq. 6. Conversely, every symmetric 4×4 matrix with $\mathbf{A}$, Eq. 6 as its 3×3 lower right minor represents a quadric of revolution if $b \neq 0$.

It is easy to verify that the determinants of the matrices $\mathbf{A}$ and $\mathbf{Q}$ can be written as follows:

$$\begin{aligned} \det \mathbf{A} &= b^2 \cdot \delta, \\ \det \mathbf{Q} &= b \cdot \Delta \end{aligned}$$

where

$$\delta = b + a_1^2 + a_2^2 + a_3^2, \tag{7}$$

$$\begin{aligned} \Delta = {} & b\,c_0\,\delta - c_1^2(a_2^2 + a_3^2 + b) - c_2^2(a_1^2 + a_3^2 + b) - c_3^2(a_1^2 + a_2^2 + b) \\ & + 2(a_1a_2c_1c_2 + a_1a_3c_1c_3 + a_2a_3c_2c_3). \end{aligned} \tag{8}$$

One can classify the quadrics of revolution where $b \neq 0$ by means of δ and Δ as shown in Table 1.

Table 1: Quadrics of revolution; classification

$\Delta \neq 0,\ \delta \neq 0:$	*ellipsoids or hyperboloids of revolution*
$\Delta \neq 0,\ \delta = 0:$	*paraboloids of revolution*
$\Delta = 0,\ \delta \neq 0:$	*cones of revolution*
$\Delta = 0,\ \delta = 0:$	*cylinders of revolution*

[4]Of course, pairs of parallel planes can also be considered as *special* quadrics of revolution.

3. Problem formulation

The aim is to find quadrics of revolution through n given points P_i, $i = 1, \ldots, n$. This means to find a solution to the equation system

$$[1, p_{ix}, p_{iy}, p_{iz}] \cdot \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_1 & a_1^2 + b & a_1 a_2 & a_1 a_3 \\ c_2 & a_1 a_2 & a_2^2 + b & a_2 a_3 \\ c_3 & a_1 a_3 & a_2 a_3 & a_3^2 + b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ p_{ix} \\ p_{iy} \\ p_{iz} \end{bmatrix} = 0, \;\; i = 1, \ldots, n \tag{9}$$

in the 8 variables $c_0, c_1, c_2, c_3, b, a_1, a_2, a_3$, where p_{ix}, p_{iy}, p_{iz} denote the Cartesian coordinates of any given point P_i. This system can also be written in the form of the *fundamental equation system*

$$\mathbf{P}_{n,10} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ a_1^2 + b \\ a_2^2 + b \\ a_3^2 + b \\ a_2 a_3 \\ a_1 a_3 \\ a_1 a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \tag{10}$$

where

$$\mathbf{P}_{n,10} = \begin{bmatrix} 1 & 2p_{1x} & 2p_{1y} & 2p_{1z} & p_{1x}^2 & p_{1y}^2 & p_{1z}^2 & 2p_{1y}p_{1z} & 2p_{1x}p_{1z} & 2p_{1x}p_{1y} \\ \vdots & & & & & & & & & \vdots \\ 1 & 2p_{nx} & 2p_{ny} & 2p_{nz} & p_{nx}^2 & p_{ny}^2 & p_{nz}^2 & 2p_{ny}p_{nz} & 2p_{nx}p_{nz} & 2p_{nx}p_{ny} \end{bmatrix}. \tag{11}$$

It will be convenient to denote the sub-matrix of $\mathbf{P}_{n,10}$ composed of the first m lines and the first k columns, by $\mathbf{M}_{m,k}$ and the one, consisting of the first m lines and the last l columns, by $\mathbf{N}_{m,l}$.

There are n constraint equations in 8 homogeneous variables. Hence one may, in general, expect

a) an infinite number of solutions if $n < 7$,

b) a finite number of solutions in case of $n = 7$ and

c) no solution if $n > 7$.

The case where $n = 4$ is satisfied by a 3 parameter set is dealt with in Section 4. Section 5 addresses problems pertaining to 5 given points. Section 6 focuses on finding right cylinders on 5 points. Section 7 treats the case $n = 6$. An algorithm to determine the right cones on 6 points is presented in Section 8. Finally, how to find the general quadrics of revolution on $n = 7$ points is exposed in Section 9.

To avoid digression too far into the maze of special or pathological cases it is assumed that the n given points are in *general position* defined as follows.

