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Notes on Infinitesimal Bending of a Toroid Formed by Revolution of a Polygonal Meridian

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Abstract. Deformation theory focuses on the examination of rigidity conditions of surfaces. In this paper we present our tools for the examination of torus-like surfaces with a polygonal meridian in the Euclidean 3-space E^3. Based on Cohn-Vossen's method we check infinitesimal bendings of the generated surfaces. Starting from given nodes of a meridian we perform the analysis and display the obtained toroids and their deformed shapes. We use C++ and OpenGL to carry out all underlaying calculations and the 3D model visualization.

Key Words: infinitesimal bending, infinitesimal deformation, rigidity, toroid, polygon, OpenGL

MSC 2000: 53A05, 53C45, 68U05

1. Introduction

The surface bending theory considers the bending of surfaces, i.e., isometric deformations as well as infinitesimal bendings. It presents one of the main parts of global differential geometry. Any *bending* transforms a surface into a continuous family of *isometric* surfaces, i.e., such that angles and the arc length of curves on the surface are preserved. On the other hand, an *infinitesimal bending* of a surface is — roughly speaking — an approximation of an isometric deformation; we only require that for each curve the arc length remains stationary.

The basic aim of deformation theory is to find classes of rigid or non-rigid surfaces. The main task at infinitesimal bending problems is to check the flexibility of a surface with respect to the given class of infinitesimal deformations. In this paper torus-like surfaces are considered which are obtained by revolving a polygonal meridian in E^3.

In the last century the bending theory was developed thanks to the work of leading mathematicians in the considered area like D. Hilbert, H. Weyl, W. Blaschke, S. Cohn-Vossen, A.D. Alexandrov, N.V. Efimov, A.V. Pogorelov, I.N. Vekua, V.T. Fomenko, I.Kh. Sabitov, I.I. Karatopraklieva, R. Connelly, R. Bishop,

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H. Stachel, H. Gluck, V.A. Alexandrov. The first result on infinitesimal bendings of a non-convex surface is due to H. Liebmann [12, 13]. He has proved that the torus and analytic surfaces containing a convex strip are rigid with respect to infinitesimal bendings, i.e., *infinitesimally rigid*.

In 1938 A.D. Alexandrov [1] extended the results of Liebmann [12, 13]. He considered closed surfaces, subdivided by piecewise smooth curves into a finite number of regions with constant Gaussian curvature. He proved that these surfaces are infinitesimally rigid, i.e., they do not admit any nontrivial infinitesimal bending. Later, T. Rado and T. Minagawa enforced the results of H. Liebmann. They proved the rigidity of the torus [15, 16] and of surfaces of revolution of class C^1, containing a convex strip of class C^2, under the presumption that the bending field is of class C^1. It is also well-known [6, 5] that a sphere is infinitesimally rigid.

The above mentioned results naturally lead to the question whether there exist non-rigid closed surfaces. The first answer to this question was given by S. Cohn-Vossen [4, 5]. He proved that from each plane curve we can get the meridian of a non-rigid surface of revolution of genus 0. This result of S. Cohn-Vossen and his method influenced many papers on infinitesimal bendings of non-convex surfaces of revolution. Surfaces of revolution of genus 0 or 1 generated by rotation of a polygon were considered by Cohn-Vossen, Bublik, K.M. Belov [3], and N.G. Perlova [17]. Cohn-Vossen considered surfaces of genus 0 generated by a polygonal line and argued about the non-rigidity of some of them. K.M. Belov [3] presented a class of flexible toroids that are topologically equivalent to a torus. At one class the meridians have the shape of a special quadrangle (with mutually perpendicular diagonals — one parallel to the axis of rotation).

Toroid surfaces containing no planar part and generated by a triangular meridian [21] or by a parallelogram a meridian [26] are rigid. Generalizations of the investigations presented in [3] were given in [22]–[25].

2. The basic facts of infinitesimal bending theory

We start with the basic facts of the theory of infinitesimal bendings of surfaces according to [6] and [5]. The basic concept used in this work can be defined in different ways.

2.1. Infinitesimal deformations of surfaces

Let's consider a surface S in E^3 of class C^α, $\alpha \geq 3$.

