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On the Representation of Dupin Cyclides in Lie Sphere Geometry with Applications

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Abstract. Dupin cyclides are canal surfaces defined as envelopes of a family of oriented spheres which touch three given oriented spheres. With respect to their attractive geometric properties they are often used in Computer Aided Geometric Design and in many engineering applications. In this paper, we study these surfaces from the point of view of Lie sphere geometry. This representation enables to solve many complicated problems through simple and well known methods of linear algebra. As for applications, we present an algorithm for computing their rational parametrizations and demonstrate a construction of blends between two canal surfaces using methods of Lie geometry.

Key Words: Quadratic spaces, Lie sphere geometry, Dupin cyclides, rational parametrizations, PN surfaces, blending surfaces

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1. Introduction

The Dupin cyclides belong to a larger family of surfaces referred to as *canal surfaces*. They were introduced in the nineteenth century by the French mathematician and engineer C. Dupin (1784–1873) as envelopes of a one-parameter set of spheres tangent to three given spheres. In addition to the classical definition, they can also be looked at in many other different ways — as the envelopes of spheres with centres on a conic and touching a sphere; in terms of the two extremal circles lying in the plane of their symmetry; as the inverse of a torus; etc. — for more details see [5, 9, 6, 21, 23, 13, 10] and the references cited therein.

From the early beginning, the family of classical Dupin cyclides as a subfamily of general cyclides has attracted the interest of several researchers. Thanks to their geometric properties they have been considered as useful surfaces for modelling purposes. In CAGD, Dupin cyclides were introduced in [6, 21]. Nowadays, they play an important role in pipe joining, blending and motion planning — see [1, 2, 11, 23, 27].

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Dupin cyclides are envelopes of a system of spheres touching three given spheres and thus it is no problem to compute their implicit representation if we use a suitable elimination technique. However, since rational descriptions of geometric objects (NURBS objects) have become a universal standard in technical applications such as CAD or CAM it is necessary to find their rational parametrizations — see, e.g., [6, 21, 25, 13, 12].

One effective way for studying Dupin cyclides uses the so-called Laguerre sphere geometry — see [13, 20]. The great advantage of this approach lies in the fact that cyclides are represented as special curves (PE circles) in $\mathbb{R}^4$ and hence well-known curve modelling techniques can be used. These results were consequently applied for constructing joins and blends of surfaces (cf. [20]). Another sphere geometry, Möbius geometry, was used in [16, 15] where quadratic series of spheres in $\mathbb{R}^4$ were studied. A separate investigation of cyclides in both above mentioned geometries motivated us to study these surfaces in *Lie sphere geometry* (see [7]), in which Laguerre and Möbius geometries are subgeometries. Recently, Lie geometry was used in [17] for studying bisectors; some investigations of Dupin cyclides in Lie geometry can be found in [5, 10].

In this paper, we use the projective model of Lie sphere geometry based on the quadratic vector space $\mathbb{R}^{4,2}$. The significant advantage of the presented approach lies in the fact that all results are obtained with the tool of linear algebra and thus they have direct and simple implementations. Hence, the main aim of this paper is to show that some chosen problems of CAGD can be easily solved by applying well known methods of linear algebra if we apply a Lie sphere geometry model and the associated techniques.

Studying Dupin cyclides through Lie sphere geometry enables us to reformulate algorithms for the computation of their rational parametrizations, which are, in addition, PN and principal (cf. [3, 19, 22]). Moreover it also gives a possibility to solve the operation of blending canal surfaces — see, e.g., [26, 1, 2, 23, 24, 27, 10, 11]. The significant advantage of using Lie sphere geometry lies in the fact that the operation of blending becomes considerably simple since it is based only on dealing with special subspaces of the quadratic space $\mathbb{R}^{4,2}$.

The rest of the paper is organized as follows: Section 2 recalls some basic facts concerning a quadratic space $\mathbb{R}^{4,2}$, its subspaces, orthogonal transformations and the projective model of Lie geometry. Section 3 is devoted to introducing and studying cyclides in Lie sphere geometry. The algorithm for computing a principal PN parametrization of Dupin cyclides is presented. In Section 4, we provide a method for the construction of blends of two canal surfaces using Dupin cyclides in Lie sphere geometry. Finally, we conclude the paper.

2. Preliminaries

2.1. Quadratic spaces and orthogonal transformations

This section recalls some basic facts and notions which are necessary for better understanding methods of Lie geometry and related problems dealing with its application for Dupin cyclides.

Definition 2.1. Let V be a finite-dimensional real vector space and $\Phi\colon\ V \times V \to \mathbb{R}$ a symmetric bilinear form. Then the pair (V, Φ) is called a *quadratic space*. We say that the quadratic space is *regular* (or *singular*) if the quadratic form given by the associated bilinear form Φ is regular (or singular).

Readers who are more interested in the theory of quadratic spaces can find more details in [4, 8]. Let the triple (p, r, q) denote the *signature* of the symmetric bilinear form Φ, i.e., the number of positive, zero, and negative eigenvalues, respectively. A space of signature (p, r, q) will be denoted by $\mathbb{R}^{p,r,q}$. If $r = 0$ or $r = q = 0$ we write briefly $\mathbb{R}^{p,q}$ or $\mathbb{R}^{p}$, respectively. Further, the symbol "$\cdot$" is used for the inner product given by Φ, and $x \cdot x$ will be abbreviated to x^2. Vectors fulfilling $x^2 > 0$, $x^2 = 0$, or $x^2 < 0$ are called *positive*, *neutral*, or *negative*, respectively.

In what follows, we will deal with linear mappings maintaining inner products given by symmetric bilinear forms which determine corresponding quadratic spaces.

Definition 2.2. The mapping $\theta\colon (V, \Phi) \to (W, \Psi)$ between two quadratic spaces is called an *isometry* if

1. θ is an isomorphism of vector spaces,
2. $\forall x, y \in V:\ \Phi(x, y) = \Psi(\theta(x), \theta(y))$.

Two spaces V, W are called *isometric*, denoted by $V \cong W$, if there exists an isometry between them. An isometry $\theta\colon V \to V$ is called *orthogonal transformation*. The group of orthogonal transformations will be denoted by $\mathbf{O}(V, \Phi)$.

Any real quadratic space is uniquely determined up to isometries by its signature (p, r, q). The orthogonal group of $\mathbb{R}^{p,r,q}$ is denoted by $\mathbf{O}_{p,r,q}$; similarly we will use the notations $\mathbf{O}_{p,q}$ and $\mathbf{O}_p$ for $r = 0$ and $r = q = 0$, respectively.

Theorem 2.3. (**Witt's extension theorem**)
Let $\mathbb{R}^{p,q}$ be a regular quadratic space V and W be a subspace of $\mathbb{R}^{p,q}$. Then any isometry $\theta\colon W \to \mathbb{R}^{p,q}$ can be extended to an orthogonal transformation of $\mathbb{R}^{p,q}$.

A proof of Witt's extension theorem can be found, e.g., in [4, 8].

2.2. Orthogonal complements of $\mathbb{R}^{4,2}$

In the rest of this paper, we will deal mainly with subspaces of the quadratic space $\mathbb{R}^{4,2}$ and their orthogonal complements. Let us recall that for any subspace $A \subset \mathbb{R}^{p,r,q}$, the *orthogonal complement* of A is defined as

$$A^{\perp} = \{\mathbf{x} \in \mathbb{R}^{p,r,q} \mid \forall \mathbf{y} \in A\colon\ \mathbf{x} \cdot \mathbf{y} = 0\}. \tag{1}$$

Basic properties of orthogonal complements are summarized in the following theorem (see [8] for more details).

