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Dynamic Differential Geometry in Education

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Abstract. We present an augmented reality application which introduces differential geometry in educational dynamic geometry software. New functionality such as a Frenet frame, center and circle of curvature in arbitrary curve points, and others were implemented. Dynamic geometry allows to study differential geometric properties under movement. Using this tool we developed examples which enable teachers and learners to intuitively explore properties of interesting curves, to visualize contact of higher order between curves and surfaces, to construct Meusnier's sphere, Dupin's indicatrix and more.

Key Words: Differential geometry, dynamic three-dimensional geometry, augmented reality, geometry education.

MSC 2000: 53A04, 53A05, 68U05

1. Introduction

Concepts and findings that originate from differential geometry [7, 30, 4, 2] are applicable in many areas such as physics [10], economics, computer graphics (e.g. [8]), engineering in general and geology. To understand the potential of differential geometry and its areas of application the topic is taught as part of many higher education curricula of technical studies worldwide. In this work we present a dynamic geometry software application to aid teaching the basics of differential geometry of space curves in a very visual and interactive way (Fig. 1).

Therefore an *augmented reality* (AR) [1] application for dynamic geometry education has been extended to support operations such as the creation of Frenet frames in points on curves, the plane, center and circle of curvature and others. This software tool is designed to support teaching and learning of basic geometric principles and properties of different types of curves and surfaces. The main advantage of using AR is that students actually see three-dimensional objects which they until now had to calculate and construct with traditional methods. Due to evaluations and observations [17] we hypothesize that by working directly in 3D space, complex spatial problems and spatial relationships can be comprehended better and faster than with traditional methods.

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After briefly summarizing related work in the area of dynamic geometry software and differential geometry we present our software tool Construct3D and its extension together with practical, intuitive examples for teaching differential geometry in higher education. They demonstrate which flexibility and potential three-dimensional dynamic geometry holds in teaching differential geometry.

Figure 1: A student working with Construct3D in our standard AR lab setup with a head mounted display

2. Related work

In Austrian schools the use of commercial 3D computer-aided design (CAD) software such as AutoCAD, Pro/ENGINEER, MicroStation, CATIA and others is wide spread in modern geometry education for teaching principles of 3D modeling. In addition there are excellent educational 3D modeling programs such as CAD3D [25] or GAM [27] (developed by Austrian geometers specifically for students) which are frequently used.

In addition to classical educational CAD tools such as CAD3D and GAM a new category of educational geometry software emerged in recent years.

2.1. Dynamic 2D geometry software

Since a computer can record the way we construct geometric objects the software is able to quickly redo constructions after changing some parameters. A fundamental property of dynamic geometry software is that the dynamic behavior of a construction can be explored by interactively moving individual defining elements such as control points of a Bézier curve: pick a point, move it and see immediately how the construction changes. This dragging capability is a fundamental improvement compared to drawings on paper or static CAD models.

Comprehensive work on dynamic geometry was done by Kortenkamp in “Foundations of Dynamic Geometry” [20]. The first software packages for dynamic geometry were Geometer’s Sketchpad [14], which appeared first in 1989, and Cabri Geometry [21], dating back to 1988. Since then dynamic geometry software has spread in education. Today, there are more than 40 packages for dynamic geometry. The most popular ones are Cinderella [28], Euklid [23], Geometer’s Sketchpad or Cabri Geometry. All of them support two-dimensional geometry only.

2.2. Dynamic 3D geometry software

In late 2004 the first three-dimensional dynamic geometry desktop application Cabri 3D was presented [6]. The current version supports basic 2D and 3D objects and the intersection of lines and planes with these objects but lacks support for general intersection curves between surfaces, Boolean operations and more complex geometric primitives such as surfaces of revolution which are present in Construct3D. Lenghts, angles, areas and volumes can be measured and further calculations can be performed with these results. Animations can be used for modeling physical phenomena. A tool replays the user's previously performed construction steps. Unfolding of all polyhedra into a printable net is supported as well.

Archimedes Geo3D [11] is a cross-platform 3D dynamic geometry application which is under development by Andreas Goebel since 2006. Similar to 2D dynamic geometry software Archimedes Geo3D supports the creation of loci which are traces of points, i.e. curves. In addition it is also possible to create traces of curves in 3D, which are surfaces. Points and basic shapes can be used as input but curves and surfaces can also be defined using mathematical parametrizations. Further available features are texturing, animation creation and shadow generation. Macros can be used to record and replay multiple construction steps and can also be called recursively. Archimedes Geo3D supports stereoscopic output either by anaglyph images or by using shutter glasses [11].