Definition 2.1. The surface S_ε is a *deformation* of the piecewise regular surface S if it is included in a continuous family of surfaces

$$S_\varepsilon\colon\ \bar r = (u, v, \varepsilon) = \bar r_\varepsilon(u, v), \quad (u, v) \in D \subset \mathbb{R}^2, \quad \varepsilon \in [0, 1], \quad \text{and} \quad \bar r_\varepsilon\colon\ D \times [0, 1] \to \mathbb{R}^3,$$

and we obtain S for $\varepsilon = 0$.

Here we consider a kind of continuous family of surfaces which is defined according to [6]:

Definition 2.2. Let the surface

$$S\colon\ \bar r = \bar r(u, v), \quad (u, v) \in D, \quad D \subset \mathbb{R}^2 \tag{2.1}$$

be included in a family of surfaces

$$S_\varepsilon :\ \bar{r}_\varepsilon = \bar{r}_\varepsilon(u,v,\varepsilon), \quad \varepsilon \geq 0 \tag{2.2}$$

depending continuously on the parameter ε and with $S_\varepsilon = S$ for $\varepsilon = 0$. If

$$S_\varepsilon :\ \bar{r}_\varepsilon = \bar{r}(u,v) + \sum_{j=1}^{m} \varepsilon^j \overset{(j)}{\bar{z}}(u,v), \quad m \geq 1, \tag{2.3}$$

where $\overset{(j)}{\bar{z}}(u,v) \in C^\alpha$ with $\alpha \geq 3$ for $j = 1,\ldots,m$ are given vector fields, then the family S_ε is called an *infinitesimal deformation of order* m of the surface S.

The theory considering geometric objects in connection with S_ε up to the precision of order m with respect to ε for $\varepsilon \to 0$ is called *infinitesimal deformation theory of surfaces of order* m. Different and more special conditions give rise to different kinds of surface deformations. Higher order deformations of polyhedral surfaces were, e.g., considered in [19] and [2].

2.2. Infinitesimal bending of first order

Let the regular surface S of class C^α, $\alpha \geq 3$, be given in vector form by (2.1) and included in the family of surfaces

$$S_\varepsilon : \bar{r}_\varepsilon(u,v,\varepsilon) = \bar{r}(u,v) + \varepsilon\, \bar{z}(u,v), \tag{2.4}$$

where $\varepsilon \to 0$, $(u,v) \in D \subset \mathbb{R}$ and $\bar{r}_0(u,v,0) = \bar{r}(u,v)$.

Definition 2.3. The surfaces (2.4) are *infinitesimal bendings of first order* of the surface S if

$$ds_\varepsilon^2 - ds^2 = o(\varepsilon), \tag{2.5}$$

i.e., if the difference of the squares of line elements of these surfaces is of order higher than one. The field $\bar{z}(u,v)$ for which

$$\left.\frac{\partial \bar{r}(u,v,\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = \bar{z}(u,v) \tag{2.6}$$

is the *velocity field* or *infinitesimal bending field* of the infinitesimal bending.

According to [6, 5] this definition is equivalent to what is stated in the next theorem:

Theorem 2.1. *A necessary and sufficient condition for the surface S_ε in (2.4) to be an infinitesimal bending of the surface S in (2.1) is*

$$d\bar{r} \cdot d\bar{z} = 0\,, \tag{2.7}$$

with $\bar{z}(u,v)$ as the velocity field at the initial instant of deformation and $\cdot$ denoting the scalar product.

Equation (2.7) is equivalent to the following three partial differential equations:

$$\bar{r}_u \cdot \bar{z}_u = 0, \quad \bar{r}_u \cdot \bar{z}_v + \bar{r}_v \cdot \bar{z}_u = 0, \quad \bar{r}_v \cdot \bar{z}_v = 0. \tag{2.8}$$