Theorem 2.4. *Let $\mathbb{R}^{p,q}$ be a regular quadratic space and V, W its arbitrary subspaces. Then the following statements hold:*

1. $(V^{\perp})^{\perp} = V$,
2. $\dim V + \dim V^{\perp} = \dim \mathbb{R}^{p,q} = p + q$,
3. $V \cong W$ *iff* $V^{\perp} \cong W^{\perp}$.

Now, let A be a k-dimensional subspace of $\mathbb{R}^{4,2}$ isometric with $\mathbb{R}^{p,r,q}$, $p+q+r = k$. Obviously, not all triples (p, r, q) can describe potential subspaces of $\mathbb{R}^{4,2}$. Eliminating impossible combinations of p, q, r (for the sake of brevity we omit this procedure), we arrive at all pairs of subspaces of $\mathbb{R}^{4,2}$ and their orthogonal complements.

Proposition 2.5. *There exist exactly* 15 *pairs of nontrivial subspaces of* $\mathbb{R}^{4,2}$ *and their orthogonal complements, which are summarized in Table 1.*

Properties of these subspaces and their relation to representations of Dupin cyclides in Lie sphere geometry will be discussed in Section 3.

Table 1: All pairs of nontrivial orthogonal complements in $\mathbb{R}^{4,2}$ of dimensions a) 1 and 5, b) 2 and 4, c) 3 and 3.

(a)

$\mathcal{A}$	$\mathbb{R}^1$	$\mathbb{R}^{0,1,0}$	$\mathbb{R}^{0,0,1}$
$\mathcal{A}^\perp$	$\mathbb{R}^{3,2}$	$\mathbb{R}^{3,1,1}$	$\mathbb{R}^{4,1}$

(b)

$\mathcal{A}$	$\mathbb{R}^2$	$\mathbb{R}^{0,2,0}$	$\mathbb{R}^{0,0,2}$	$\mathbb{R}^{1,1,0}$	$\mathbb{R}^{1,1}$	$\mathbb{R}^{0,1,1}$
$\mathcal{A}^\perp$	$\mathbb{R}^{2,2}$	$\mathbb{R}^{2,2,0}$	$\mathbb{R}^4$	$\mathbb{R}^{2,1,1}$	$\mathbb{R}^{3,0,1}$	$\mathbb{R}^{3,1,0}$

(c)

$\mathcal{A}$	$\mathbb{R}^3$	$\mathbb{R}^{2,1,0}$	$\mathbb{R}^{2,1}$	$\mathbb{R}^{1,2,0}$	$\mathbb{R}^{1,2}$	$\mathbb{R}^{1,1,1}$
$\mathcal{A}^\perp$	$\mathbb{R}^{1,2}$	$\mathbb{R}^{1,1,1}$	$\mathbb{R}^{2,1}$	$\mathbb{R}^{1,2,0}$	$\mathbb{R}^3$	$\mathbb{R}^{2,1,0}$

2.3. Projective model of Lie sphere geometry

In this subsection, we introduce in brief some fundamentals of *Lie sphere geometry* — for more details see [5] or [7]. Lie sphere geometry is a geometry of oriented spheres, oriented planes and points, called *Lie spheres* altogether. Let us recall that under oriented spheres or planes we understand standard Euclidean spheres or planes equipped with the orientation of associated normal vector fields, e.g., the orientation of a sphere is given by its signed radius — the positive sign of the radius means that the normal vectors are pointing outside.

A crucial invariant of Lie geometry is the so-called *oriented contact*. Two Lie spheres are said to be in oriented contact if they are tangent and moreover if the corresponding normal vectors at the contact point have the same orientation (cf. Fig. 1). For a point and a sphere/plane, the oriented contact means just the incidence.

As a suitable model of Lie sphere geometry we will use the projective space P^5 associated

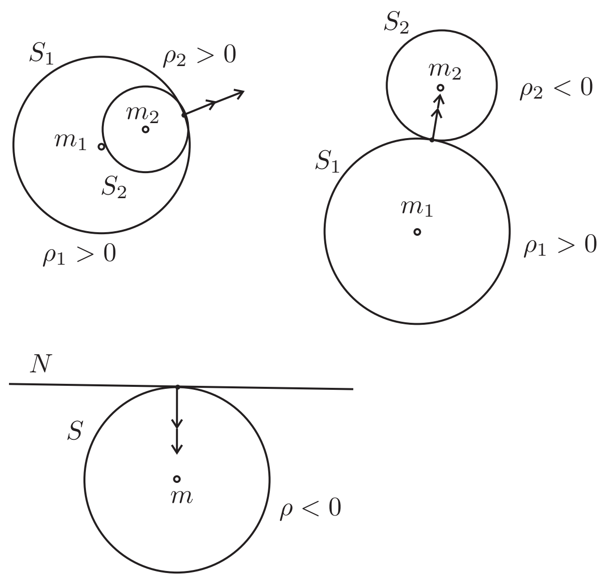

Figure 1: Oriented contact of Lie spheres – for the sake of simplicity demonstrated in $\mathbb{R}^2$

with the quadratic space $\mathbb{R}^{4,2}$. Points $\mathbf{x}\mathbb{R}$ in P^5 are represented by vectors $\mathbf{x} \neq \mathbf{o}$ in $\mathbb{R}^{4,2}$. Analogously, we introduce the correspondence between $A \subset \mathbb{R}^{4,2}$ and $A\mathbb{R} \subset P^5$ for any homogeneous set A. In the projective model, Lie spheres are in bijective correspondence with *neutral points* from P^5, i.e., with points $\mathbf{x}\mathbb{R} \in P^5$ fulfilling $\mathbf{x}^2 = 0$. The neutral vectors in $\mathbb{R}^{4,2}$ constitute the so-called *Lie quadric*

$$\mathcal{L}^4 = \{\mathbf{x}\mathbb{R} \in P^5 \mid \mathbf{x}^2 = 0\}. \tag{2}$$

Next, we consider an orthogonal basis $\{e_1, e_2, e_3, e_+, e_-, e_r\}$ in $\mathbb{R}^{4,2}$, where

$$e_1^2 = e_2^2 = e_3^2 = e_+^2 = 1 \quad \text{and} \quad e_-^2 = e_r^2 = -1. \tag{3}$$

After denoting $n_\infty = e_+ + e_-$ and $n_0 = e_+ - e_-$, we can easily describe the correspondence between Lie spheres and projective points of $\mathcal{L}^4$.

First, let S be an oriented sphere with the Euclidean centre $m = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \in \mathbb{R}^3$ and the oriented radius $\rho \in \mathbb{R}$. This sphere is represented by the point

$$\mathbf{m}\mathbb{R} = [2m + (m^2 - \rho^2)n_\infty - n_0 + 2\rho e_r]\mathbb{R} \tag{4}$$

(for more details see [7]). The vector $\mathbf{m}$ is called a *normalized representative of the sphere S*. If $\rho = 0$, we can consider the sphere S as a point. Clearly, the representatives of all points lie in the hyperplane

$$\mathcal{B}\colon\ e_r \cdot \mathbf{x} = 0. \tag{5}$$

If a non-normalized representative $\mathbf{x}$ of an oriented sphere (or a point) is given, we can easily normalize it

$$\mathbf{x} \ \mapsto\ -2\,\frac{\mathbf{x}}{\mathbf{x} \cdot n_\infty}\,. \tag{6}$$

Second, let N be an oriented plane given in $\mathbb{R}^3$ by the equation $x \cdot n + \lambda = 0$, where $n = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 \in \mathbb{R}^3$ is the unit normal vector. This plane corresponds to the point on Lie quadric

$$\mathbf{n}\mathbb{R} = [n + \lambda n_\infty - e_r]\mathbb{R} \tag{7}$$

(for more details see [7]). Analogously to the previous case, the vector $\mathbf{n}$ is called a *normalized representative of the plane N*. Representatives of all oriented planes lie in the hyperplane

$$\mathcal{N}\colon\ n_\infty \cdot \mathbf{x} = 0. \tag{8}$$

Furthermore, we can normalize an arbitrary representative of an oriented plane

$$\mathbf{x} \ \mapsto\ \frac{\mathbf{x}}{\mathbf{x} \cdot e_r}\,. \tag{9}$$

Remark 2.6. Since there is a bijective correspondence between Lie spheres in $\mathbb{R}^3$ and their representatives in $\mathcal{L}^4$, we will not distinguish between them if there is no danger of confusion.