Definition 1. *Let $P_1, \ldots, P_n$ be points in the Euclidean 3-space then we say that $P_1, \ldots, P_n$ are* in general position *if in case of $n = 4$ the four points are not coplanar and in case of $n \geq 5$ the n points are mutually distinct and neither coplanar nor cospherical.*

4. Quadrics of revolution on four non coplanar points

Let P_1, P_2, P_3, P_4 be non coplanar points then we rewrite Eq. 10 as

$$\mathbf{M}_{4,4} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = -\mathbf{N}_{4,6} \cdot \begin{bmatrix} a_1^2 + b \\ a_2^2 + b \\ a_3^2 + b \\ a_2 a_3 \\ a_1 a_3 \\ a_1 a_2 \end{bmatrix}. \tag{12}$$

Since P_1, P_2, P_3, P_4 are non coplanar the matrix

$$\mathbf{M}_{4,4} = \begin{bmatrix} 1 & 2p_{1x} & 2p_{1y} & 2p_{1z} \\ 1 & 2p_{2x} & 2p_{2y} & 2p_{2z} \\ 1 & 2p_{3x} & 2p_{3y} & 2p_{3z} \\ 1 & 2p_{4x} & 2p_{4y} & 2p_{4z} \end{bmatrix}$$

is regular and thus Eq. 12 is equivalent to

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = -\mathbf{M}_{4,4}^{-1} \cdot \mathbf{N}_{4,6} \cdot \begin{bmatrix} a_1^2 + b \\ a_2^2 + b \\ a_3^2 + b \\ a_2 a_3 \\ a_1 a_3 \\ a_1 a_2 \end{bmatrix}. \tag{13}$$

So we have the following

Result 1. *If P_1, P_2, P_3, P_4 are four non coplanar points, then for any choice of an axis direction $[a_1, a_2, a_3]^\top$ and any $b \neq 0$ one obtains exactly one quadric of revolution on P_1, P_2, P_3, P_4.*

Remark 2. Note the following two items:

a) Choosing a fixed axis direction $[a_1, a_2, a_3]^\top$ and varying b produces a pencil of quadrics of revolution on P_1, P_2, P_3, P_4. This case is described in detail by Strobel [16, pp. 11–16].
b) The unique sphere determined by P_1, P_2, P_3, P_4 is obtained by setting $a_1 = a_2 = a_3 = 0$ and choosing an arbitrary value $b \neq 0$ in Eq. 13.

5. Quadrics of revolution on five given points in general position

Lemma 1. *Let $P_1, \ldots, P_5$ be five points in general position and let $\mathbf{M}_{5,5}^{(5)}$, $\mathbf{M}_{5,5}^{(6)}$ and $\mathbf{M}_{5,5}^{(7)}$ denote the sub-matrices of $\mathbf{P}_{5,10}$ established by the first four and one of the fifth, sixth and seventh columns, respectively, then at least one of these matrices is regular.*

Proof. Suppose that all three matrices are singular; then there exist vectors

$$\begin{aligned} [\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4]^\top &\neq [0,0,0,0,0]^\top, \\ [\beta_0, \beta_1, \beta_2, \beta_3, \beta_4]^\top &\neq [0,0,0,0,0]^\top, \\ [\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4]^\top &\neq [0,0,0,0,0]^\top, \end{aligned}$$

so that

$$\left.\begin{array}{ccccccccccccc}\alpha_0 & + & \alpha_1 p_{ix} & + & \alpha_2 p_{iy} & + & \alpha_3 p_{iz} & + & \alpha_4 p_{ix}^2 & & & = & 0,\\ \beta_0 & + & \beta_1 p_{ix} & + & \beta_2 p_{iy} & + & \beta_3 p_{iz} & + & & \beta_4 p_{iy}^2 & & = & 0,\\ \gamma_0 & + & \gamma_1 p_{ix} & + & \gamma_2 p_{iy} & + & \gamma_3 p_{iz} & + & & & \gamma_4 p_{iz}^2 & = & 0\end{array}\right\} \tag{14}$$

for $i = 1, 2, 3, 4, 5$.

If one of the values α_4, β_4 or γ_4 is zero then the five points lie in a plane contradicting the assumption of being not coplanar. Hence $\alpha_4, \beta_4, \gamma_4 \neq 0$ and we may multiply the three equations of Eq. 14 by α_4^{-1}, β_4^{-1} and γ_4^{-1}, respectively, and then add them to show that $P_1, \ldots, P_5$ belong to a sphere which again contradicts our assumption of the points being in general position. □

Given five points $P_1, \ldots, P_5$ in general position, then according to the Lemma 1 above, one of the matrices $\mathbf{M}_{5,5}^{(5)}$, $\mathbf{M}_{5,5}^{(6)}$ or $\mathbf{M}_{5,5}^{(7)}$, say, $\mathbf{M}_{5,5} = \mathbf{M}_{5,5}^{(5)}$, is invertible. As a consequence the fundamental equation system Eq. 10 can be rewritten in the form