Theorem 2.2. *Under an infinitesimal bending of a surface each line element is transformed according to*

$$ds_\varepsilon - ds = o(\varepsilon) \geq 0. \tag{2.9}$$

Proof: Based on (2.4) and (2.5) we have

$$ds_\varepsilon^2 = d\bar{r}^2 + \varepsilon^2 d\bar{z}^2 \implies ds_\varepsilon^2 = ds^2 \left[1 + \varepsilon^2 \left(\frac{d\bar{z}}{ds}\right)^2\right],$$

i.e.,

$$ds_\varepsilon = ds \left[1 + \varepsilon^2 \left(\frac{d\bar{z}}{ds}\right)^2\right]^{1/2}.$$

If we apply the development of the function $f(x) = (1+x)^{1/2}$ into a Maclaurin series at $x = \varepsilon^2 \left(\frac{d\bar{z}}{ds}\right)^2$, we obtain (2.9). □

3. Infinitesimal bending of toroids with a polygonal meridian

Now we consider an infinitesimal deformation of a surface of revolution with a simple polygonal meridian. We will give necessary and sufficient conditions for such a toroid to be non-rigid. And we explain the procedure for generating a bending field.

Let P_n be the simple polygon with apices $A_i(u_i, \rho_i)$, $i = 1, 2, \dots, n$, in a meridian plane equipped with a cartesian coordinate system $uO\rho$ with u as axis of rotation. The sides of P_n obey the equations

$$A_m A_{m+1}\colon\ \rho_{(m)} = \rho_m + \frac{\rho_{m+1} - \rho_m}{u_{m+1} - u_m}(u - u_m), \tag{3.1}$$

$$\rho'_{(m)} = \frac{\rho_{m+1} - \rho_m}{u_{m+1} - u_m} = k_m, \quad m = 1, 2, \dots, n; \quad A_{n+1} \equiv A_1,$$

where $\rho_{(m)}$ is the value of ρ on $A_m A_{m+1}$. In order to consider an infinitesimal bending of this surface of revolution with a closed piecewise smooth meridian, we will use Cohn-Vossen's method [5]. The radius vector of the surface is

$$\bar{r}(u, v) = u\,\bar{e} + \rho(u)\bar{a}(v),$$

where $\rho = \rho(u)$ is the equation of the meridian. If $\bar{e}$ is the unit vector of the axis of rotation, $\bar{a}(v)$ the unit vector of the ρ-axis, v the angle between the initial meridian plane including $\bar{a}(v)$, then $\bar{a}'(v)$ is perpendicular to $\bar{a}(v)$ and to $\bar{e}$ (see [6, p. 90] or [5, p. 253]).

We try to find a fundamental infinitesimal bending field of the surface S in the form

$$\begin{aligned}\bar{z}(u, v) = \bar{z}_k(u, v) = {} & [\varphi_k(u)e^{ikv} + \tilde{\varphi}_k(u)e^{-ikv}]\,\bar{e} + \\ & [\,\psi_k(u)e^{ikv} + \tilde{\psi}_k(u)e^{-ikv}]\,\bar{a}(v) + [\chi_k(u)e^{ikv} + \tilde{\chi}_k(u)e^{-ikv}]\,\bar{a}'(v).\end{aligned}$$

The functions $\varphi_k(u)$, $\psi_k(u)$ and $\chi_k(u)$ satisfy the equations

$$\begin{aligned}\varphi'_k(u) + \rho'(u)\,\psi'_k(u) &= 0,\\ \psi_k(u) + ik\,\chi'_k(u) &= 0,\\ ik\,\varphi_k(u) + \rho'(u)[\,ik\,\psi_k(u) - \chi_k(u)] + \rho(u)\,\chi'_k(u) &= 0.\end{aligned}$$

The differential equation of the second order

$$\rho(u)\lambda''(u) + (k^2-1)\rho''(u)\lambda(u) = 0 \tag{3.2}$$

is satisfied for $\lambda(u) = \psi_k(u)$, $\chi_k(u)$. We omit the index k and denote with $\psi_{(i)}$ the value of the function ψ on $A_i\, A_{i+1}$ for $i = 1, 2, \ldots, n$ under $A_{n+1} \equiv A_1$.

