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Towards van der Laan's Plastic Number in the Plane

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Abstract. In 1960 D.H. van der Laan, architect and member of the Benedictine order, introduced what he calls the *"Plastic Number"* ψ, as an ideal ratio for a geometric scale of spatial objects. It is the real solution of the cubic equation $x^3 - x - 1 = 0$. This equation may be seen as example of a family of trinomials $x^n - x - 1 = 0$, $n = 2, 3, \ldots$. Considering the real positive roots of these equations we define these roots as members of a "Plastic Numbers Family" (PNF) comprising the well known Golden Mean $\phi = 1,618...$, the most prominent member of the Metallic Means Family [12] and van der Laan's Number $\psi = 1,324...$ Similar to the occurrence of ϕ in art and nature one can use ψ for defining special 2D- and 3D-objects (rectangles, trapezoids, ellipses, ovals, ovoids, spirals and even 3D-boxes) and look for natural representations of this special number.

Laan's Number ψ and the Golden Number ϕ are the only "*Morphic Numbers*" in the sense of Aarts et al. [1], who define such a number as the common solution of two somehow dual trinomials. We can show that these two numbers are also distinguished by a property of log-spirals. Laan's Number ψ cannot be constructed by using ruler and compass only. We present a planar graphic construction of a segment of length ψ using a dynamical graphics software as well as a computer-independent solution by intersecting a circle with an equilateral hyperbola. This allows to deduce and analyse "Laan-Number figures" like ψ*-rectangles* with side length ratio $1 : \psi$ and a ψ*-pentagons* with sides of ratio $1 : \psi : \psi^2 : \psi^3 : \psi^4$. To this ψ-pentagon we also find a "ψ*-Pythagoras Theorem*".

Key Words: Golden Mean, Plastic Number, Morphic Number, gnomons, spirals
MSC 2000: 51M04, 51M25

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1. Introduction

The *Metallic Means Family* (MMF), see [12], is formed by the positive solutions of quadratic equations of the type

$$x^2 - px - q = 0, \quad p, q \in \mathbb{N}.$$

Its most prominent member is the well known *Golden Mean* $\phi = 1,618...$ for $p = q = 1$. Correspondingly, the van der Laan *"Plastic Number"* $\psi = 1,324717958...$, see [7], is the real solution of the cubic equation

$$x^3 - x - 1 = 0. \tag{1}$$

Generalizing (1) to higher degree n we may call the real positive solution of equations of this type a *"Plastic Number"* and collect these numbers in the *"Plastic Numbers Family"* (PNF), [2]. Aarts et al. [1] showed that there exist only two members of the PNF which solve an additional equation of similar type $x^{-m} - x + 1 = 0$ (or $x^{m+1} - x^m - 1 = 0$), with m a natural number, namely the Golden Number ϕ and Laan's Number ψ. They called such numbers *"Morphic Numbers"* (MN). The definition of such a MN is therefore

$$\xi \text{ is a Morphic Number} \iff \xi + 1 = \xi^n \wedge \xi - 1 = \xi^{-m},\ n, m \in \mathbb{N},\ n \neq 1,\ \xi > 0. \tag{2}$$

The Golden Mean satisfies (2) for $n = 2$ and $m = 1$ and is therefore one of the only two MN. From the algebraic identity

$$(x^5 - x^4 - 1) = (x^3 - x - 1) \cdot (x^2 - x + 1),$$

which vanishes for ψ, we conclude that

$$\xi^5 - \xi^4 - 1 = 0 \iff \xi - 1 = \xi^{-4} \implies m = 4 \quad \text{and} \quad \xi = \psi \in \text{MN}. \tag{3}$$

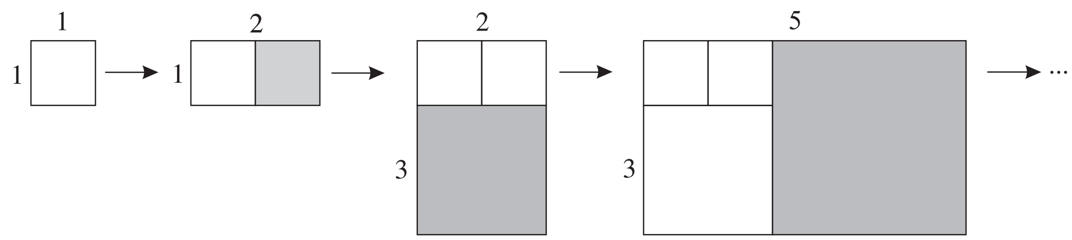

Figure 1: Sequence to generate a golden rectangle as a limes figure

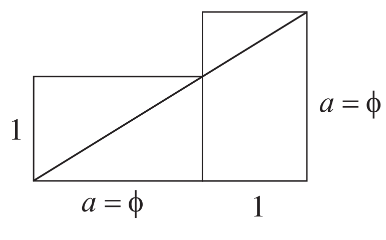

Figure 2: Geometrical model of the Golden Ratio: Given two congruent rectangles $1 \times a$, then the vector $(1, a-1)^{\mathrm{T}}$ and the diagonal vector $(a, 1)^{\mathrm{T}}$ are linearly dependent iff $a = \phi$.