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ a_1^2 + b \end{bmatrix} = -\mathbf{M}_{5,5}^{-1} \cdot \mathbf{N}_{5,5} \cdot \begin{bmatrix} a_2^2 + b \\ a_3^2 + b \\ a_2 a_3 \\ a_1 a_3 \\ a_1 a_2 \end{bmatrix}. \tag{15}$$

The last line of these five equations has the structure

$$kb + q(a_1, a_2, a_3) = 0 \tag{16}$$

where k is a real non-zero[5] number and $q(a_1, a_2, a_3)$ is a non-trivial[6] quadratic homogeneous polynomial in a_1, a_2, a_3. In conclusion we can determine b uniquely from Eq. 16 for every prescribed axis direction $[a_1, a_2, a_3]^\top$ and then c_0, c_1, c_2, c_3 via Eq. 15. So we have the following

Result 2. *If $P_1, \ldots, P_5$ are points in general position then for any choice of an axis direction $[a_1, a_2, a_3]^\top$ one gets exactly one quadric of revolution containing the five points. Only the directions which null the polynomial $q(a_1, a_2, a_3)$ (Eq. 16) yield parabolic cylinders or pairs of parallel planes instead of quadrics of revolution.*

Remark 3. Since we have assumed that the points are in general position we excluded the cases where they lie on a common sphere or in a common plane. In case of $P_1, \ldots, P_5$ lying in a common plane ε each quadric passing through the five points has to intersect ε along the second order curve determined by the points. This case was studied in detail by Narasinga [7] and Röschel [12].

Remark 4. Laguerre [4] proved the following theorem for a generic quintuple P_1, P_2, P_3, P_4, P_5 of points: The axis of each quadric of revolution passing through the five points is an asymptote of a twisted cubic containing the centers M_k of the five circumspheres of $P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}$, ($i_j \in \{1, 2, 3, 4, 5\}$ mutually distinct).

Conversely, any point on the plane of infinity determines an axis direction and together with the five sphere centers M_k a twisted cubic c, in general. The tangent of c at M_k, *i.e.*, asymptote of c, is the axis of a quadric of revolution on the five given points P_1, P_2, P_3, P_4, P_5.

[5]The assumption $k = 0$ leads to a contradiction as follows: If $k = 0$ then Eq. 16 is fulfilled for $a_1 = a_2 = a_3 = 0$ and any choice of $b \neq 0$. Substitution of these values in Eq. 15 yields unique values of c_0, c_1, c_2 and c_3. Therefore the 5 points lie on a sphere and are not in the required general configuration.

[6]The polynomial $q(a_1, a_2, a_3)$ is non-trivial because the coefficient of a_1^2 is 1.

6. Right cylinders on five given points in general position

Cylinders of revolution are characterized by the conditions $\Delta = \delta = 0$ where δ and Δ are determined according to Eq. 7 and Eq. 8. If $\delta = 0$ we have

$$b = -a_1^2 - a_2^2 - a_3^2 \tag{17}$$

and after substitution into the condition $\Delta = 0$:

$$a_1c_1 + a_2c_2 + a_3c_3 = 0 \tag{18}$$

Hence, the right cylinders among all quadrics of revolution can be extracted by requiring that the two Eqs. 17 and 18 are fulfilled.

Determining the right cylinders on five given points $P_1, \ldots, P_5$ in general position is accomplished as follows. Substituting the condition expressed by Eq. 17 into Eq. 15 yields

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ -a_2^2 - a_3^2 \end{bmatrix} = -\mathbf{M}_{5,5}^{-1} \cdot \mathbf{N}_{5,5} \cdot \begin{bmatrix} -a_1^2 - a_3^2 \\ -a_1^2 - a_2^2 \\ a_2a_3 \\ a_1a_3 \\ a_1a_2 \end{bmatrix}. \tag{19}$$

The last of these five equations has the form

$$-a_2^2 - a_3^2 + k_1(a_1^2 + a_3^2) + k_2(a_1^2 + a_2^2) + k_3a_2a_3 + k_4a_1a_3 + k_5a_1a_2 = 0. \tag{20}$$

The left hand side of this equation is a quadratic homogeneous polynomial in a_1, a_2, a_3 which again is non-trivial as one can easily check. Hence, Eq. 20 represents a second order curve c_∞ on the plane at infinity. Real solutions can occur only in the case where c_∞ is a pair of real lines or a real double line or a unipartite conic. In the first two cases one can always find linear homogeneous parametrizations

$$a_i(s,t) = \alpha_{i0}s + \alpha_{i1}t, \quad i = 1,2,3 \tag{21}$$

and in the case where c_∞ being a regular conic with real points a quadratic homogeneous parametrization

$$a_i(s,t) = \alpha_{i0}s^2 + \alpha_{i1}st + \alpha_{i2}t^2, \quad i = 1,2,3 \tag{22}$$

of c_∞. The parameters are s and t while α_{ij} are constants.