From the equations (3.1) and (3.2) follows also the linearity of the functions $\psi_i(u)$

$$\psi_{(i)} = M_i\, u + N_i\,, \quad i = 1, 2, \ldots, n \tag{3.3}$$

At the points $u = \sigma$ of the meridian, where $\rho(\sigma - 0) = \rho(\sigma + 0)$, i.e., at the apices of the polygon, we get, supposing the continuity of the function $\psi_{(i)}(u)$,

$$\psi_{(i)}(u_i) = \psi_{(i-1)}(u_i), \quad i = 2, \ldots, n; \quad \psi_{(1)}(u_1) = \psi_{(n)}(u_1),$$

and from there based on (3.3)

$$M_i\, u_i + N_i = M_{i-1}u_i + N_{i-1} \quad i = 1, 2, \ldots, n; \quad M_0 \equiv M_n, \quad N_0 \equiv N_n.$$

If we consider this system as a system with respect to unknowns N_i, $i = 1, 2, \ldots, n$, we get

$$\begin{array}{llll}
N_1 & & -N_n & = -M_1 u_1 + M_n u_1 \\
N_1 - N_2 & & & = -M_1 u_2 + M_2 u_2 \\
& \ldots\ldots\ldots\ldots\ldots\ldots & & \\
& N_{m-1} - N_m & & = -M_{m-1} u_m + M_m u_m \\
& \ldots\ldots\ldots\ldots\ldots\ldots & & \\
& & N_{n-1} - N_n & = -M_{n-1} u_n + M_n u_n
\end{array} \tag{3.4}$$

At the apices of the polygon we have according to [5] the next equation

$$\rho(\sigma)[\psi_k'(\sigma+0) - \psi_k'(\sigma-0)] + (k^2-1)\psi_k(\sigma)[\rho'(\sigma+0) - \rho'(\sigma-0)] = 0.$$

Applying this equation to the apices M_i, $i = 1, 2, \ldots, n$, we get the system of equations

$$\begin{gathered}
\rho_i(M_i - M_{i-1}) + (k^2-1)((M_i\, u_i + N_i)(k_i - k_{i-1}) = 0, \\
i = 1, 2, \ldots, n; \quad M_0 \equiv M_n, \quad k_0 \equiv k_n.
\end{gathered} \tag{3.5}$$

The equations (3.4) and (3.5) represent a system of linear equations for the unknowns M_i and N_i, $i = 1, 2, \ldots, n$. Let A denote the matrix of the system and P the extended matrix.

The system is compatible if and only if $\operatorname{rank} A = \operatorname{rank} P$, i.e., if and only if

$$M_n = \frac{1}{u_1 - u_n} \sum_{i=1}^{n-1} (u_i - u_{i+1}) M_i. \tag{3.6}$$

According to (3.5) and (3.6) we get the reduced system

$$\begin{array}{lllll}
N_1 & & & -N_n & = (M_n - M_1)u_1 \\
& -N_2 & & +N_n & = (M_1 - M_n)u_1 + (M_2 - M_1)u_2, \\
& & & \ldots\ldots\ldots\ldots & \\
& & -N_m & +N_n & = (M_1 - M_n)u_1 + \sum_{l=2}^{m}(M_l - M_{l-1})u_l, \\
& & & \ldots\ldots\ldots\ldots & \\
& & & N_{n-1} + N_n & = (M_1 - M_n)u_1 + \sum_{l=2}^{n-1}(M_l - M_{l-1})u_l, \\
& & & \text{for} \quad m = 3, \ldots, n-2. &
\end{array} \tag{3.7}$$

After introducing the notation

$$u_i - u_j = u_{i,j}, \quad k_i - k_j = k_{i,j}$$

we get from (3.7)

$$\begin{aligned} N_1 &= N_n + \frac{u_1}{u_{1,n}}\left[u_{n,2}M_1 + \textstyle\sum_{i=2}^{n-1} u_{i,i+1}M_i\right] \\ &\cdots\cdots\cdots \\ N_m &= N_n + \frac{u_1}{u_{1,n}}\textstyle\sum_{i=1}^{m-1} u_{i,i+1}M_i + \dfrac{u_m u_n - u_1 u_{m+1}}{u_{1,n}}\, M_m + \dfrac{u_1}{u_{1,n}}\textstyle\sum_{i=m+1}^{n-1} u_{i,i+1}M_i \\ &\cdots\cdots\cdots \\ N_{n-1} &= N_n + \frac{u_n}{u_{1,n-2}} + \textstyle\sum_{i=1}^{n-2} u_{i,i+1}M_i + \dfrac{u_n u_{n-1,1}}{u_{1,n}}\, M_{n-1} \\ &\qquad \text{for} \quad m = 2,\ldots,n-2. \end{aligned} \tag{3.8}$$