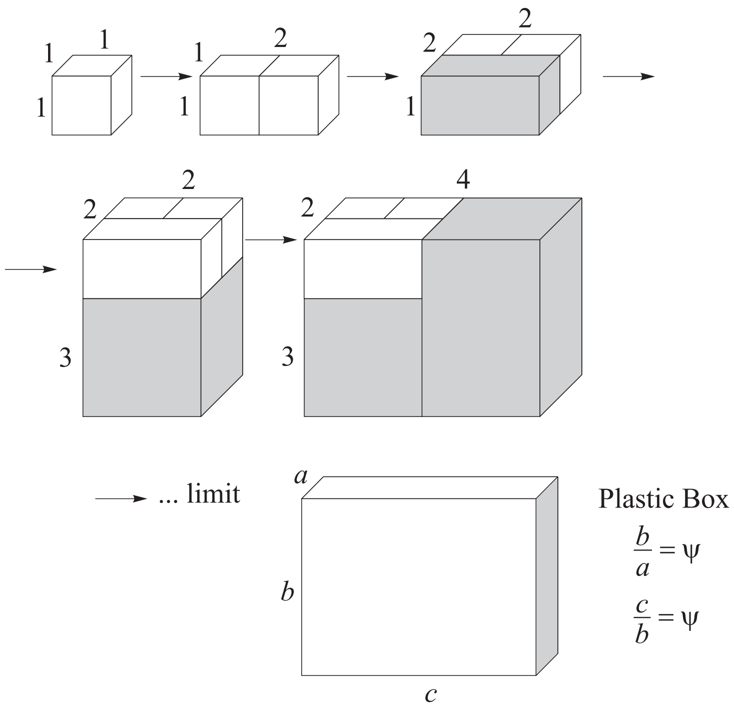

Figure 3: Construction of a *"ψ-box"* as a limit figure

The analogies between the two only members of the MNF are primarily of algebraic nature. Both of them originate in recurrent sequences, the *Fibonacci sequence* (4) for the Golden Number ϕ

$$\left.\begin{array}{l} a_0 = a_1 = 1 \\ a_{n+1} = a_n + a_{n-1}, \ (n \geq 2) \\ 1, 1, 2, 3, 5, 8, 13, \ldots \end{array}\right\} \rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \phi \tag{4}$$

and the *Padovan sequence* (5) for Laan's Number ψ

$$\left.\begin{array}{l} b_0 = b_1 = b_2 = 1 \\ b_{n+1} = b_{n-1} + b_{n-2}, \ (n \geq 3), \\ 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, \ldots \end{array}\right\} \rightarrow \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \psi. \tag{5}$$

From a geometrical point of view we may connect the degree n of the generalized equation (1) to the dimension of a figure in a Euclidean space and we consider the notion of proportion and pseudo-gnomonical growth as is indicated in Figs. 1 and 2 for the case $n = 2$, i.e., the Golden Mean (compare also [14, 6, 5]), and in Figs. 3 and 4 for $n = 3$, for Laan's Number.

Note that we can construct the elements of the Padovan sequence as a planar model using equilateral triangles, too, like it is shown in Fig. 5: This spiral of equilateral triangles generates a sequence of irregular pentagons with four angles of 120° and one of 60°, which tend to a limit figure like the one shown in Fig. 6 . We call it the *"ψ-pentagon"* and if it has one edge of unit length we call it *"unitary"*. Analogously, a rectangle $a \times b$ such that $b/a = \psi$ will be called a *"ψ-rectangle"* and for $a = 1$ *"unitary"*.

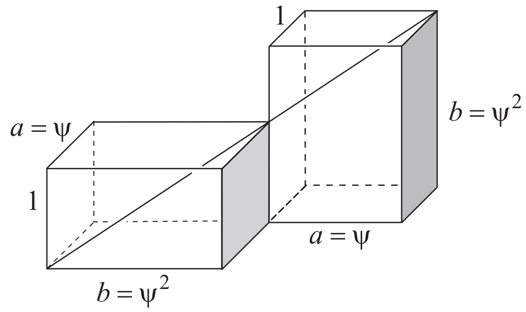

Figure 4: 3D-model of Laan's Ratio: Given two congruent boxes $1 \times a \times b$, then the vector $(a, 1, b-1)^{\mathrm{T}}$ and the diagonal vector $(b, a, 1)^{\mathrm{T}}$ are linearly dependent iff $a = \psi$

2. ψ-decomposition of a square

Instead of calculating the root of (1) numerically, we may look for constructions of a segment of length ψ well knowing that such a construction cannot be performed by compass and ruler alone. Basic for the following is

Property 1. *A square can be decomposed into three similar rectangles R_1, R_2, R_3, all of them with edges in proportion ψ^2 and the same similarity factors $R_1 : R_2 = R_2 : R_3$ (Fig. 7).*

Indeed, we start with the unitary square $ABCD$ and choose points F and H on the sides BC resp. CD (see Fig. 7) such that the similarity conditions

$$\frac{x+1}{x} = y = \frac{1}{x+1-y}$$

hold, then we get $y = 1 + \frac{1}{x}$ and finally $x^3 - x - 1 = 0$. Hereby we used the abbreviations $d(B, F) =: x$, $d(F, C) =: 1$ and $d(C, H) =: y$ for the distances of F, H to the corners B resp. C of the square.