By substitution of this parametrization into the first four lines of Eq. 19 one either gets a quadratic parametrization

$$c_i(s,t) = \gamma_{i0}s^2 + \gamma_{i1}st + \gamma_{i2}t^2, \quad i = 0,1,2,3 \tag{23}$$

or a quartic parametrization

$$c_i(s,t) = \sum_{j=0}^{4} \gamma_{ij}\, s^{4-j}\, t^j, \quad i = 0,1,2,3 \tag{24}$$

of the unknowns c_i.

After substituting the (linear or quadratic) parametrization of a_i and the (quadratic or quartic) parametrization of c_i into the condition Eq. 18 one obtains a homogeneous polynomial in s, t whose degree is ≤ 6. The real zeroes s^*, t^* of this polynomial yield possible values for $a_1, a_2, a_3, c_0, c_1, c_2, c_3$ with the help of Eq. 21 or Eq. 22 and Eq. 23 or Eq. 24. This uniquely determines the equation of the corresponding solution cylinder.

Result 3. *The task of determining the right cylinders on 5 given points in general position is a sextic problem.*

This confirms a result of Zsombor-Murray and El Fashny [19].
Applying this method to finding the right cylinders on the five points

$$\begin{array}{lll} P_1 \dots [0.0, 0.0, 0.0]^\top, & P_2 \dots [1.0, 0.0, 0.0]^\top, & P_3 \dots [0.0, 1.0, 0.0]^\top, \\ P_4 \dots [0.0, 0.0, 1.0]^\top, & P_5 \dots [0.8, 0.8, 1.0]^\top & \end{array}$$

produces a solution with the six cylinders shown in Fig. 1.

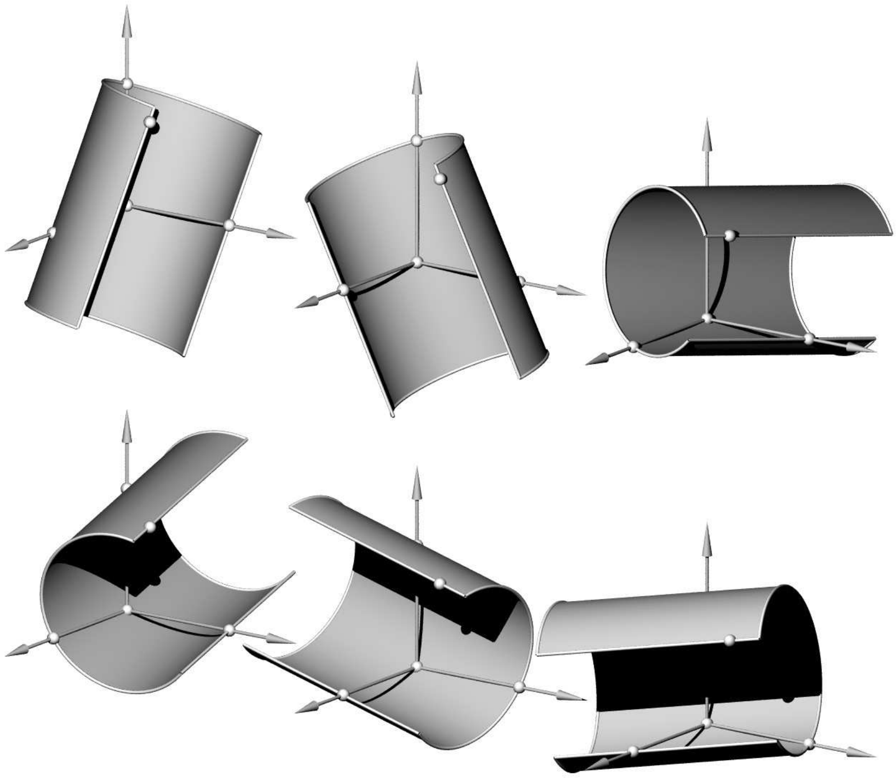

Figure 1: Six right cylinders on five given points

7. Quadrics of revolution on six given points in general position

Let now six points $P_1, \dots, P_6$ in general position be given. In this section we will discuss how to determine all quadrics of revolution on these points.

Since the six points are in general position (mutually distinct and neither coplanar nor cospherical) one can re-enumerate them always in a way that the quadruple P_1, P_2, P_3, P_4 is not coplanar and the points of the quintuples $\{P_1, P_2, P_3, P_4, P_5\}$ and $\{P_1, P_2, P_3, P_4, P_6\}$ are also in general position. (This can be easily checked by elementary considerations.)

Because P_1, P_2, P_3, P_4, P_5 are in general position we have

$$b = q_1(a_1, a_2, a_3) \tag{25}$$

where q_1 is a quadratic homogeneous and non-trivial polynomial according to the results of Section 5. Analogously, we find

$$b = q_2(a_1, a_2, a_3) \tag{26}$$

with another quadratic homogeneous and non-trivial polynomial q_2 by applying the results of Section 5 to the quintuple P_1, P_2, P_3, P_4, P_6.