Then the system (3.5) with unknowns $M_1,\ldots,M_{n-1},N_n$ reduces to

$$\left[\rho_1 u_{2,n} + (k^2-1)k_{1,n}u_1u_{1,2}\right]\, M_1 = \sum_{i=1}^{n-1} u_{i,j+1}\left[(k^2-1)k_{1,n}u_1u_{1,2}\right] M_1 \tag{3.9.1}$$

$$\begin{gathered} [(k^2-1)k_{2,1}u_nu_{1,2} - \rho_2 u_{1,n}]\, M_1 + [\rho_2 u_{1,n} + (k^2-1)k_{2,1}u_1u_{2,3}]\, M_2 \\ +(k^2-1)k_{2,1}\left[u_{1,2}N_n + u_1\textstyle\sum_{i=3}^{n-1} u_{i,i+1}M_i\right] = 0 \end{gathered} \tag{3.9.2}$$

$$\cdots\cdots\cdots$$

$$\begin{gathered} (k^2-1)k_{m,m-1}u_n\textstyle\sum_{i=1}^{m-2} u_{i,j+1}M_i + [(k^2-1)k_{m,m-1}u_nu_{m-1,m} + \rho_m u_{n,1}]\, M_{m-1} \\ +\,(k^2-1)k_{m,m-1}u_1\textstyle\sum_{i=m+1}^{n-2} u_{i,j+1}M_i + (k^2-1)k_{m,m-1}u_{1,n}N_n = 0 \end{gathered} \tag{3.9.m}$$

$$\cdots\cdots\cdots$$

$$\begin{gathered} [\rho_n + (k^2-1)k_{n,n-1}u_n]\textstyle\sum_{i=1}^{n-1} u_{i,i+1}M_i + [\rho_n u_{n-1,1} + (k^2-1)k_{n,n-1}u_nu_{n-1,n}]\, M_{n-1} \\ +\,[\rho_m u_{1,n} + (k^2-1)k_{m,m-1}u_1u_{m,m+1}]\, M_m + (k^2-1)k_{n,n-1}u_{1,n}N_n = 0 \end{gathered} \tag{3.9.n}$$

Let B denote the matrix of the system (3.9). A necessary and sufficient condition for a nontrivial solution of this system of homogenous linear equations is

$$\det B = 0. \tag{3.10}$$

After transforming B in triangular form we get the condition

$$B_{n,n} = 0. \tag{3.11}$$

In this way, the next theorem is proved.

Theorem 3.1. *A necessary and sufficient condition for the non-rigidity of the surface of revolution with the polygonal meridian with apices $A_i(u_i,\rho_i)$, $\rho_i > 0$, $u_i \neq u_{i+1}$, $i = 1,2,\ldots,n$, is given by (3.10), which is equivalent to (3.11), where B is the matrix of the system (3.9) and*

$$u_{i,j} = u_i - u_j, \quad k_{i,j} = k_i - k_j, \quad k_i = \frac{\rho_{i+1}-\rho_i}{u_{i+1}-u_i}, \quad k \geq 2.$$

The procedure used here offers a possibility to determine the field of infinitesimal bending. Under condition (3.10) we obtain from (3.9) the reduced system

$$\begin{aligned} b_{1,1}N_n + b_{1,2}M_{n-1} + \cdots + b_{1,n}M_1 &= 0 \\ b_{2,2}M_{n-1} + \cdots + b_{2,n}M_1 &= 0 \\ \cdots\cdots\cdots \\ b_{n-1,n-1}M_2 + b_{n-1,n}M_1 &= 0 \end{aligned}$$

and from there $M_2 = \frac{b_{n-1,n}}{b_{n-1,n-1}} M_1$, provided $b_{n-1,n-1} \neq 0$, and $M_3, \ldots, M_{n-1}, N_n$ expressed in terms of M_1 (undefined constant). Further we get $\psi_i(u)$ based on (3.3). In this way we get the bending field.