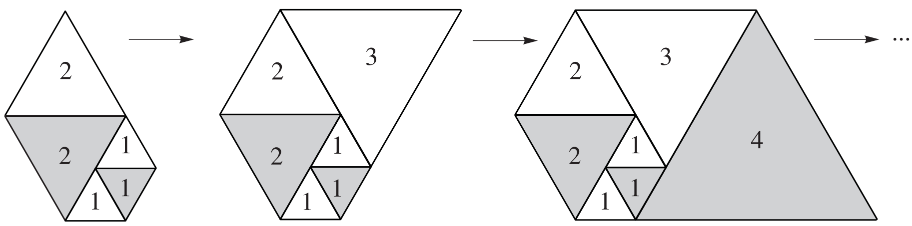

Figure 5: Planar model of the Padovan sequence

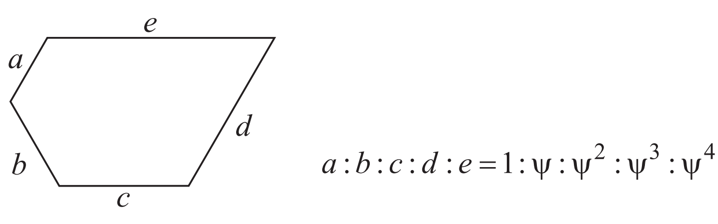

Figure 6: ψ-pentagon

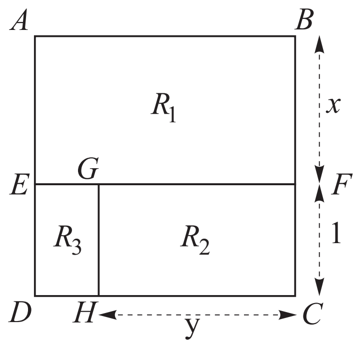

Figure 7: Decomposition of a square into three ψ-rectangles

Consequently, $x = \psi$ is the only (real) solution and it is easy to verify that the three rectangles $BFEA$, $FCHG$ and $HDEG$ are $\psi \times \psi^3$, $1 \times \psi^2$, $\psi^{-2} \times 1$, respectively. The decomposition is unique up to automorphisms of $ABCD$.

We note that $d(HD) = d(CD) - d(CH) = \psi^3 - \psi^2 = \psi^{-2}$ and therefore we automatically have $\psi^5 - \psi^4 - 1 = 0$, which characterizes ψ as a Morphic Number, too. We construct these rectangles in the following way (Figs. 8 and 9):

1. Given the square $ABCD$, choose point F on BC and draw the line $r := AF$ and the line EF parallel to AB.
2. Draw a line parallel to r through C and intersect it with EF, get point G determining a rectangle $GFCH$, similar to $ABFE$.
3. When F is moving on BC, the remaining rectangle $EGHD$ (Fig. 8) changes its shape. Exactly when DG is orthogonal to r, this rectangle gets similar to the other two, and we obtain three similar rectangles with sides in ratio ψ^2 (Fig. 9).

Using a *dynamical graphics software* like, e.g., Cabri géomètre or Cinderella, this construction easily leads to the "graphically correct" point G.

A *computer independent* construction looks for the locus of the intersection points G between the lines FE and the lines through C parallel to r, while F is moving. The pencil of lines CG is parallel-congruent to the pencil r with carrier A, which is perspective to the point-set F. Therefore the pencil of parallel lines FE must be projectively related to the pencil CG. So, according to J. Steiner (c.f. [4]), G must be a point of a *conic* h generated by two projective pencils. h passes through C and through the ideal points of AB and BC and is therefore an equilateral hyperbola. Because of the orthogonality between r and DG

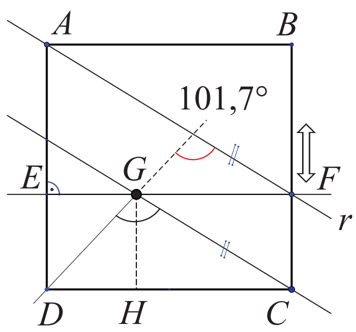

Figure 8: Construction of G by use of a dynamic graphics software

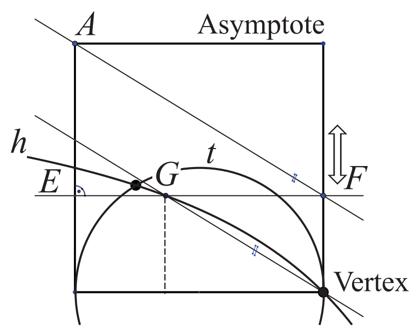

Figure 9: Construction of G by intersecting a Thales circle and an equilateral hyperbola