In conclusion the axis direction numbers a_1, a_2, a_3 have to fulfill the homogeneous quadratic condition

$$f(a_1, a_2, a_3) \quad = \quad q_2(a_1, a_2, a_3) - q_1(a_1, a_2, a_3) \quad = \quad 0$$

which represents a second order curve c_∞ on the plane at infinity. The only exception occurs if the two polynomials $q_1(a_1, a_2, a_3)$ and $q_2(a_1, a_2, a_3)$ are identical in which case the condition on a_1, a_2, a_3 vanishes.

As in the previous section c_∞ can be linearly parameterized as shown by Eq. 21 when c_∞ is a pair of real lines or a double line or quadratically as shown by Eq. 22 if c_∞ is a unipartite conic.

Substituting this parametrization into Eq. 25 or Eq. 26 yields a quadratic or quartic homogeneous parametrization of b in s and t:

$$b \quad = \quad b(s, t) \tag{27}$$

Finally, since P_1, P_2, P_3, P_4 are not coplanar Eq. 13 can be used to obtain quadratic or quartic parameterizations of $c_0, \ldots, c_3$ by substituting Eq. 21 or Eq. 22 and Eq. 27:

$$\begin{bmatrix} c_0(s,t) \\ c_1(s,t) \\ c_2(s,t) \\ c_3(s,t) \end{bmatrix} \quad = \quad -\mathbf{M}_{4,4}^{-1} \cdot \mathbf{N}_{4,6} \cdot \begin{bmatrix} a_1^2(s,t) + b(s,t) \\ a_2^2(s,t) + b(s,t) \\ a_3^2(s,t) + b(s,t) \\ a_2(s,t) \cdot a_3(s,t) \\ a_1(s,t) \cdot a_3(s,t) \\ a_1(s,t) \cdot a_2(s,t) \end{bmatrix} \tag{28}$$

Thus a homogeneous parametrization of the set of all quadrics of revolution on the given six points $P_1, \ldots, P_6$ is available.

Result 4. *In general, there is a one-parametric set of quadrics of revolution on six points $P_1, \ldots, P_6$ in general position. The points at infinity on the axes of these quadrics lie on a second order curve c_∞.*

8. Right cones on six given points in general position

Based on Section 7 the quadrics of revolution on six given points in general position will usually establish a family determined by the parameterizations Eq. 21 or Eq. 22 of a_1, a_2, a_3, Eq. 27 of b and Eq. 28 of c_0, c_1, c_2, c_3. In order to select the right cones from this one degree of freedom family one must substitute these parameterizations into the condition

$$\begin{aligned} \Delta \quad = \quad & b\, c_0\, (b + a_1^2 + a_2^2 + a_3^2) - c_1^2(a_2^2 + a_3^2 + b) - c_2^2(a_1^2 + a_3^2 + b) - c_3^2(a_1^2 + a_2^2 + b) \\ & + 2(a_1 a_2 c_1 c_2 + a_1 a_3 c_1 c_3 + a_2 a_3 c_2 c_3) = 0. \end{aligned} \tag{29}$$

This yields a homogeneous bivariate polynomial in s, t of degree ≤ 12. Every real solution s^*, t^* of this polynomial yields a solution cone.

Result 5. *The problem of finding the right cones on 6 given points in general position can be reduced to the task of determining the zeroes of a univariate polynomial of degree 12, in general. Every real solution of this polynomial corresponds to exactly one solution cone.*