2.3. Professional CAD software

Similarities exist between variational or parametric CAD modelling and dynamic geometry software. In general small changes of parameters in a CAD construction do not cause stringent topological changes in the construction. This can be used for instance to customize a single prototype construction quickly or in case of data compression for storage of a large number of similar objects. The problems that occur in parametric CAD [12, 13] are similar to those of dynamic geometry. Parts of these problems are discussed and solved in [20].

Only few commercial 3D CAD software packages provide differential geometry functionality and are therefore related to this work. Rhino3D (`www.rhino3d.com`) and Pro/ENGINEER are two such examples. Professional CAD packages are usually not interactive in a sense that changes are applied in real time in comparison to dynamic geometry software which always provides immediate feedback to the learner. Because of their fields of application they are not necessarily optimized to deliver real time results. CAD modelling tools fulfill stringent accuracy requirements and are typically used for models of higher complexity compared to those used in education when learning about surface properties.

Rhino3D provides features for the analysis of curves on surfaces in order to visualize Gaussian curvature, mean curvature, and the minimum or maximum radius of curvature. Pro/Engineer and other CAD packages offer similar curve and surface analyzing tools. None of the above presented tools allows to study differential geometric properties of curves and surfaces in a real-time dynamic — in the sense of dynamic 3D geometry — way.

In the following we present Construct3D which is the first 3D dynamic geometry application that provides functions to explore curves and surfaces using dynamic differential geometry. In addition we demonstrate through a series of educational examples which 'added value' dynamic geometry provides to teaching differential geometry and how it enhances understanding of fundamental geometric knowledge.

3. Construct3D

Construct3D [18, 16] is a three-dimensional dynamic geometry construction tool which has been designed for educational use. Three usability studies with more than 100 students have been conducted since 2000 [17] and guidelines have been formulated regarding how to design AR applications for (geometry) education [19]. A collaborative augmented reality (AR) setup is utilized with the main advantage that students actually see three dimensional objects in 3D. The setup supports two collaborating users wearing stereoscopic see-through head mounted displays (HMDs) (Sony Glasstron D100BE) providing a common, shared virtual construction space. One PC with two graphic ports renders stereoscopic views for both users. Head and hands are tracked with millimeter accuracy using an iotracker [26] optical tracking system. This allows students to walk around objects and to view them from different perspectives.

Construct3D's menu system is mapped to a hand-held pen and panel interface, the Personal Interaction Panel (PIP) [31] (Fig. 2). The pen is used for operating the menu on the panel as well as for direct manipulation of the scene in 3D space. Augmented reality is used so that both users see the same virtual objects as well as each others' pens and menus, therefore a user can provide help to another user if desired. The face-to-face setting allows for traditional pedagogic communication between teacher and students. Other setups for educational use have been reported in [16].

Construct3D is based on the Studierstube software platform [29] as a runtime environment and for multi-user synchronization. The current version of Construct3D offers functions for the construction of 3D points and geometric objects. It provides planar and spatial geometric operations on objects, measurements, and structuring of elements into '3D layers'. It supports generation of and operation on these object types: Points (either freely positioned in space or fixed on curves and surfaces), lines, planes, circles, ellipses, cuboids, spheres, cylinders, cones, B-Splines curves, NURBS surfaces up to 8x8 control points and variable degree, and surfaces of revolution. To mention just a few, the following geometric operations are implemented: Boolean operations (union, difference, intersection) on 3D objects, intersections between all types of 2D and 3D objects resulting in intersection points and curves as first class objects, planar slicing of objects, rotational sweeps, helical sweeps, general sweeps along a path, surface normals, tangential planes, tangents and many more. The system features support for 3D dynamic geometry. All points can be picked and dragged at any given time. Experiencing what happens under movement allows better insight into a particular construction and geometry in general.

A comprehensive overview of Construct3D is given in [18, 16].

3.1. Geometry kernel

Construct3D utilizes the ACIS geometry kernel for a wide range of calculations. The 3D ACIS Modeler [5] is Spatial's 3D modeling development technology used by developers worldwide, in industries such as CAD/CAM/CAE, AEC, animation, and shipbuilding. It is the geometry kernel of Autocad and many other well known CAD applications. ACIS is under development for more than 15 years and features an object-oriented C++ architecture that enables robust, 3D modeling capabilities. It integrates wireframe, surface, and solid modeling functionality with both manifold and non-manifold topology, and a rich set of geometric operations.