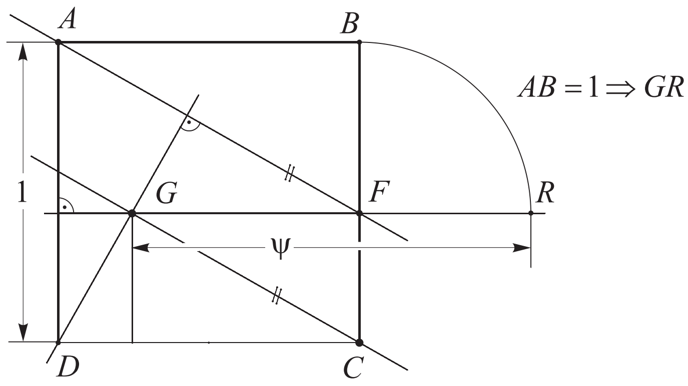

Figure 10: Line segment of length ψ to a unit square

the point G also belongs to the *Thales circle* c over CD. Combining these conditions we find $G \in c \cap h$, $G \neq C$, and — as everybody knows — this intersection problem of two curves of 2^{nd} order is not solvable by ruler and compass alone.

Starting with a (graphically found) solution G it is now very easy to obtain segments of length ψ^{-m} for $m \in \mathbb{N}$. In Fig. 10 we indicate, how to construct a segment of length ψ with respect to the unit square: Because of $\psi = \psi^{-1} + \psi^{-2}$ (per definition) we just have to sum up the segments $GF = \psi^{-1}$ and $FB = \psi^{-2}$.

Starting with the unit square 1×1 and the plastic rectangle $1 \times \psi$, it is very simple to get all the negative integer powers of ψ (see Fig. 11): By help of a "perspectivity centre" P, which is the intersection point of the diagonals of the unit square and the ψ-rectangle, we derive a *"ψ-rectangle grid"*.

This grid offers a possibility of decomposing the square into rectangles of ratio ψ^n. As an example we decompose a square of side length ψ^5, beginning with the first row of the three rectangles $\psi^5 \times 1$, $\psi^5 \times \psi$ and $\psi^5 \times \psi^2$. Figure 12 also provides a geometric proof of the obvious relation

$$1 + \psi^{-1} + \psi^{-2} + \cdots = \frac{1}{1 - \psi^{-1}} = \psi^5, \tag{6}$$

which is equivalent to (2).

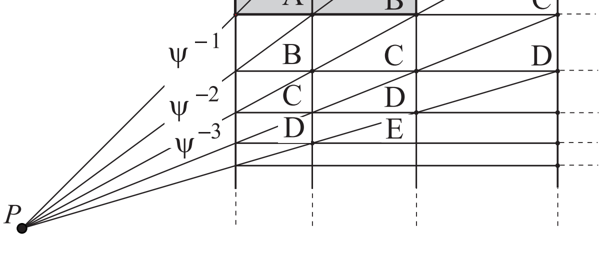

Figure 11: ψ-rectangle grid

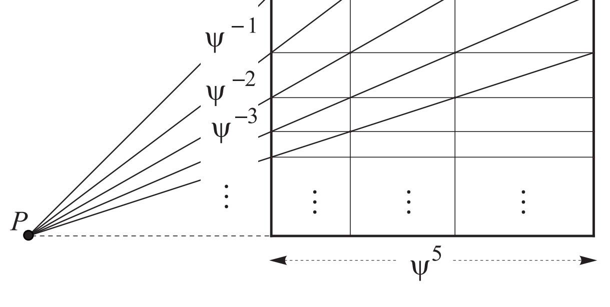

Figure 12: Powers of ψ in the square $\psi^5 \times \psi^5$

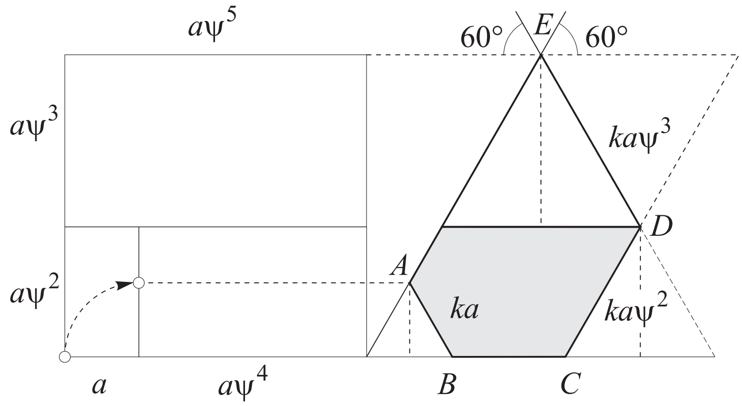

Figure 13: Plastic pentagon derived from a decomposed square

3. The ψ-pentagon

According to Fig. 6 a ψ-pentagon has four angles of 120° and one of 60°. Therefore we can embed it into an equilateral triangle. Starting with the plastic decomposition of the square Fig. 9 we derive a plastic pentagon $ABCDE$ as shown in Fig. 13 such that consecutive sides have the ratio ψ:

$$\frac{BC}{AB} = \frac{CD}{BC} = \frac{DE}{CD} = \frac{AE}{DE} = \psi. \tag{7}$$

In fact, in Fig. 13 we can observe that

$$\begin{aligned} BC &= ED - AB = ka(\psi^3 - 1) = ka\psi, \\ AE &= ED + DC - AB = ka(\psi^3 + \psi^2 - 1) = ka(\psi + \psi^2) = ka\psi^4, \end{aligned} \tag{8}$$

where $k = \dfrac{1}{\sin(\pi/3)} = \dfrac{2\sqrt{3}}{3}$ and a is a unit.