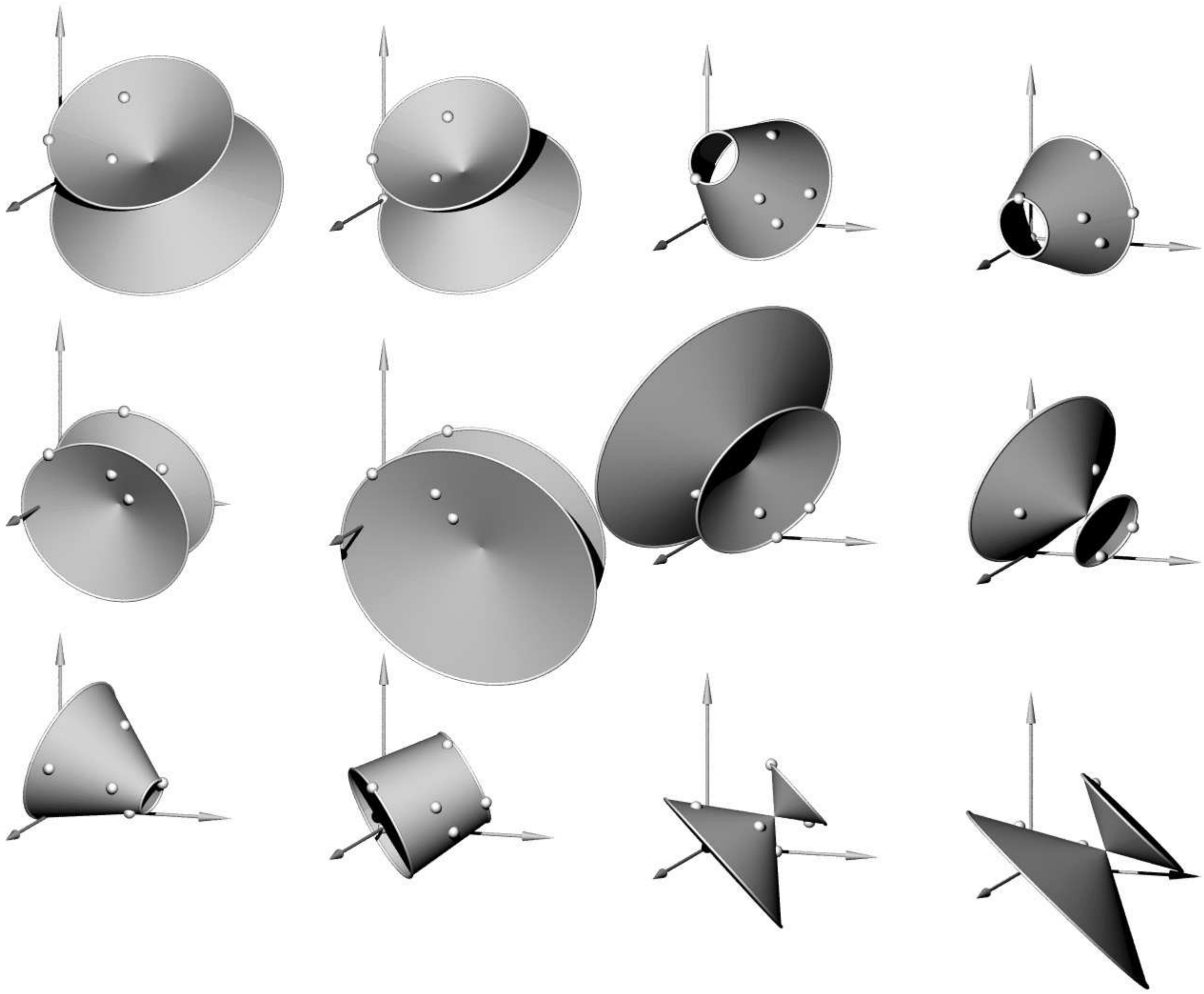

Figure 2: Twelve right cones on six given points

Remark 5. E. Müller [6] studied the problem of finding the right cones with prescribed axis direction on four points.

The algorithm described above was applied to the following six given points:

$$P_1 \ldots [0.0, 0.0, 0.0]^\top, \quad P_2 \ldots [2.0, 2.0, 1.0]^\top, \quad P_3 \ldots [1.0, 3.0, 1.0]^\top,$$
$$P_4 \ldots [1.0, 2.0, 2.4]^\top, \quad P_5 \ldots [2.5, 0.5, 1.5]^\top, \quad P_6 \ldots [1.5, 2.3, 0.3]^\top$$

In this case the twelve solution cones displayed in Fig. 2 were obtained.

9. Quadrics of revolution on seven given points in general position

If 7 points $P_1, \ldots, P_7$ in general position are given then it is easy to verify that one can always find sextuples of this set, say, $\{P_1, P_2, P_3, P_4, P_5, P_6\}$ and $\{P_1, P_2, P_3, P_4, P_5, P_7\}$ whose points are also in general position.

Then, according to Section 7 the axes' points at infinity of the quadrics of revolution on $P_1, P_2, P_3, P_4, P_5, P_6$ are situated on a second order curve

$$c_{1,\infty}: \quad f_1(a_1, a_2, a_3) \;=\; 0, \tag{30}$$

in general. Analogously, each quadric of revolution on $P_1, P_2, P_3, P_4, P_5, P_7$ has an axis whose point at infinity generally lies on another second order curve

$$c_{2,\infty}: \quad f_2(a_1, a_2, a_3) \;=\; 0 \tag{31}$$

Hence, we have

Result 6. *The points at infinity of the axes of the quadrics of revolution on seven given points in general position lie in general on the intersection of two second order curves $c_{1,\infty}$ and $c_{2,\infty}$.*

In conclusion there are at most four possible intersection points if $c_{1,\infty}$ and $c_{2,\infty}$ are not coincident in whole or in part; the latter in the case of distinct line pairs sharing a line.