One of the main advantages of the presented model consists in the possibility to describe the oriented contact of two Lie spheres using the bilinear form which defines the corresponding quadratic space.

Lemma 2.7. *Let $\mathbf{x}, \mathbf{y} \in \mathcal{L}^4$ be two representatives of Lie spheres. Then these spheres are tangent iff* $\mathbf{x} \cdot \mathbf{y} = 0$.

We denote the *group of the Lie sphere transformations* by Lie(3). Since the oriented contact is the fundamental invariant of Lie sphere geometry, it is convenient to recall (see [7]), that the following correspondence holds.

Theorem 2.8. $\mathrm{Lie}(3) \cong \mathbf{O}_{4,2}/\{\pm \mathrm{id}\}$.

Among important subgeometries of Lie geometry, we can find Laguerre and Möbius geometry. *Laguerre geometry* studies properties which are invariant under Laguerre transformations consisting of two bijective mappings which preserve the oriented contact — one acts in the set of oriented spheres and the other in the set of oriented planes. Readers who are interested in a detailed survey of this kind of geometry are referred to [5, 7, 18, 20, 13]. Hence, Laguerre transformations are exactly those Lie transformations mapping the hyperplane $\mathcal{N}$ given by (8) to itself. So the group of Laguerre transformations Lag(3) is a subgroup of Lie(3). On the other hand, *Möbius geometry* studies properties invariant under Möbius transformations which preserve points and non-oriented Möbius spheres (planes or Euclidean spheres). It can be shown that the group of Möbius transformations is double covered by Lie sphere transformations mapping a hyperplane

$$\mathcal{B}\colon\ e_r \cdot \mathbf{x} = 0 \tag{10}$$

to itself. For mores details see [5, 7, 16, 15].

3. Dupin cyclides in Lie sphere geometry

3.1. PE subspaces and Dupin cyclides representation

A *Lie canal surface* is the envelope of a one-parameter family of Lie spheres. This envelope can be constructed as the union of all circles/lines of intersection of infinitesimally neighbouring pairs of Lie spheres. These circles/lines are called *composing curves*. In this paper, we will deal with a special class of Lie canal surfaces, namely with Dupin cyclides — see Definition 3.2. Modelling canal surfaces with Dupin cyclides in the cyclographic model of Laguerre geometry was thoroughly studied in [20]. It was shown that any Dupin cyclide is the cyclographic image of the so-called *PE* (*Pseudo-Euclidean*) *circle*. Next in [13], a theory of Dupin cyclides using associated *isotropic hypersurfaces* was developed. Recently, the correspondence between Lie canal surfaces and curves $\mathbf{a}(\xi)\mathbb{R}$ on the Lie quadric has been presented in [17]; the condition for a real Lie canal surface has the form

$$\frac{\mathrm{d}\,\mathbf{a}}{\mathrm{d}\,\xi} \cdot \frac{\mathrm{d}\,\mathbf{a}}{\mathrm{d}\,\xi} \geq 0. \tag{11}$$

In addition, the equality holds for composing curves being degenerated to points.

Remark 3.1. If the strong inequality in (11) holds for some ξ_0 then the tangent space

$$\mathrm{span}\left(\frac{\mathrm{d}\,\mathbf{a}}{\mathrm{d}\,\xi}(\xi_0),\ \mathbf{a}(\xi_0)\right) \tag{12}$$

is isometric to $\mathbb{R}^{1,1,0}$.

We start with studying families of Lie spheres tangent to three fixed Lie spheres. Consider three distinct Lie spheres represented by neutral vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^{4,2}$ and denote

$$A = \operatorname{span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)\,. \tag{13}$$

Then applying Lemma 2.7, the set of all Lie spheres touching three given spheres is exactly the set of Lie spheres contained in the subspace $A^\perp$.

Since we have assumed that the Lie spheres $\mathbf{a}_i$ are distinct then we can obtain only the following possible signatures of $A^\perp$:

Case I. If $\dim A = 2$ then the projective line $A\mathbb{R}$ intersects $\mathcal{L}^4$ in three distinct points $\mathbf{a}_i\mathbb{R}$, which is possible only for $A\mathbb{R} \subset \mathcal{L}^4$. This is equivalent to $A \cong \mathbb{R}^{0,2,0}$ and by Table 1(b) we have $A^\perp \cong \mathbb{R}^{2,2,0}$. This case represents a pencil of tangent Lie spheres whose envelope is not well-defined.

Case II. If $\dim A = 3$ then $\mathbf{a}_i$ are three linearly independent neutral vectors and thus A has to be isometric to one of the following spaces:

1. $A \cong \mathbb{R}^{1,2}$. Using Table 1(c) we obtain $A^\perp \cong \mathbb{R}^3$. Hence $A^\perp\mathbb{R} \cap \mathcal{L}^4 = \emptyset$, and no Lie sphere exists which touches all $\mathbf{a}_i$.
2. $A \cong \mathbb{R}^{1,1,1}$. In this case $A^\perp \cong \mathbb{R}^{2,1,0}$ and therefore it contains exactly one Lie sphere.
3. $A \cong \mathbb{R}^{2,1}$ is the most interesting situation when the orthogonal complement $A^\perp$ is also isometric to $\mathbb{R}^{2,1}$. In this case we obtain a one-parameter set of Lie spheres touching $\mathbf{a}_i$, i.e., it gives a Lie canal surface as its envelope.

Definition 3.2. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be three Lie spheres such that $A = \operatorname{span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \cong \mathbb{R}^{2,1}$. Then the envelope of all Lie spheres tangent to $\mathbf{a}_i$, denoted by $\mathcal{C}_A$, is called *Dupin cyclide*. Further, any subspace isometric to $\mathbb{R}^{2,1}$ will be called *PE-subspace*.

Remark 3.3. The name *PE subspace* emphasizes a relation to the pseudo-Euclidean (PE) inner product studied in [20, 13]. Clearly, for any PE-subspace A the corresponding projective set $A\mathbb{R} \cap \mathcal{L}^4$ is a regular conic section.

Since we deal in this paper only with Dupin cyclides, we call them "cyclides" only, provided there is no danger of confusion.

In [13], the so-called isotropic hypersurface was defined as the linear join of two dual PE-circles. We use this construction analogously for PE-subspaces in Lie sphere geometry.

Definition 3.4. For an arbitrary PE-subspace A we define the set

$$\mathbf{C}_A := \{\Delta = \operatorname{span}(\mathbf{a}, \mathbf{a}_\perp) \mid \mathbf{a} \in A,\ \mathbf{a}_\perp \in A^\perp,\ \mathbf{a}^2 = \mathbf{a}_\perp^2 = 0\}. \tag{14}$$

As mentioned above, the set $\mathbf{C}_A$ is in a direct correspondence with the isotropic hypersurfaces introduced in [13] and thus any $\Delta = \operatorname{span}(\mathbf{a}, \mathbf{a}_\perp) \in \mathbf{C}_A$ represents a bundle of Lie spheres tangent to the given cyclide at the point of contact of $\mathbf{a}$ and $\mathbf{a}_\perp$. Hence, we can formulate:

Proposition 3.5. *For all $\Delta \in \mathrm{C}_A$, every $\mathbf{x} \in \Delta$ represents a Lie sphere tangent to the cyclide $\mathcal{C}_A$.*

Corollary 3.6. *Any Dupin cyclide is an envelope of two one-parameter sets of Lie spheres, one represented by A and the other by $A^\perp$, i.e., the subspaces A and $A^\perp$ determine the same Dupin cyclide.*

3.2. Classes of Dupin cyclides

Definition 3.7. Let G be a subgroup of Lie(3). Then two cyclides $\mathcal{C}_A$ and $\mathcal{C}_B$ are said to be *G-equivalent* if there exists $\theta \in G$ such that $\theta(A) = B$, or $\theta(A) = B^{\perp}$.