Let us consider a band of rectangles A, B, C, D, F along the circumference of a ψ-pentagon (Fig. 14) all having the same "height" a, the length of the shortest side. Without loss of generality we set $a = 1$. Then it is easy to prove:

Property 2. *"ψ-Pythagoras theorem": The sum of areas of the unit-rectangles* A, B, C, D, F *erected externally over sides of a ψ-pentagon equals the area of the square over the largest side of this ψ-pentagon (Fig. 14).*

Proof: From Fig. 12 follows

$$1 + \psi + \psi^2 = \psi^5. \tag{9}$$

Therefore

$$\begin{aligned} \mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D} + \mathrm{F} &= 1 + \psi + \psi^2 + \psi^3 + \psi^4 = \psi^5 + \psi^3 + \psi^4 = \psi^3(1 + \psi + \psi^2) = \\ &= \psi^3\psi^5 = \psi^8 = \mathrm{F} + \mathrm{E}. \end{aligned} \qquad \square$$

4. Gnomon properties of ψ-figures and ψ-spirals

The term *"gnomon"* denotes a geometric figure $\mathcal{G}$, which after addition to an initially given figure $\mathcal{F}$ forms a figure similar to the given $\mathcal{F}$. For example, a square $\mathcal{G}$ is the gnomon of a Golden Rectangle $\mathcal{F}$.

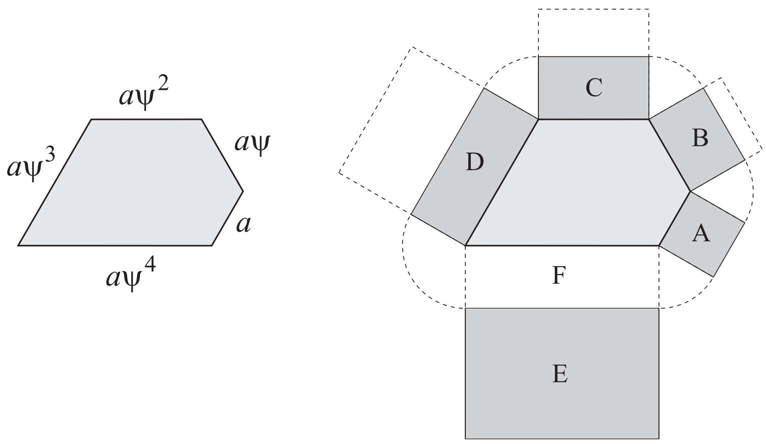

Figure 14: ψ-Pythagoras theorem to a ψ-pentagon

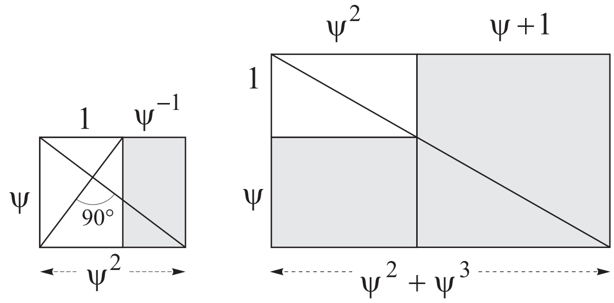

Figure 15: (a) Gnomon of a ψ-rectangle; (b) gnomon of rectangle $1 \times \psi^2$

Property 3. *The gnomon of the plastic rectangle* $1 \times \psi$ *is the rectangle of ratio* ψ^2 *(Fig. 15a). The gnomon of the rectangle* $1 \times \psi^2$ *is the figure formed by the union of the plastic rectangle* $\psi \times \psi^2$ *and a square* $\psi^2 \times \psi^2$ *(Fig. 15b).*

These assertions follow from

$$1+\psi^{-1} = \frac{\psi+1}{\psi} = \frac{\psi^3}{\psi} = \psi^2 \quad \text{and} \quad \frac{\psi^2+\psi+1}{\psi+1} = \frac{\psi^2+\psi^3}{\psi+1} = \psi^2. \tag{10}$$

Both expressions are a direct consequence of (1). In a simplified form, these properties are represented in Fig. 16, which can be seen as front and side view of the ψ-boxes of Fig. 4:

Property 4. *The gnomon of the* ψ*-pentagon is an equilateral triangle, the edge of which is equal to the biggest edge of the pentagon (Fig. 17).*