In Construct3D the ACIS geometry kernel has been integrated especially for calculating Boolean operations, intersections, tangents and tangential planes, sweep and helical surfaces as well as NURBS and B-Spline surfaces. ACIS uses mathematical boundary representations

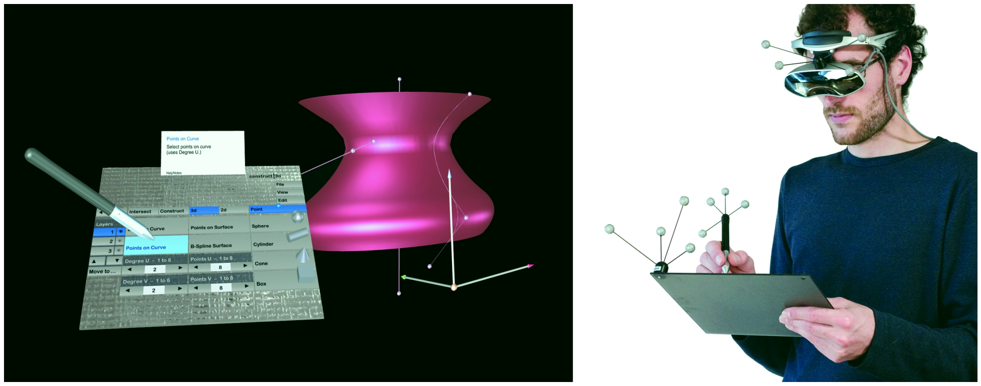
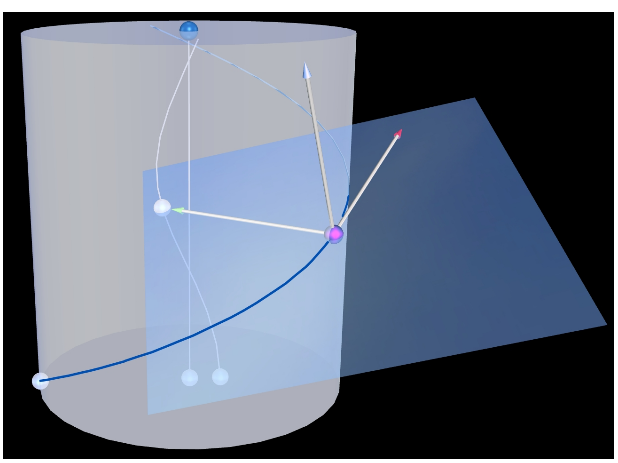


Figure 2: Right: A student working with Construct3D holds a wireless pen and a panel which are optically tracked using retro-reflective markers. Left: The current view of the student in the head-mounted display. The menu system on the panel is visible.

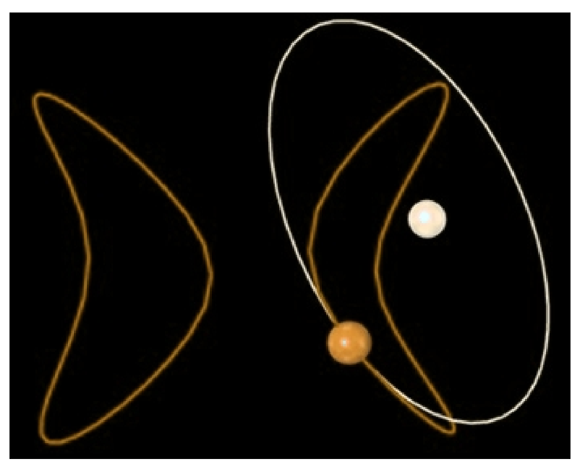
internally and provides methods to calculate derivatives of arbitrary order in a given point on curves and surfaces (as long as they are differentiable). The ACIS documentation states that "a certain number of derivatives are evaluated directly and accurately; higher derivatives are automatically calculated by finite differencing. The accuracy of these decreases with the order of the derivative, as the cost increases." This functionality allowed the straightforward extension of Construct3D to visualize basics of differential geometry in three-dimensional space.

3.2. Differential geometry functions

New features were implemented in Construct3D to support the creation of Frenet frames in points of curves, the plane, center and circle of curvature and the osculating sphere (sphere of curvature). In the following basic differential geometric objects and their properties are described but for further details we refer to standard monographs such as [7, 30, 4, 2].



(a) Frenet frame in a point P of a helix together with the plane of curvature in P. The center of curvature moves along a helix as well (Section 4.1).



(b) Center and the corresponding circle of curvature (white) in a point on the intersection curve between two cylinders.