Since Dupin cyclides are envelopes of two one-parameter families of Lie spheres contained in some PE-subspace, it is seen that two cyclides are equivalent if and only if there exists a transformation from G which maps these families at each other. Using Theorem 2.3, we can construct for every pair of PE-subspaces A, B a certain transformation $\theta \in \mathrm{Lie}(3) \cong \mathbf{O}_{4,2}/\{\pm\mathrm{id}\}$ such that $\theta(A) = B$. This implies the following observation.

Theorem 3.8. *All Dupin cyclides are Lie-equivalent.*

Since there is only one type of Dupin cyclides in Lie geometry it is useful to study them also in a suitable subgeometry, i.e., using some subgroup of Lie(3), to get a better insight into their geometric properties. We take $\mathrm{Lag}(3) \subset \mathrm{Lie}(3)$ and study classes of equivalent Dupin cyclides in Laguerre geometry. Our approach will lead to the well-known classification of these surfaces (degenerate, quartic, cubic or parabolic cyclides — see, e.g., [13, 10]). Let us recall that transformations from Lag(3) map oriented planes to oriented planes, i.e., the hyperplane $\mathcal{N}$ given by (8) is invariant with respect to all Laguerre transformations.

We consider PE-subspace A and denote

$$D = A \cap \mathcal{N} \quad \text{and} \quad D_{\perp} = A^{\perp} \cap \mathcal{N}, \tag{15}$$

whose dimensions can be restricted to

$$2 \leq \dim D, \dim D_{\perp} \leq 3. \tag{16}$$

The neutral vectors in these subspaces correspond to the oriented planes and hence, their signatures indicate the number of planes contained in the generating family of the given cyclide.

Since $\mathrm{span}(n_\infty) \cong \mathbb{R}^{0,1,0}$ we obtain $\mathcal{N} \cong \mathbb{R}^{3,1,1}$ — see Table 1(a). Hence, D and $D_{\perp}$ have to be isometric to one of the following subspaces

$$\text{a) } \mathbb{R}^{2,1}, \quad \text{b) } \mathbb{R}^{2}, \quad \text{c) } \mathbb{R}^{1,1}, \quad \text{d) } \mathbb{R}^{1,1,0}. \tag{17}$$

We know that n_∞ is an element of $\mathcal{N}$ (the hyperplane of all representatives of Euclidean planes), on the other hand it is not a representative of any plane. Thus, the case $n_\infty \in A$ must be investigated separately.

Lemma 3.9. *If $n_\infty \in A$ then $D \cong \mathbb{R}^{1,1,0}$.*

Proof. We assume that the statement is false. Then there exists a neutral vector $\mathbf{a} \in D$ such that $\dim(\mathrm{span}(\mathbf{a}, n_\infty)) = 2$. From the definition of D in (15) follows that $\mathbf{a} \cdot n_\infty = 0$ which gives $\mathrm{span}(\mathbf{a}, n_\infty) \cong \mathbb{R}^{0,2,0}$. However, a space of signature $(0, 2, 0)$ cannot be a subspace of a PE-subspace A. This completes the proof. □

Next, we will see that the signatures of the subspaces D and $D_{\perp}$ from (15) are closely related.

Lemma 3.10. *The following statements hold:*

I. $D \cong \mathbb{R}^{1,1,0}$ *and* $n_\infty \in D$ *iff* $D_{\perp} \cong \mathbb{R}^{2,1}$.

II. $D \cong \mathbb{R}^{1,1,0}$ *and* $n_\infty \notin D$ *iff* $D_\perp \cong \mathbb{R}^{1,1,0}$ *and* $n_\infty \notin D_\perp$.

III. $D \cong \mathbb{R}^2$ *iff* $D_\perp \cong \mathbb{R}^{1,1}$.

Proof. For the sake of brevity, we prove only the first part of Lemma 3.10 the remaining statements can be proved in a similar way.

Case I. Let $n_\infty \in D$. Since all neutral vectors in $A^\perp$ are Lie spheres tangent to n_∞, it follows that $A^\perp$ contains only representatives of oriented planes. Hence $D_\perp = A^\perp \cap \mathcal{N} \cong \mathbb{R}^{2,1}$. On the contrary, $D_\perp \cong \mathbb{R}^{2,1}$ implies that $A^\perp \subset \mathcal{N}$ contains only oriented planes and $n_\infty \in A$. Hence, by Lemma 3.9 we get $D \cong \mathbb{R}^{1,1,0}$ and $n_\infty \in D$. □

Clearly, Lemma 3.10 divides the set of Dupin cyclides into three subsets. The following theorem says that these subsets are exactly the classes of Laguerre eqivalence.

Theorem 3.11. (**Laguerre classification**)
The three types of cyclides given by Lemma 3.10 give the classes of Laguerre-equivalence.

Proof. Let $\mathcal{C}_A$ and $\mathcal{C}_B$ be two Dupin cyclides of the same type. Denote $E = B \cap \mathcal{N}$, $E_\perp = B^\perp \cap \mathcal{N}$.

Case I. Since $n_\infty \in D \subset A$ and $n_\infty \in E \subset B$, there exists an isometry $\psi\colon A \to B$ such that $\psi(n_\infty) = n_\infty$. Extending this isometry to the orthogonal transformation $\hat{\psi} \in \mathbf{O}_{4,2}$, we get the required transformation.

Case II. In this case $n_\infty \in D \oplus D_\perp$, where the symbol $\oplus$ denotes the direct sum. If not, then with respect to the definition of $\mathcal{N}$ (see (8)) we obtain

$$\mathcal{N} = D \oplus D_\perp \oplus \operatorname{span}(n_\infty) \;\cong\; \mathbb{R}^{1,1,0} \oplus \mathbb{R}^{1,1,0} \oplus \mathbb{R}^{0,1,0} \;\cong\; \mathbb{R}^{2,3,0}. \tag{18}$$

However, we know that $\mathcal{N} \cong \mathbb{R}^{3,1,1}$ — see Table 1(a). The same argument holds for $E \oplus E_\perp$. Hence, the isometries

$$\varphi\colon D \to E, \text{ and } \varphi_\perp\colon D_\perp \to E_\perp \tag{19}$$

can be extended by Witt's extension theorem (Theorem 2.3) to the isometries

$$\hat{\varphi}\colon A \to B, \text{ and } \hat{\varphi}_\perp\colon A^\perp \to B^\perp. \tag{20}$$

Since A and $A^\perp$ are orthogonal complements, any vector $\mathbf{x} \in \mathbb{R}^{4,2}$ can be written uniquely in the form $\mathbf{x} = \mathbf{y} + \mathbf{y}_\perp$, where $\mathbf{y} \in A$ and $\mathbf{y}_\perp \in A^\perp$. Further, the orthogonal transformation $\psi \in \mathbf{O}_{4,2}$, defined by

$$\psi(\mathbf{x}) := \hat{\varphi}(\mathbf{y}) + \hat{\varphi}_\perp(\mathbf{y}_\perp) \tag{21}$$

maps the hyperplane $\mathcal{N}$ to itself, and thus any two cyclides of type II are Laguerre-equivalent.