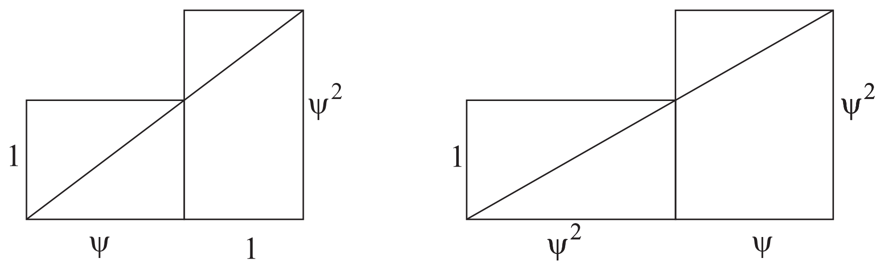

Figure 16: ψ-proportions and powers of ψ

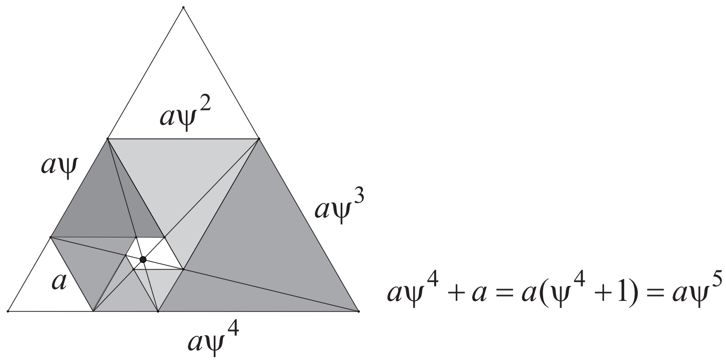

Figure 17: Equilateral triangle as gnomon of a ψ-pentagon

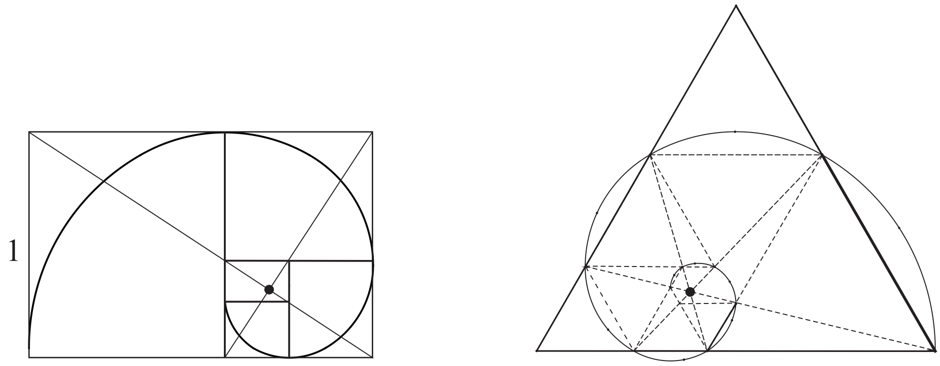

Figure 18: (a) Golden spiral based on golden rectangles and squares;
(b) ψ-spiral based on ψ-pentagons and equilateral triangles

By adding the gnomon to the former figure we get similar figures, which increase or decrease by the same factor ψ resp. ψ^{-1}. This gives rise to *logarithmic spiral polygons* which can be "smoothened" by *circular arcs* quite analogue to the well known "*golden spiral construction*" shown in Fig. 18a, b. (Starting with a golden unit rectangle, the total length of these infinitely many decreasing circular arcs is

$$\frac{\pi}{2} \cdot (\phi + 1) = \frac{\pi}{2} \cdot \phi^2 \quad (\phi \ \ldots \ \text{Golden Mean}).$$

The gnomon of the ψ-pentagon is, by Property 4, the equilateral triangle of an edge added to the biggest edge of the pentagon. Then the circular arcs, which have length $\pi/3$ times radius and decreasing by factor ψ^{-1} (see Fig. 12) form an infinite sequence of total length $L = \frac{\pi}{3} \cdot \psi^5$.

Remark: Evidently, one can also construct spirals associated to a generalized Laan's "Plastic Number" λ, which is the (real and positive) solution of equation

$$x^n - x - 1 = 0, \quad n \in \mathbb{N}, \quad n \geq 2\,. \tag{11}$$

From a figure analogue to Fig. 12 we can extract

$$1 + \lambda^{-1} + \lambda^{-2} + \lambda^{-3} + \cdots = \frac{1}{1 - \lambda^{-1}} = 1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1}. \tag{12}$$

For even n this can be written as

$$\frac{\lambda}{\lambda - 1} = (1 + \lambda)(1 + \lambda^2 + \cdots + \lambda^{n/2}) = \lambda^n(1 + \lambda^2 + \cdots + \lambda^{n/2}) \tag{13}$$

and for odd n as

$$\frac{\lambda}{\lambda-1} = (1+\lambda)(1+\lambda^2+\cdots+\lambda^{(n-1)/2}) + \lambda^{n-2} = \lambda^{n-2}(1+\lambda^2+\cdots+\lambda^{(n+1)/2}). \qquad (14)$$

So, for $n = 2, 3, 4, 5, \dots$ we find

$$L = \frac{\lambda}{\lambda-1} = \lambda^2 \text{ resp. } \lambda^5 \text{ resp. } \lambda^4(1+\lambda^2) \text{ resp. } \lambda^3(1+\lambda^2+\lambda^4) \text{ resp. } \dots \qquad (15)$$

Note that only for the cases $n = 2$ and $n = 3$ the geometric series (9) sum up to a power of λ, which means that also the defining equation of the morphic numbers (3) is fulfilled. Thus we can state