Figure 3: Frenet frame, plane, center and circle of curvature in Construct3D

3.2.1. Frenet frame

The Frenet frame or Frenet trihedron is a reference frame, a rectilinear coordinate system attached to a point of a space curve consisting of the tangent $\mathfrak{t}$, normal $\mathfrak{n}$, and the binormal vector $\mathfrak{b}$ which are defined as

$$\begin{aligned}\mathfrak{t} &= \mathfrak{x}'(s),\\ \mathfrak{n} &= \frac{\mathfrak{x}''(s)}{|\mathfrak{x}''(s)|},\\ \mathfrak{b} &= \mathfrak{t}\times\mathfrak{n}\end{aligned}$$

with a non-degenerate curve $\mathfrak{x}$, parametrized by its arclength s. In Construct3D the Frenet frame can be attached to as many points of a curve as desired. During movement of such points along the curve (or while changing the curve itself, e.g., by moving control points of a B-spline curve) the trihedron travels along which allows to study tangent, normal and binormal of the curve in any point at any time (Fig. 3(a)).

3.2.2. Plane, center and circle of curvature

The plane of curvature, or osculating plane, in a curve point contains the circle of curvature of a space curve and is spanned by the normal vector $\mathfrak{n}$ and the tangent $\mathfrak{t}$. The circle of curvature osculates the curve (Fig. 3(b)). Its midpoint $\mathfrak{M}$ — the center of curvature — lies in direction of the normal vector of the curve. The distance between center of curvature and curve point is the radius r of the circle of curvature. In case of arc length parametrization of the curve $\mathfrak{x}$ the corresponding curvature is computed by $\kappa(s) = |\mathfrak{x}''(s)|$. With the help of the Frenet formulas the position of the center of curvature can be derived $\mathfrak{M}(s) = \mathfrak{x}(s) + \frac{1}{\kappa}\mathfrak{n}(s)$. The radius of curvature is therefore inversely proportional to the curvature $\rho(s) = \frac{1}{\kappa(s)}$.

3.2.3. Osculating sphere

An osculating sphere, or sphere of curvature has contact of at least third order with a curve $\mathfrak{x}$. The osculating sphere in P can also be defined as the limit of a variable sphere passing through four points of $\mathfrak{x}$ as these points approach P — a property that is used in example 4.2.

The center $\mathfrak{M}$ of any sphere which has contact of (at least) second order with $\mathfrak{x}$ at point P, where the curvature $\kappa > 0$, lies on the axis of curvature (also called polar axis) which is parallel to the binormal passing through the center of curvature corresponding to P. The torsion of a curve point can be regarded a measure of the rotation of the corresponding plane of curvature around the tangent. The osculating sphere has center

$$\mathfrak{M}(s) = \mathfrak{x}(s) + \rho(s)\mathfrak{n}(s) + \frac{\rho'(s)}{\tau(s)}\mathfrak{b}(s).$$

All respective derivatives are calculated by the ACIS kernel in real time whenever the position of a curve point P changes in order to update the center of the osculating sphere while moving P.

4. Teaching contents for dynamic 3D differential geometry

To demonstrate Construct3D's potential in dynamic differential geometry we present teaching contents. Previous evaluation studies identified the main strengths of Construct3D as an augmented reality teaching aid: The biggest advantages compared to traditional software tools are obvious if using Construct3D for teaching content which utilizes three-dimensional dynamic geometry and requires the visualization of abstract problems. We noticed that students need to be challenged to use dynamic functionality. Otherwise some of them are satisfied with constructing static models and do not intend to explore on their own. Therefore examples are introduced which require to study constructions under movement to foster active exploration. Our approach of active, explorative learning is in accordance with pedagogic theories such as activity theory [9] and constructivism [15, 22, 24].

The examples range in difficulty from higher grade high school to basic university mathematics and geometry education. For each example we provide brief background knowledge as a quick summary of the topic and highlight properties which are most relevant and most interesting in regard to dynamic geometry.

4.1. Tangent, normal, binormal

A good starting point is to study Frenet frames in various curve points. We construct a helix and display the corresponding cylinder that contains the helix. When moving a Frenet frame along a helix (Fig. 3(a)) diverse curve properties can be studied. Students will soon notice that the slope of the tangent does not change when moving the point P along the curve. Construct3D allows to measure angles and updates the measurement in real time. Measuring the angle between a tangent in P and a generator line in P (which both move with P) or between the tangent and a plane normal to the axis of the helix through P returns a constant value in all positions of P. It helps to realize that the slope of the tangent stays constant.