Case III. Now, we have $D \oplus D_\perp \cong \mathbb{R}^{3,0,1}$. There is no neutral vector in this space and therefore $n_\infty \notin D \oplus D_\perp$. Hence, there exist $\mathbf{a} \in A \backslash D$ and $\mathbf{a}_\perp \in A^\perp \backslash D_\perp$ such that $n_\infty = \mathbf{a} + \mathbf{a}_\perp$. Moreover, since $n_\infty \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in \mathcal{N}$, the following statements hold

$$A = D \oplus \operatorname{span}(\mathbf{a})\,, \quad A^\perp = D_\perp \oplus \operatorname{span}(\mathbf{a}_\perp)\,, \quad \mathbf{a}^2 + \mathbf{a}_\perp^2 = 0. \tag{22}$$

The analogous claims are fulfilled for B and $B^\perp$. Now, the construction of the Laguerre transformation mapping $\mathcal{C}_A$ on $\mathcal{C}_B$ is trivial. □

With respect to the previous classification, we obtain one class of degenerate and two classes of nondegenerate Dupin cyclides; in classical geometry the latter are called quartic and parabolic. Representatives of all these Laguerre types I, II and III are shown in Figs. 2, 3 and 4.

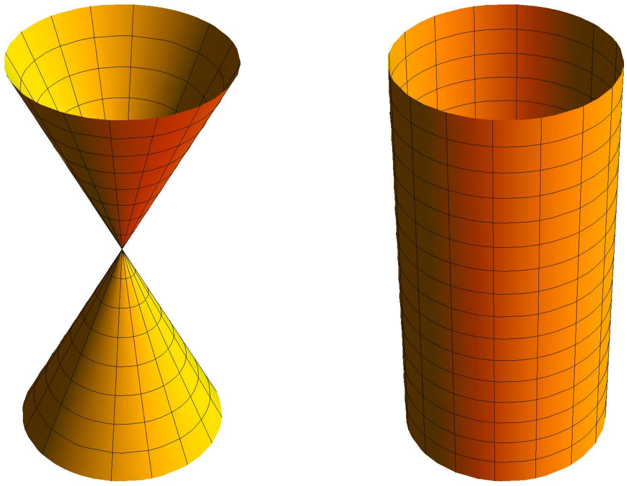

Figure 2: Type I — degenerate cyclides: Cylinder and cone

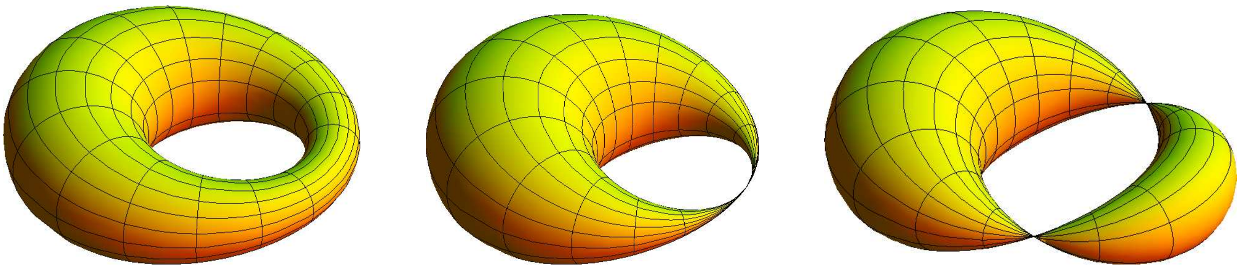

Figure 3: Type II — nondegenerate cyclides: Quartic cyclides

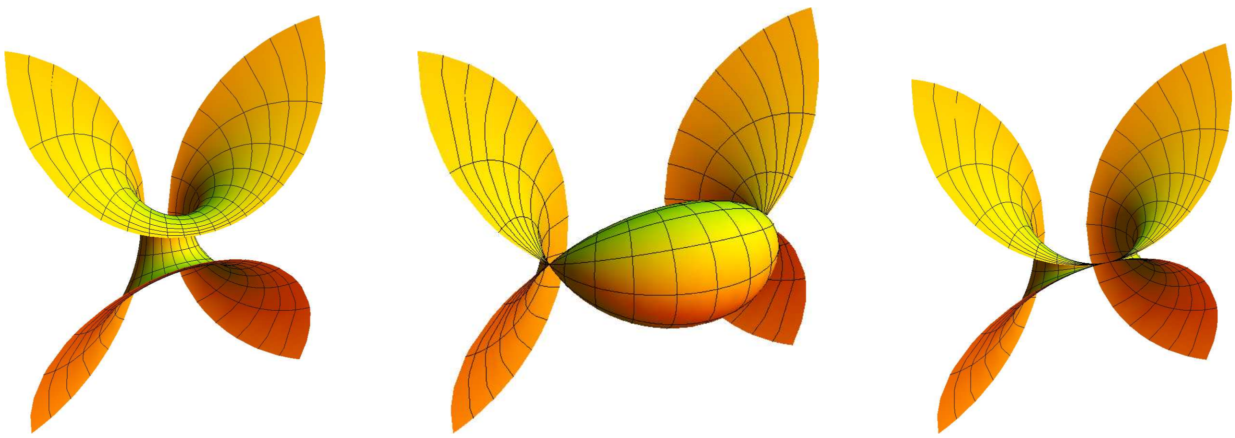

Figure 4: Type III — nondegenerate cyclides: Parabolic cyclides

3.3. Algorithm for computing rational parametrizations of Dupin cyclides

Rational descriptions of geometric objects (NURBS objects) have become a universal standard in most of technical applications. From this reason, it is necessary to find rational parametrizations of cyclides to use them for modelling in CAD and CAM. In this paper, we formulate a parametrization algorithm based on the representation of Dupin cyclides in Lie sphere geometry (i.e., using suitable methods of linear algebra).

Let $\mathcal{C}_A$ be a cyclide determined by a pair of PE-subspaces $(A, A^\perp)$. For arbitrary neutral vectors $\mathbf{a} \in A$, $\mathbf{a}_\perp \in A^\perp$ it holds $\Delta = \mathrm{span}(\mathbf{a}, \mathbf{a}_\perp) \in \mathbf{C}_A$ by Definition 3.4. Proposition 3.5 says that Δ is corresponding to the bundle of all Lie spheres tangent to $\mathcal{C}_A$. Obviously, any point sphere in this bundle is a point of the cyclide $\mathcal{C}_A$. Since $\mathbf{a}$ and $\mathbf{a}_\perp$ are tangent and distinct, there must exist exactly one point sphere $\mathbf{c}$ in Δ. This point can be computed as follows

$$\mathbf{c} = (e_r \cdot \mathbf{a})\mathbf{a}_\perp - (e_r \cdot \mathbf{a}_\perp)\mathbf{a}. \tag{23}$$

Indeed, $\mathbf{c} \in \Delta$ because it is the linear combination of $\mathbf{a}$ and $\mathbf{a}_\perp$ and it is a point sphere because $\mathbf{c} \cdot e_r = 0$, cf. (5).

Now, let $\mathbf{a}(\xi)\mathbb{R}$ be a parametrization of $A\mathbb{R} \cap \mathcal{L}^4$ and $\mathbf{a}_\perp(\zeta)\mathbb{R}$ be a parametrization of $A^\perp\mathbb{R} \cap \mathcal{L}^4$. Then

$$\Delta(\xi, \zeta) := \mathrm{span}(\mathbf{a}(\xi), \mathbf{a}_\perp(\zeta)) \tag{24}$$

is a parametrization of $\mathbf{C}_A$. Using (23), we arrive at

$$\mathbf{c}(\xi, \zeta) := [e_r \cdot \mathbf{a}(\xi)]\mathbf{a}(\zeta)_\perp - [e_r \cdot \mathbf{a}(\zeta)_\perp]\mathbf{a}(\xi). \tag{25}$$

To compute a parametrization of $\mathcal{C}_A$ in $\mathbb{R}^3$ from (25), we have to find the corresponding normalized representative (6) of $\mathbf{c}(\xi, \zeta)$ in the form

$$\mathbf{c}(\xi, \zeta) \ \mapsto \ -2\,\frac{\mathbf{c}(\xi, \zeta)}{\mathbf{c}(\xi, \zeta) \cdot n_\infty}\,. \tag{26}$$

Finally, using (4) for $\rho = 0$ we arrive at

$$\mathbf{c}(\xi, \zeta) = 2c(\xi, \zeta) + c^2(\xi, \zeta)n_\infty - n_0, \tag{27}$$

where $c(\xi, \zeta)$ is a parametrization of $\mathcal{C}_A$. In addition, if $\Delta(\xi, \zeta)$ is rational then the parametrization $c(\xi, \zeta)$ is rational, too. Algorithm 1 summarizes the presented method.