Property 5. *The only generalized Laan's Plastic Numbers $\lambda(n)$, for which the length L of the logarithmic spiral polygon (starting from an edge of length 1 and decreasing to the pole) is a power of λ, are the morphic numbers, namely the Golden Number ϕ ($L = \phi^2$) and Laan's Number ψ ($L = \psi^5$).*

Be aware that for the length L of *any* spiral through the vertices of the spiral polygon we have to multiply the length of the spiral polygon with the arc length of the spiral over the unit segment where the spiral polygon starts. For the Golden Circular Arc Spiral (Fig. 17a) this factor is $\pi/2$ and for the Circular Arc ψ-Spiral (Fig. 17b) it is $\pi/3$, but of course one could replace the circular arcs by any rectifiable arc, e.g., by a (smooth) logarithmic spiral. (If we use — contrary to a rectifiable curve — a fractal curve, e.g., a von Koch curve, in order to replace the spiral polygon's unit edge, an arc length would not be defined.)

5. ψ-ellipses

Property 6. *The area of the two crescent formed regions limited by the ellipses*

$$\frac{x^2}{\psi^4} + \frac{y^2}{\psi^2} = 1 \quad \text{and} \quad x^2 + \frac{y^2}{\psi^2} = 1$$

is equal to π. Consequently, this region between both ellipses has an area equal to the area of the circle of radius 1 inscribed in this region (Fig. 19).

This property is obvious by (1). Because of $\psi^5 - \psi^4 = 1$ two analogous results are obtained, which are shown in Fig. 20:

There is another interesting "ψ-ellipse", namely the uniquely defined ellipse e circumscribed to a ψ-pentagon.

6. ψ-trapezoids and ψ-ovals

Definition: A *ψ-trapezoid* is an isosceles trapezoid of angles 60° and 120° such that one pair of its edges is in ratio ψ. It is easily seen that there are only four ψ-trapezoids derived from the equalities $\psi^3 - \psi - 1 = 0$ and $\psi^5 - \psi^4 - 1 = 0$, as is shown in Fig. 21.

Property 7. *The ψ-pentagon can be decomposed into two ψ-trapezoids, one of type A and the other of type C. In turn, as C is decomposable into in one of type A and one of type B, a ψ-pentagon can be decomposed into two similar trapezoids of type A and one of type B (Fig. 22).*

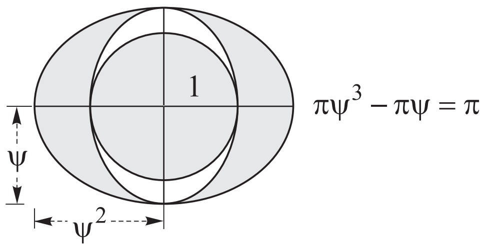

Figure 19: ψ-ellipses

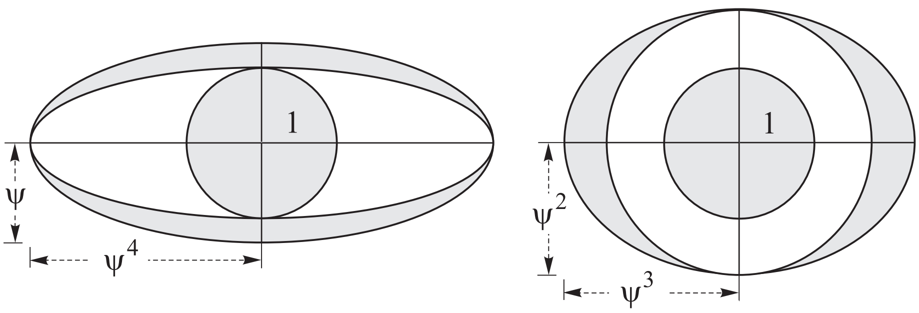

Figure 20: Other types of ψ-ellipses

A direct consequence of this property is that the area of a unit ψ-pentagon with sides 1, ψ, ψ^2, ψ^3, ψ^4 equals $\psi^9\sqrt{3}/4$.

The four ψ-trapezoids generate four types of *"ψ-ovals"* formed by circular arcs. One of them is shown in Fig. 23. The upper half of such an oval belongs to the so-called "*three centered curves*", used in architecture to replace the more complicated elliptic arcs in vaults. This oval provides a good approximation of an ellipse with axis ratio $\psi^5 : (2\psi^3 - 1) = \psi^5 : (2\psi + 1)$ (Fig. 23).