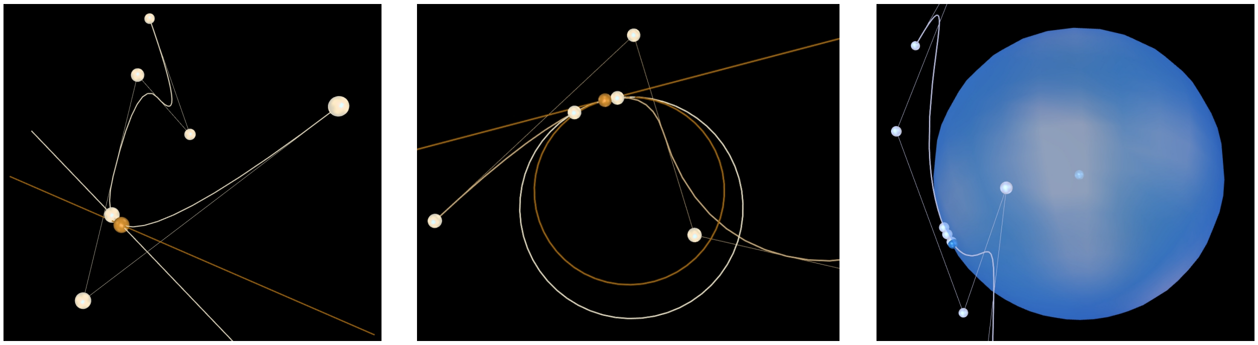
The curvature and the torsion of a helix are constant. Conversely, any space curve with constant non-zero curvature and constant torsion is a helix. Constant curvature can again be observed: Since the curvature is constant the center of curvature moves on an offset curve to the original helix with the same axis which is a helix itself. This can easily be seen in dynamic geometry by moving a point along the helix and studying its center of curvature during movement (Fig. 3(a)). Equivalentely the centres of the osculating spheres of a helix are on a helix which can be visualized with Construct3D as well.

In this context it might also be reasonable to discuss the geodesic property of a helix on a cylinder as well as the helix as a loxodrome of the cylinder and as a line of constant slope. This is just one example of how Frenet frames can be used in dynamic geometry to learn about properties of curves.

4.2. Tangency and contact of m-th order

Introducing differential calculus in high school frequently starts by introducing the difference quotient. Graphically the slope of a tangent to a function graph is calculated by choosing two points on a curve. One is the point T of interest where the slope of the tangent needs to be calculated, the other one is an arbitrary point P on the curve. The slope of their chord is computed. When moving P closer to T the chord converges to the tangent and the difference quotient becomes the derivative in the limit case. We call the tangent to be in first order contact with the curve. This can be quickly visualized in Construct3D using an arbitrary curve, e.g., a B-Spline curve such as in Fig. 4(a).

In general a curve $\mathfrak{c}$ touches a surface $\Phi : F(x_1, x_s, x_3) = 0$ in point $\mathfrak{c}(s_0)$ in $(m+1)$ points (which is contact of order m) if the function $g := F \circ \mathfrak{c}$, i.e., $g(s) = F(x_1(s), x_2(s), x_3(s))$



(a) The tangent (orange) as the limit case of the chord (white) of two converging points.

(b) Second order contact: A circle of curvature (orange) to a curve (orange) and the approximating circle (white) through three converging points.

(c) Third order contact: The approximation of an osculating sphere (blue) through four converging points (white) on a B-Spline curve is shown.

Figure 4: Visualizing contact of higher order in dynamic geometry

possesses the root of multiplicity $(m+1)$, i.e.,

$$g(s_0) = g'(s_0) = \cdots = g^{(m)}(s_0), \quad g^{(m+1)}(s_0) \neq 0$$

Given the possibility of constructing the circle of curvature and the sphere of curvature in a curve point, together with the option of dynamically moving points on curves we can visualize the principle of higher order contact in Construct3D.

Utilizing the 'circle of curvature'-feature the circle in a point P has been constructed in Fig. 4(b) (orange) to the given curve. It serves as a reference and represents the limit case. In addition two points on the curve were chosen and moved close to P. The circle passing through all three points is displayed in white. Students can move these points and compare the circle to the limit case of second order contact. In the limit case all three points have identical position and the circles coincide.