If a_1^*, a_2^*, a_3^* are the coordinates of an axis' point at infinity satisfying Eq. 30 and Eq. 31 then the corresponding values for b and c_0, c_1, c_2, c_3 can be determined uniquely by applying the method described in Section 7.

Result 7. *In general there are at most four quadrics of revolution on seven given points in general position.*

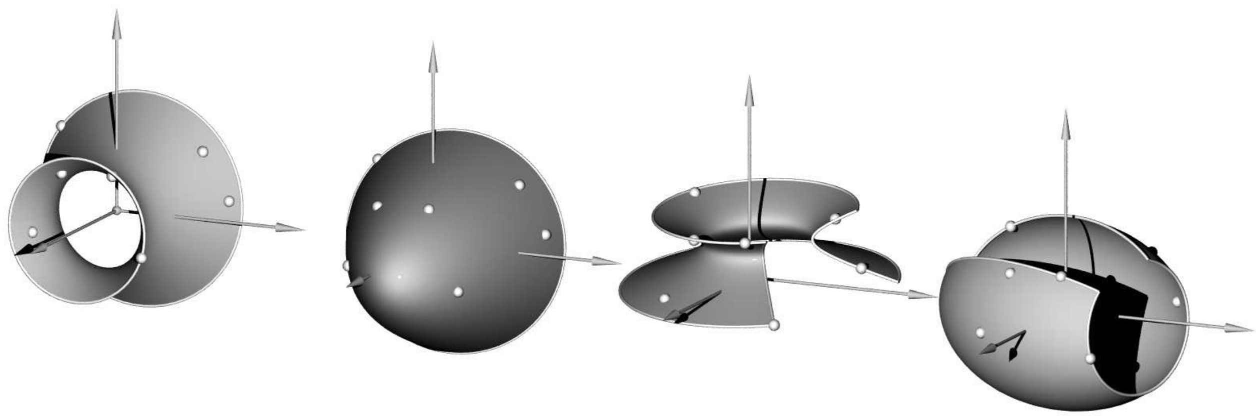

Figure 3: Four quadrics of revolution on seven given points

With the following seven given points one obtains four solution quadrics of revolution, namely a one-sheet hyperboloid, a two-sheet hyperboloid, another one-sheet hyperboloid and an ellipsoid (from left to right in Fig 3; only one sheet of the two-sheet hyperboloid is displayed.)

$$\begin{array}{lll} P_1 \ldots [3.1, -1.2, 2.2]^\top, & P_2 \ldots [3.5, 1.4, 2.4]^\top, & P_3 \ldots [3.3, -2.5, -0.5]^\top, \\ P_4 \ldots [4.2, 3.1, -0.9]^\top, & P_5 \ldots [-2.2, 4.3, 0.36]^\top, & P_6 \ldots [-1.9, 3.2, 2.5]^\top, \\ P_7 \ldots [-2.6, -3.8, 2.8]^\top & & \end{array}$$

10. Conclusions and unresolved problems for future research

The following list describes some relevant topics that remain to be investigated.

- Omitted cases where points lie on a sphere might be considered in the light of Remark 4, *i.e.*, LAGUERRE's work [4].

- Quadrics of revolution on $n = 5$, 6 or 7 given *coplanar* points exist only if these lie on c a proper conic in that plane. This is generally the case where $n = 5$. It was treated by Narasinga [7] and Röschel [12] but the special case where c degenerates to a line pair or a double line has not received attention.
- In Section 7 (quadrics of revolution on 6 points) the case where the two quadratic polynomials $q_1(a_1, a_2, a_3)$ and $q_2(a_1, a_2, a_3)$ are identical was excluded. How can this case be geometrically characterized?
- An infinite number of quadrics might contain 7 given points (Section 9) if the two second order curves $c_{1,\infty}$ and $c_{2,\infty}$ have a common component. What are the geometric characteristics of this case?
- Moreover methods to discriminate among the types of quadrics of revolution, *i.e.*, ellipsoids, paraboloids and hyperboloids of one and two sheets, have yet to be formulated.
- Another task, pertinent to application in camera aided inspection, is to find the most suitable quadric of revolution through an overdetermined set of points $\{P_1, \ldots, P_n\}$, $n > 7$. Using a setup with the parametrization Eq. 6 it can be shown that this problem can be reduced to a pair of homogeneous quartic equations in the axis direction numbers a_1, a_2, a_3.
  Alternatively, one could choose various subsets of 7 points from a given set of $n > 7$ points and then apply the method described in Section 9 to determine a quadric of revolution. This quadric could then be taken as a starting point in, *e.g.*, a least-squares Gauss-Newton fit optimization procedure. Deficiencies or defects stemming from singularity of the involved matrices $\mathbf{M}_{k,k}$ are easy to detect. In such a case the chosen 7-tupel of points can be easily replaced by another. After all, camera data contain a great many points.