Algorithm 1 Parametrization of Dupin cyclide

Input: Three Lie spheres $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$

Output: A parametrization $c(\xi, \zeta)$ of the Dupin cyclide $\mathcal{C}_A$

$A \leftarrow \mathrm{span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$
if $A \cong \mathbb{R}^{2,1}$ **then**
  $\mathbf{a}(\xi)\mathbb{R} \leftarrow$ a parametrization of $A\mathbb{R} \cap \mathcal{L}^4$
  $\mathbf{a}_\perp(\zeta)\mathbb{R} \leftarrow$ a parametrization of $A^\perp\mathbb{R} \cap \mathcal{L}^4$
  $\hat{\mathbf{c}}(\xi, \zeta) \leftarrow [e_r \cdot \mathbf{a}(\xi)]\mathbf{a}(\zeta)_\perp - [e_r \cdot \mathbf{a}(\zeta)_\perp]\mathbf{a}(\xi)$
  $\mathbf{c}(\xi, \zeta) \leftarrow 2\,\frac{\hat{\mathbf{c}}(\xi, \zeta)}{\hat{\mathbf{c}}(\xi, \zeta) \cdot n_\infty} = 2c(\xi, \zeta) + c^2(\xi, \zeta)n_\infty - n_0$
  return $c(\xi, \zeta)$
else
  A does not determine a Dupin cyclide.
end if

Remark 3.12. Instead of parametrizing curves $A\mathbb{R} \cap \mathcal{L}^4$ and $A^{\perp}\mathbb{R} \cap \mathcal{L}^4$ it could be more convenient to start with some referential pair of PE-subspaces $(A_0, A_0^{\perp})$, say

$$A_0 = \mathrm{span}(e_1, e_2, e_-) \text{ and } A_0^{\perp} = \mathrm{span}(e_3, e_+, e_r)\,, \tag{28}$$

and find a transformation $\theta \in \mathrm{Lie}(3)$ such that $\theta(A_0) = A$. Now, the rational parametrizations of $A_0\mathbb{R} \cap \mathcal{L}^4$ and $A_0^{\perp} \cap \mathcal{L}^4$ can be computed by

$$a^0(\xi)\mathbb{R} := [2\xi e_1 + (1-\xi^2)e_2 + (1+\xi^2)e_-]\mathbb{R} \tag{29}$$

and

$$a^0_{\perp}(\zeta)\mathbb{R} := [2\zeta e_3 + (1-\zeta^2)e_+ (1+\zeta^2)e_r]\mathbb{R}. \tag{30}$$

Hence $\mathbf{a}(\xi)\mathbb{R} = \theta(\mathbf{a}^0(\xi))\mathbb{R}$ and $\mathbf{a}_{\perp}(\zeta)\mathbb{R} = \theta(\mathbf{a}^0_{\perp}(\zeta))\mathbb{R}$.

Under some conditions, cyclides may degenerate to curves, namely if A or $A^{\perp}$ is a subset of $\mathcal{B}$. For practical reasons, we will omit this possibility in the remainder of this paper and mean by cyclides only 2-surfaces.

Any parametrization obtained by the described method is *principal*, i.e., parametric curves of $c(\xi, \zeta)$ are principal (their velocities always point in a principal direction). Hence, by the presented approach we cannot obtain an arbitrary parametrization of the given cyclide. Nevertheless, if we choose $\mathbf{a}(\xi)\mathbb{R}$ and $\mathbf{a}(\zeta)\mathbb{R}$ being birational then $c(\xi, \zeta)$ is birational, too. Therefore, any rational parametrization of the given cyclide can be obtained by some rational reparametrization of $c(\xi, \zeta)$.

Finally, we recall that rational surfaces with rational offsets, called *Pythagorean Normal vector (PN) surfaces* (cf. [19, 18, 14]), are such surfaces which possess rational parametrizations (*PN parametrizations*) providing rational associated unit normal vector fields.

Theorem 3.13. *The rational parametrizations computed by Algorithm 1 are PN parametrizations.*

Proof. Let the parametrization of $\mathcal{C}_A$ be computed from (27) and consider the function

$$\mathbf{n}(\xi, \zeta) = [n_{\infty} \cdot \mathbf{a}(\xi)]\mathbf{a}(\zeta)_{\perp} - [n_{\infty} \cdot \mathbf{a}(\zeta)_{\perp}]\mathbf{a}(\xi). \tag{31}$$

By Proposition 3.5, $\mathbf{n}(\xi_0, \zeta_0)$ is the plane tangent to the cyclide at the point $\mathbf{c}(\xi_0, \zeta_0)$ for any arbitrary ξ_0 and ζ_0, except for both $\mathbf{a}(\xi_0)$ and $\mathbf{a}_{\perp}(\zeta_0)$ being planes (in this case $\mathbf{n}(\xi_0, \zeta_0)\mathbb{R} = \mathbf{c}(\xi_0, \zeta_0)\mathbb{R} = n_{\infty}\mathbb{R}$, i.e., the tangent planes are not defined for points at infinity). Using (9) we can normalize $\mathbf{n}(\xi, \zeta)$:

$$\mathbf{n}(\xi, \zeta) \;\mapsto\; \frac{\mathbf{n}(\xi, \zeta)}{\mathbf{n}(\xi, \zeta) \cdot e_r} = n(\xi, \zeta) + \lambda(\xi, \zeta) n_{\infty} - e_r\,. \tag{32}$$

Thus, for rational $c(\xi, \zeta)$ we have a rational unit normal field $n(\xi, \zeta)$. □

4. Blending two canal surfaces with Dupin cyclides

The most used application of Dupin cyclides in geometric modelling is G^1-*blending* of canal surfaces, which is a G^1-continuous transition between two given canal surfaces along prescribed curves. Next, we show how this practical problem can be solved via methods of Lie sphere geometry.

Definition 4.1. Two Lie canal surfaces $\mathcal{R}$ and $\mathcal{S}$ are said to be *glued* along the curve $\mathcal{K}$ iff $\mathcal{R}$ is tangent to $\mathcal{S}$ along $\mathcal{K}$ and $\mathcal{K}$ is a composing curve of both surfaces.

Remark 4.2. Tangent planes of any Lie canal surface along a composing circle envelope the so-called *tangent cone*, which may happen to be a cone of revolution, a cylinder of revolution or a plane. Hence, instead of blending general canal surfaces, we may focus only on blending cones. If no confusion occurs we will just speak of a "cone" instead of a "tangent cone".

The projective model of Lie sphere geometry translates the operation of gluing to the tangency of projective curves corresponding to one-parameter families of generating spheres. More exactly, let $\mathbf{r}(\xi)\mathbb{R}$ be the curve corresponding to the first canal surface and $\mathbf{s}(\zeta)\mathbb{R}$ the curve corresponding to the other one. Then the composing curve is determined by $\mathbf{r}(\xi_0)\mathbb{R}$ and $\mathbf{s}(\zeta_0)\mathbb{R}$ for some ξ_0 and ζ_0. The gluing condition is then equivalent to

$$\mathbf{r}(\xi_0)\mathbb{R} = \mathbf{s}(\zeta_0)\mathbb{R} \quad \text{and} \quad \operatorname{span}\left(\mathbf{r}(\xi_0),\ \frac{\mathrm{d}\,\mathbf{r}}{\mathrm{d}\,\xi}(\xi_0)\right) = \operatorname{span}\left(\mathbf{s}(\zeta_0),\ \frac{\mathrm{d}\,\mathbf{s}}{\mathrm{d}\,\zeta}(\zeta_0)\right). \tag{33}$$

In what follows, we will deal only with non-degenerate cases (see Remark 3.1), when tangent spaces are isometric to $\mathbb{R}^{1,1,0}$. Since these spaces correspond to tangent cones, we may take $R, S \cong \mathbb{R}^{1,1,0}$ as the input of our blending algorithm.