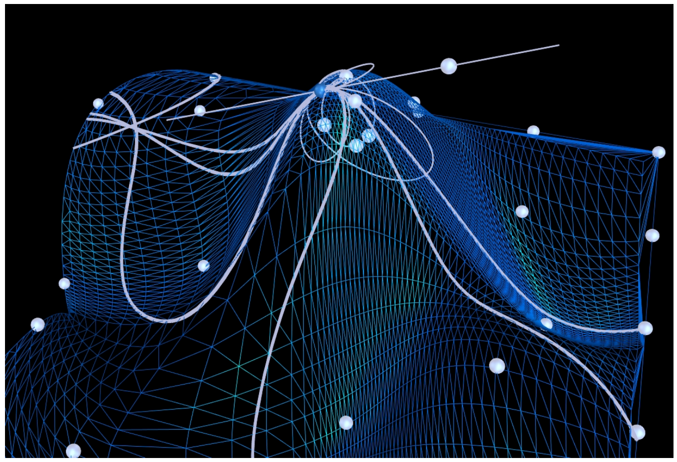
The osculating plane in P possesses second order contact with the curve as well. Likewise the plane of curvature can be approximated by a plane through three converging curve points [3].

Third order contact is established between an osculating sphere and a curve in point P. For a curve $\mathfrak{c}$, the limiting sphere is obtained by taking the sphere that passes through P (drawn blue in Fig. 4(c)) and three other points on $\mathfrak{c}$ and then letting these three points converge towards P independently along $\mathfrak{c}$.

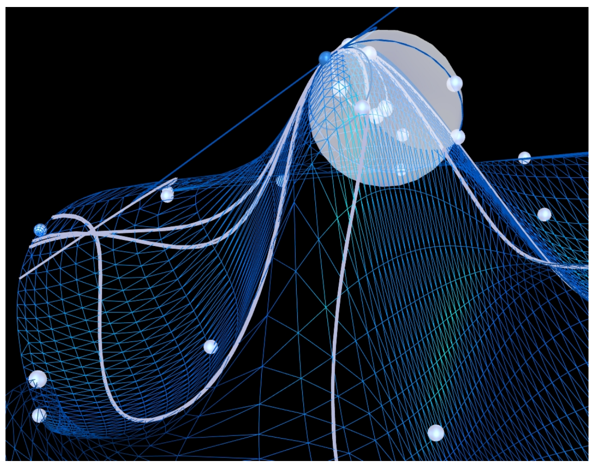
4.3. Meusnier point

Jean Baptiste Meusnier's Theorem (1779) states that all curves lying on a surface Φ and having at a given point $P \in \Phi$ the same tangent t have the same normal curvature κ_n in this point P. Therefore the normal curvature κ_n is a property of the line element (P, t). Meusnier's Theorem further implies that the circles of curvature in (P, t) of all curves through P with tangent t lie on a common sphere called Meusnier sphere. The midpoint of the Meusnier sphere is the Meusnier point. The centers of curvature of all circles of curvature lie on a common circle.

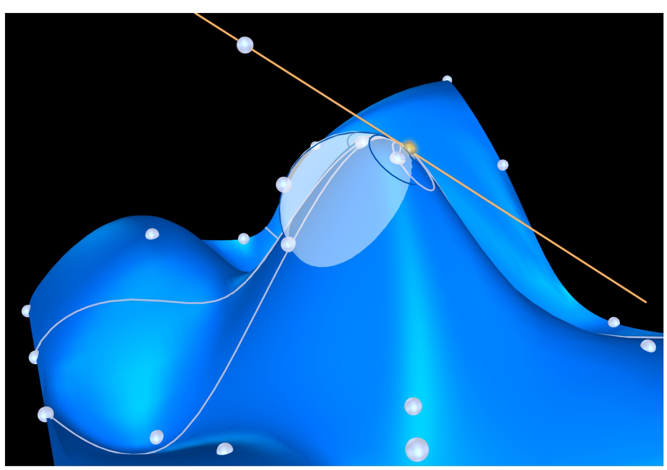
To construct Meusnier's point and sphere in Construct3D we take an arbitrary surface Φ — a NURBS surface with a 5×5 control patch was chosen in Fig. 5. We pick an arbitrary



(a) Intersection curves of four planes through (P, t) (P in blue, t white) with the NURBS surface Φ (blue wireframe). The circles of curvature to three curves in P are visible (white) as well.



(b) The circle containing all centers of curvature (white) to (P, t) also contains the Meusnier point. The Meusnier sphere is displayed transparent white.



(c) Meusnier sphere in (P, t) (orange) containing the circles of curvature (blue).