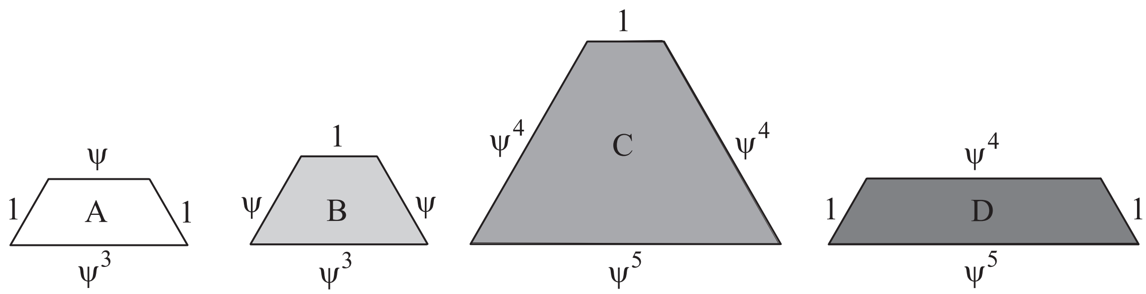

Figure 21: The four possible ψ-trapezoids

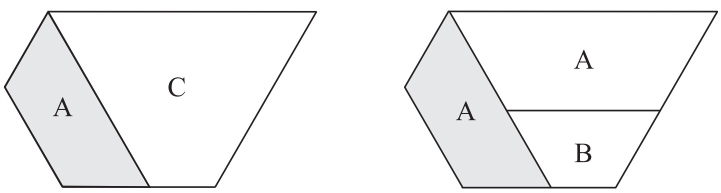

Figure 22: Dissections of a ψ-pentagon

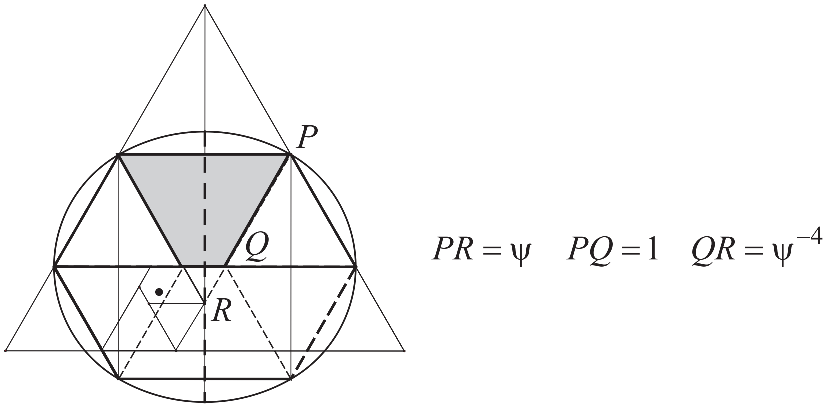

Figure 23: Centrally symmetric ψ-oval of type C

Besides, starting with a unit ψ-pentagon, we may reflect it at the symmetry axis of the second side (which has length ψ) and obtain a hexagon (Fig. 24). This hexagon has the side lengths ψ, ψ^2, ψ^3, ψ^{-1}, ψ^3, ψ^2, and the shortest diagonal (which is orthogonal to the symmetry axis) has length ψ^4. Therefore the hexagon consists of two ψ-trapezoids of type A and C glued together along this shortest diagonal (see Fig. 23). But one can dissect these trapezoids again into equilateral triangles and trapezoids of all types. Now one can circumscribe a *"ψ-oval"* to this special "ψ-hexagon", which consists, e.g., of six circular arcs of 60° (Fig. 23). But the ψ-hexagon would also allow to circumscribe an ellipse as a possible *"ψ-oval"*.

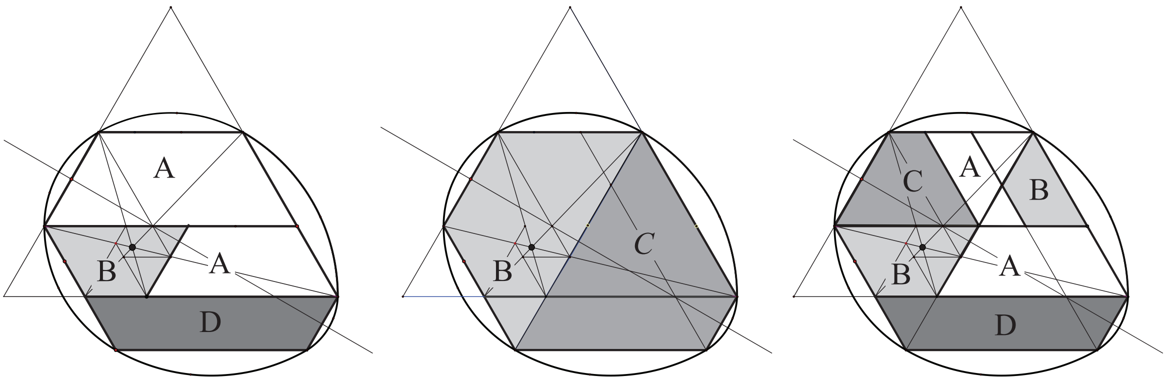

Figure 24: Axial symmetric *"ψ-hexagon"* together with dissections into ψ-trapezoids and a circumscribed ψ*-oval*

7. Conclusion

Obviously, one can "play" much more with this interesting number ψ and the presented objects and figures. In this paper we mainly considered planar figures and their properties, but the ψ-boxes in Figs. 3 and 4 indicate that it would be worthwhile to investigate *3D-objects* with relationship to ψ, too. The definition of ψ by a cubic equation, which is somehow related to that connected with the angle trisection problem, opens up to the following task:

Find a (simple) paper folding process to gain point G in Fig. 9.

We hope that the given examples will stimulate further research and generalizations to the class of "plastic" and "morphic" numbers, as these families have non-empty intersections with the Metallic Mean Family, whose members allow interpretations, e.g., in physics.

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