Remark 4.3. For the rest of this paper let $R = \operatorname{span}(\mathbf{r}_0, \mathbf{r}_+)$ and $S = \operatorname{span}(\mathbf{s}_0, \mathbf{s}_+)$, where $\mathbf{r}_0^2 = \mathbf{s}_0^2 = 0$, $\mathbf{r}_+^2, \mathbf{s}_+^2 > 0$, are two subspaces isometric to $\mathbb{R}^{1,1,0}$.

Definition 4.4. A Dupin cyclide $\mathcal{C}$ blends two cones R and S if and only if $\mathcal{C}$ and R are glued along a curve $\mathcal{K}_1$, $\mathcal{C}$ and S are glued along a curve $\mathcal{K}_2$ and $\mathcal{K}_1, \mathcal{K}_2$ belong to the same family of composing curves of $\mathcal{C}$.

However, a construction of a *single cyclide blend* (cf. Fig. 5) is in most input situations infeasible. Thus, we have to take into account a *double cyclide blend* (cf. Fig. 6) where a blending Dupin cyclide $\mathcal{C}$ in Definition 4.4 is replaced by two glued Dupin cyclides $\mathcal{C}_A$ and $\mathcal{C}_B$.

It directly follows from (33):

Proposition 4.5. *Two cyclides $\mathcal{C}_A$ and $\mathcal{C}_B$ are glued along a parametric curve if and only if any of the subspaces $A \cap B$, $A \cap B^\perp$, $A^\perp \cap B$, $A^\perp \cap B^\perp$ is isometric to $\mathbb{R}^{1,1,0}$ or $\mathbb{R}^{2,1}$.*

Applying Proposition 4.5, the construction of cyclide blend between two cones is equivalent to finding PE-subspaces A and B such that:

1. $R \subset A$,
2. $S \subset B$,
3. $A \cap B \cong \mathbb{R}^{1,1,0}$ or $A \cap B \cong \mathbb{R}^{2,1}$.

The subspace A corresponds to the cyclide $\mathcal{C}_A$ which is glued to the cone $\mathcal{R}$ (cf. Condition 1) and similarly B determines the cyclide $\mathcal{C}_B$ glued to the cone $\mathcal{S}$ (cf. Condition 2). In addition, both cyclides are glued along the curve determined by $A \cap B$ (cf. Condition 3).

Next, we will investigate different possibilities for the subspace $R + S$. For this, it is convenient to prove the following lemma:

Lemma 4.6. *For $R \cap S \cong \mathbb{R}^1$ the following equivalences hold:*

(i) $R + S \cong \mathbb{R}^{1,2,0} \;\Leftrightarrow\; \mathbf{r}_0 \cdot \mathbf{s}_0 = 0$,

(ii) $R + S \cong \mathbb{R}^{2,1} \;\Leftrightarrow\; \mathbf{r}_0 \cdot \mathbf{s}_0 \neq 0$.

Proof. From the list of all possible 3-dimensional subspaces of $\mathbb{R}^{4,2}$ in Table 1(c) we conclude: $\mathbb{R}^{1,2,0}$ and $\mathbb{R}^{2,1}$ are the only subspaces which contain two subspaces isometric to $\mathbb{R}^{1,1,0}$ and intersect in a positive subspace. Obviously, $\mathbb{R}^{1,2,0}$ contains a 2-dimensional neutral subspace which is equivalent to the condition $\mathbf{s}_0 \cdot \mathbf{r}_0 = 0$. Hence, $R+S \cong \mathbb{R}^{2,1}$ if and only if $\mathbf{s}_0 \cdot \mathbf{r}_0 \neq 0$. □

Next, we consider only such configurations of given cones where the fixed generating Lie spheres are not in oriented contact, i.e., we assume $\mathbf{r}_0 \cdot \mathbf{s}_0 \neq 0$. The situation with tangent generating Lie spheres may be handled similarly; for the sake of brevity we omit this case. Now, we can continue with the discussion:

Case I. Let $\dim R + S = 3$. With respect to the assumption $\mathbf{r}_0 \cdot \mathbf{s}_0 \neq 0$, the intersection is a positive subspace. Moreover, $R + S$ is isometric to $\mathbb{R}^{2,1}$ by Lemma 4.6 and therefore the cones can be blended by a single cyclide determined be PE-subspace $R + S$ (see Fig. 5).

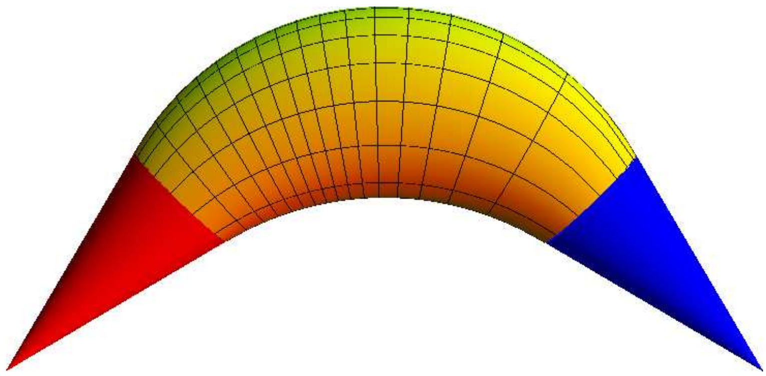

Figure 5: Single cyclide blend

Case II. If $\dim R + S = 4$ then R and S are disjoint, and thus two cyclides are necessary for blending the given cones. For any subspace $T = \mathrm{span}(\mathbf{t}_0, \mathbf{t}_+) \cong \mathbb{R}^{1,1,0}$ such that

$$T \cap R = \mathrm{span}(\mathbf{r}) \cong \mathbb{R}^1, \qquad \mathbf{t}_0 \cdot \mathbf{r}_0 \neq 0, \tag{34}$$

$$T \cap S = \mathrm{span}(\mathbf{s}) \cong \mathbb{R}^1, \qquad \mathbf{t}_0 \cdot \mathbf{s}_0 \neq 0, \tag{35}$$

the subspaces $A = R + T$ and $B = S + T$ are isometric to $\mathbb{R}^{2,1}$ by Lemma 4.6. The corresponding cyclides $\mathcal{C}_A$ and $\mathcal{C}_B$ obviously form a blend between the given cones.

To find all subspaces T fulfilling conditions mentioned above, it suffices to take all pairs $(\mathbf{r}, \mathbf{s})$, where $\mathbf{r} \in R$ and $\mathbf{s} \in S$ are positive vectors such that $T = \mathrm{span}(\mathbf{r}, \mathbf{s}) \cong \mathbb{R}^{1,1,0}$ and neutral vectors in this space are not orthogonal neither to $\mathbf{r}_0$, nor to $\mathbf{s}_0$.

Lemma 4.7. *Let (X, Φ) be a two-dimensional quadratic space. Then $(X, \Phi) \cong \mathbb{R}^{1,1,0}$ if and only if there exist two positive non-collinear vectors $x, y \in X$ such that*

$$|\Phi(x, y)| = \sqrt{\Phi(x, x)}\sqrt{\Phi(y, y)}. \tag{36}$$

Moreover, if these vectors exist the equality holds for all pairs of positive vectors.