Figure 5: Meusnier point and Meusnier sphere

point P on Φ, take an arbitrary tangent t through P in its tangential plane to Φ. For further constructions (P, t) is the line element of our choice. Three arbitrary planes through (P, t) are intersected with Φ resulting in three intersection curves. We construct the centers and circles of curvature to all these curves in P and get three centers and circles of curvature (Fig. 5(a)). The circle containing all three centers of curvature can be seen in Figs. 5(a) and (b) (a small white circle containing four points). We visually verified it by checking if a fourth center of curvature coincides with it as well.

The Meusnier sphere contains all circles of curvature in (P, t) and therefore also the constructed ones. Four points were chosen on the circles and then the sphere passing through all of them was constructed (by intersecting their symmetry planes). This gives the center of the sphere, the Meusnier point to (P, t) (Fig. 5(b)). Finally the Meusnier sphere in (P, t) is shown in Fig. 5(c).

By moving P on Φ the Meusnier sphere, the circles of curvatures, the intersection curves and all other depending elements can be studied.

In a teaching lesson students can for example investigate in which cases the Meusnier point is identical to the center of the osculating sphere in a curve point. By dynamic exploration it is straighforward to find cases where the Meusnier sphere degenerates.

4.4. Classification of points on a surface

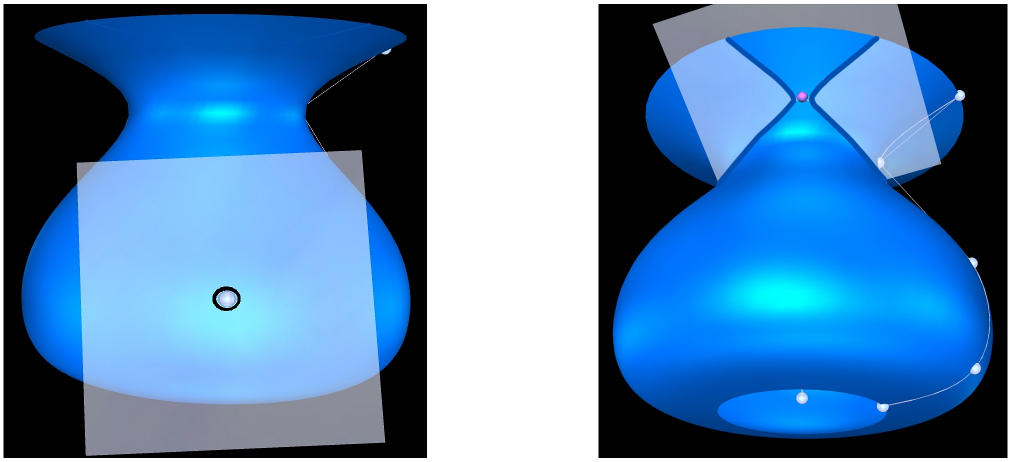
Curvature in curve points can be visualized well by drawing the circle of curvature. The curvature vector of a curve $\mathfrak{c}$ on a surface can be written as $\mathfrak{c}'' = \kappa_g \mathfrak{g} + \kappa_n \mathfrak{n}$. $\mathfrak{g}$ is a vector normal to the normal vector $\mathfrak{n}$ in the tangential plane, κ_g is called geodesic curvature and κ_n normal curvature.

Surface points can be classified regarding their curvature into elliptic, hyperbolic and parabolic points. The Dupin indicatrix visualizes and describes curvature properties at a point of a surface. It is named after Pierre Charles Francois Dupin (1813), who was the first to use this curve in the study of surfaces.

Dupin's indicatrix lies in the tangential plane to the surface Φ at point P. $\rho^N(t)$ is the radius of the normal curvature in direction t: $\rho^N(t) = 1/\kappa_n(t)$ and k is a positive number $k > 0$. If we take any tangent t in the tangential plane of point P and plot the length $\sqrt{k\rho^N(t)} > 0$ on both sides of P on t then we get a point set in the tangential plane — symmetric around P — called Dupin indicatrix $i(k)$ to the constant k. To each tangent direction the normal curvature can be read out of Dupin's indicatrix if the constant k is known.

The Dupin indicatrix is an ellipse if P is an elliptic point, it degenerates into a circle if the point is an umbilical nonplanar point. For a hyperbolic point, the Dupin indicatrix is a pair of conjugate hyperbolas. For a parabolic point, the Dupin indicatrix degenerates into a pair of parallel lines.