It is believed that the unified treatment of the three types of quadrics of revolution exposed herein is original and constitutes a solid foundation upon which to build effective, precise methods to quickly fit overdetermined point sets to such surfaces. This will be invaluable in camera aided inspection of manufactured structural and machine elements of such cross section, a fruitful extension of this research that will have important industrial applications.

Acknowledgements

This research is supported by a Natural Sciences and Engineering Research Canada "Discovery" grant. Thanks are also extended to Manfred Husty from the University of Innsbruck and Johannes Wallner and Andreas Weinmann, both from TU Graz, for some fruitful discussions during the preparation of this paper.

References

[1] C. Beder, W. Förstner: *Direct Solutions for Computing Cylinders from Minimal Sets of 3D Points*. European Conf. on Computer Vision, Graz, pp. 135–146 (2006).

[2] O. Devillers, B. Mourrain, F.P. Preparata, P. Trebuchet: *Circular Cylinders by Four or Five Points in Space*. Discrete and Computational Geometry **29**, 83–104 (2003).

[3] N. Koller, R. Ofner, P. O'Leary, E. Fauster: *Optical Servoing for Industrial Surface Machining*. Proc. of SPIE, Machine Vision Applications in Industrial Inspection XIV 6017, pp. 1–11 (2006).

[4] E. Laguerre: *Sur la courbe enveloppée par les axes des coniques qui passent par quatre points donnés et sur les axes des surfaces de révolution du second ordre qui passent par cinq points donnés. Sur les lignes spiriques*. Nouv. Ann. Math. (2) **18**, 206–218 (1879).

[5] D. Lichtblau: *Cylinders Through Five Points*. Complex and Real Enumerative Geometry, F. Botana, E. Roanes-Lozano (eds): Automated Deduction in Geometry ADG (2006).

[6] E. Müller: *Durch vier Punkte einen Drehkegel von vorgegebener Achsenrichtung zu legen*. Arch. d. Math. u. Phys. (3) **22**, 225–226 (1914).

[7] A. Narasinga Rao: *Quadrics of Revolution through a Given Conic*. J. Ind. Math. Soc. **17**, 213–222 (1928).

[8] A. Narasinga Rao, M.S. Srinivasachari: *Quadrics of Revolution through a Pair of Skew Lines*. J. Annamali Univ. **2**, 71–77 (1933).

[9] P. O'Leary, M. Harker, P.J. Zsombor-Murray: *Direct and Least Squares Fitting of Coupled Geometric Objects for Metric Vision*. Vision, Image and Signal Processing, IEEE Proceedings-Publication (6) **152**, 687–694 (2005).

[10] P. O'Leary, P. Schalk, R. Ofner, A. Gfrerrer: *Instrumentation and Analysis-Methods for the Measurement of Profiles using Light Sectioning*. Proc. of 23rd IEEE Instrumentation and Measurement Technology Conference, pp. 1108–1113 (2006).

[11] F. Pernkopf, P. O'Leary: *Image Acquisition Techniques for Automatic Visual Inspection of Metallic Surfaces*. NDT&E International **36**, 609–617 (2003).

[12] O. Röschel: *Drehflächen zweiter Ordnung durch einen Kegelschnitt*. Stud. Sci. Math. Hung. **29**, n. 3-4, 379–386 (1994).

[13] H. Schaal: *Ein geometrisches Problem der metrischen Getriebesynthese*. Sitzungsber., Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. **194**, n. 1-3, 39–53 (1985).

[14] H. Schaal: *Konstruktion der Drehzylinder durch vier Punkte einer Ebene*. Sitzungsber., Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. **195**, n. 4-7, 406–418 (1986).

[15] U. Strobel: *Über die Drehkegel durch vier Punkte, Teil I*. Sitzungsber., Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. **198**, 281–293 (1989).

[16] U. Strobel: *Die Drehkegel durch vier Punkte*, Dissertation, Universität Stuttgart, 117 pages (1990).

[17] U. Strobel: *Über die Drehkegel durch vier Punkte, Teil II*. Sitzungsber., Abt. II, österr. Akad. Wiss., Math.-Naturw. Kl. **200**, 91–109 (1991).

[18] M. Tratnig, P. O'Leary, H. Hlobil, J. Reisinger: *Camera Calibration, Data Segmentation, and Fitting Approaches for a Visual Edge Inspection System*. Proc. of SPIE, Machine Vision Applications in Industrial Inspection XII 5303, pp. 42–51 (2004).

[19] P.J. Zsombor-Murray, S. El Fashny: *A Cylinder of Revolution on Five Points*. J. Geometry Graphics **10**(2), 125–131 (2007).

Received August 3, 2009; final form November 18, 2009