Proof. Let $\{e_1, e_2\}$ be an orthogonal basis of (X, Φ) and let $x = \alpha_1 e_1 + \alpha_2 e_2$, $y = \beta_1 e_1 + \beta_2 e_2$ be two positive vectors satisfying

$$|\Phi(x, y)| = \sqrt{\Phi(x, x)}\sqrt{\Phi(y, y)}. \tag{37}$$

Substituting for x and y into (37) we obtain

$$(\alpha_1\beta_2 + \alpha_2\beta_1)^2\Phi(e_1, e_1)\Phi(e_2, e_2) = 0. \tag{38}$$

Since the vectors x and y are linearly independent we get $\Phi(e_1, e_1)\Phi(e_2, e_2) = 0$. Without loss of generality we may assume $\Phi(e_1, e_1) = 0$, then $\Phi(e_2, e_2) > 0$ (otherwise x and y are not positive).

Conversely, let $\{e_1, e_2\}$ be a basis of (X, Φ) such that $\Phi(e_1, e_1) = 0$ and $\Phi(e_2, e_2) = 1$. Then it is easily seen that the relation holds for any two positive vectors. □

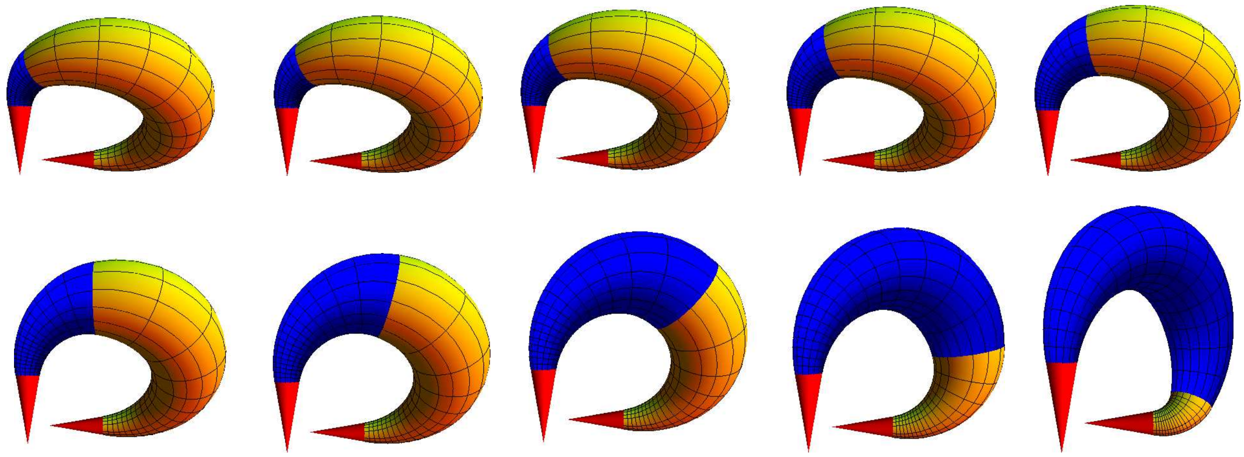

Figure 6: A system of double cyclide blends

The positive vectors in R can be parametrized by

$$\mathbf{r}(\alpha) = \alpha\mathbf{r}_0 + \mathbf{r}_+ \tag{39}$$

and analogously for S

$$\mathbf{s}(\beta) = \beta\mathbf{s}_0 + \mathbf{s}_+. \tag{40}$$

By Lemma 4.7 the space

$$T(\alpha, \beta) = \operatorname{span}(\mathbf{r}(\alpha), \mathbf{s}(\beta)) \tag{41}$$

is isometric to $\mathbb{R}^{1,1,0}$ if and only if

$$[\mathbf{r}(\alpha) \cdot \mathbf{s}(\beta)]^2 = \mathbf{r}^2(\alpha)\mathbf{s}^2(\beta). \tag{42}$$

We substitute for $\mathbf{r}(\alpha)$ and $\mathbf{s}(\beta)$ from (39), (40) and obtain

$$(\mathbf{r}_0 \cdot \mathbf{s}_0\alpha\beta + \mathbf{r}_0 \cdot \mathbf{s}_+\alpha + \mathbf{r}_+ \cdot \mathbf{s}_0\beta + \mathbf{r}_+ \cdot \mathbf{s}_+)^2 - \mathbf{r}_+^2\mathbf{s}_+^2 = 0. \tag{43}$$

This equation in α and β factorizes into two quadratic equations describing two hyperbolas. Hence, we have obtained two one-parameter families of subspaces T. It remains to show that for any α, β solving (43) the neutral vector $\mathbf{t}_0 \in T$ is orthogonal neither to $\mathbf{r}_0$ nor to $\mathbf{s}_0$ — see (34), (35). Since $T = \operatorname{span}(\mathbf{r}, \mathbf{s})$, the neutral vector $\mathbf{t}_0$ can be computed as

$$\mathbf{t}_0 := (\mathbf{s} \cdot \mathbf{s})\mathbf{r} - (\mathbf{s} \cdot \mathbf{r})\mathbf{s}. \tag{44}$$

Substituting for $\mathbf{r}$ and $\mathbf{s}$ from (39) and (40), we can see that $\mathbf{t}_0 \cdot \mathbf{s}_0 = 0$ if and only if

$$\alpha = -\frac{\mathbf{r}_+ \cdot \mathbf{s}_0}{\mathbf{r}_0 \cdot \mathbf{s}_0}\,, \tag{45}$$

and $\mathbf{t}_0 \cdot \mathbf{r}_0 = 0$ if and only if

$$\beta = -\frac{\mathbf{r}_0 \cdot \mathbf{s}_+}{\mathbf{r}_0 \cdot \mathbf{s}_0}\,. \tag{46}$$

Obviously, these are the equations of the axes of hyperbolas from (43) and thus they cannot appear as solutions of this equation. Finally, the blending procedure is summarized in Algorithm 2.

Algorithm 2 Blending two cones (canal surfaces)

Input: Two subspaces $R = \mathrm{span}(\mathbf{r}_0, \mathbf{r}_+)$, $S = \mathrm{span}(\mathbf{s}_0, \mathbf{s}_+)$ isometric to $\mathbb{R}^{1,1,0}$ with $\mathbf{r}_0 \cdot \mathbf{s}_0 \neq 0$.

Output: Single/double cyclide blend given by $\{$cyclide $\mathcal{C}_A\}/\{$cyclides $\mathcal{C}_A$, $\mathcal{C}_B\}$.

if $\dim R + S = 3$ **then**
  $A \leftarrow R + S$
  return $\mathcal{C}_A$
else
  $(\alpha_0, \beta_0) \leftarrow$ a solution of (43)
  $T \leftarrow \mathrm{span}(\alpha_0 \mathbf{r}_0 + \mathbf{r}_+, \beta_0 \mathbf{s}_0 + \mathbf{s}_+)$
  $A \leftarrow R + T$
  $B \leftarrow S + T$
  return $\mathcal{C}_A$, $\mathcal{C}_B$
end if

5. Conclusions

Spatial Lie sphere geometry gives a suitable tool for studying Dupin cyclides. These surfaces are represented in $\mathbb{R}^{4,2}$ as so-called PE-subspaces and they are equivalent under particular linear transformations. Hence, this approach enables to translate some chosen problems of geometric modelling into the language of linear algebra. The results presented in this paper relate the well-known approach based on the Lagurre geometry with more general methods based on Lie geometry. Following this, we formulated the algorithm for the computation of rational parametrizations of Dupin cyclides and discussed properties of the obtained parametrizations. Finally, one of the classical problems of CAD, namely blending of canal surfaces, was solved using the introduced representation. The operation of blending was translated into algebraic operations with subspaces of $\mathbb{R}^{4,2}$ which makes this procedure considerably simpler.

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