Dupin's indicatrix can also be interpreted as an intersection of a plane, parallel to the tangential plane in P, which is offset by an infinitesimally small amount. In order to 'visualize' Dupin's indicatrix in Construct3D the tangential plane in a surface point was offset by an epsilon value in the direction of the surface normal vector. The intersection of this minimally offset plane with the surface is a visual indication to which type — according to the above mentioned classification — the point belongs to. The intersection resembles an ellipse in an elliptic point and resembles a hyperbola in a hyperbolic point. Figure 6 shows two examples of intersection curves in points of a surface of revolution. Parabolic points can be visualized



(a) Clearly an elliptic point

(b) Intersection of the offset tangential plane with Φ in a hyperbolic point

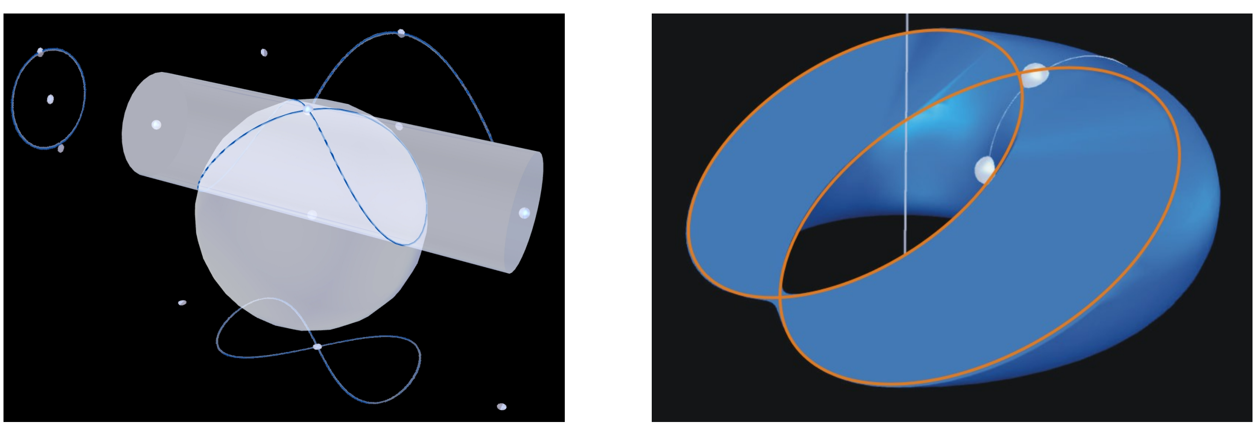
Figure 6: Elliptic and hyperbolic points on a surface of revolution

easily if the surface of choice is a cylinder for instance. All points on a cylinder are parabolic and the intersection with the offset tangential plane are two parallel generator lines.

4.5. Studying well-known curves

Some well-known curves came into mind when considering educational applications of the presented work. A curve that is frequently studied is Viviani's Window (Fig. 7(a)). Top (lemniscate), front (circle) and left side view (parabolic segment) are shown in Fig. 7(a). There are multiple interesting properties of Viviani's Window that can be studied in this context.

Another example are the Villarceau circles (Fig. 7(b)) produced by cutting a torus diagonally by a double tangential plane. The Villarceau circles are loxodromes of the torus [32]. In Construct3D this can be explored visually by moving a point along the Villarceau circles together with its Frenet frame. Observing the tangent's angle to the circles of longitude and latitude in that point shows that it stays constant during movement.



(a) Viviani's window with additional top, front and left side view

(b) Villarceau circles

Figure 7: Learning about properties of interesting curves

Without going into further detail it is obvious that there is a wide variety of content that can be studied in a dynamic geometry application such as Construct3D which provides differential geometric functionality.

5. Conclusion and future work

In this paper we introduced three-dimensional differential geometry in dynamic geometry software. We showed the applicability of Construct3D for dynamic differential geometry education in a wide range of examples. The content in section 4 has not been evaluated with students yet but previous evaluations with Construct3D have been comprehensive [17]. They provided constructive feedback that improved technological development as well as content design [19] and have been taken into account when developing the examples presented here.

Augmented or virtual reality is supposed to enrich traditional geometry education, not to substitute it. There are still major obstacles to overcome before these technologies may be used in schools which are mainly related to costs — costs of hardware but also of technical personnel to run and maintain technologically complex setups. In order to bring augmented reality to schools further technological developments are needed to lower prices of necessary hardware equipment and to develop alternative setups which enable larger groups of students to participate in the learning experience.

Regarding future work we are investigating the hypothesis that students' spatial abilities can be improved by training in augmented reality in an ongoing research project. Therefore an extensive psychological study with more than 250 students is currently under way.

6. Acknowledgements

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