4. Visualization of infinitesimal bendings of a toroid with a polygonal meridian

The computer enables to display surfaces when seen from different points of view. Furthermore, it enables to analyze the non-rigidity conditions and to compute a bending field.

Previously, we started with a family of toroids and determined their properties using the package *Mathematica*. We took points of a meridian as input, and checked the rigidity conditions. The output string defined in symbolical notation the surface of revolution together with the field of infinitesimal bending of first order. The result was the basis for the graphical output which allowed a graphical analysis. Graphical representations of deformations have been considered in [9, 10, 20].

In order to have more flexibility in the visual presentation and to speed up the basic and 3D calculations, we developed *SurfBand*. It is very useful to examine surfaces of revolution and to check their distortion under the influence of an infinitesimal bending field.

4.1. Use of ***SurfBand***

SurfBand, the program devoted to visualize infinitesimal bending of toroids, has been developed in C++ [11] and uses OpenGL [8, 14] standard to display graphics. It should therefore be portable, although it has only been tested on *Microsoft Windows* platforms. The underlaying calculations of the geometric model were done in ANSI C++, but rising control to interactive level was done using MFC [18].

It has early been presented at the ESI Conference "Rigidity and Flexibility" in Vienna, 2006. It takes as input cartesian coordinates of points of the polygon, and then performs the non-rigidity analysis. If a polygon satisfies the non-rigidity conditions, we are able to display the family of bendable surfaces. It also can show a few already found examples of infinitesimally flexible convex and non convex toroids with a polygonal meridian.

As soon as the polygon of an appropriate flexible toroid is specified, we can use a *View/Property dialog* in order to examine the shape. Here we can influence the visibility and the colors of the conical sections of the surface. At the beginning the bending parameter is set 0.0 (no bending), the angle of rotation is set $\frac{3\pi}{2}$, and the number of subdivision points of the grid is set to 20. Afterwards is can be changed via appropriate scroll bars, and the effect on the 3D model can be inspected in the main application window.

Pressing and holding down the left mouse button, while dragging the mouse, will rotate the surface. Pressing *"w"* is necessary to show the wire frame model and *"f"* for filling the model. The point of view can be positioned farer or closer to the model. There is a bright spotlight to achieve more realistic pictures of the 3D object.

The program can run in the *"drag and rotate" mode* where the rotation of the model can be repeated continuously (loop or reversed way). The apices of the selected polygon trace circles around the z-axis during the rotation. Bending deforms the circles into curves which are visible and manageable via the *View/Cone borders dialog*. Its activation pushes the program to run in a mode, which hides cones or shows them only. It is possible to adjust interactively properties like color and visibility of curves of all available polygon's apices, the minimum and maximum values of the bending parameter and the number of inner borders. The display can show curves representing borders or meshes which are more suitable for representing the surface formed by the bending borders.

We use a kind of Free-form deformation (FFD) in modeling the infinitesimal bending of a bendable toroid. FFD [7, 20] is a general method for deforming objects that provides a higher and more powerful level of control and is computationally efficient. It enables to create an animation. We are able to define the initial form and properties of the model via the *AnimationBeginProperty dialog* and the final form and properties of the model via the appropriate *AnimationEndProperty dialog*. After checking the Drag to the animate box, pressing and holding down the left button of the mouse while moving the mouse will memorize the applied rotations. Releasing the left mouse button will finish the creation of animation which shows the transformation from the initial to the final form.

5. Examples

- The first example of a non-rigid toroidal surface (Fig. 1) is given by the meridian, a convex quadrangle with apices $A(-3,\, \frac{18}{7})$, $B(0,\, \frac{65}{28})$, $C(5,\, \frac{18}{7})$, $D(0,\, \frac{32}{7})$, $k = 3$, $\varepsilon = 0.15$.
- The second example (Fig. 2) is based on the convex pentagon with apices $A(-1,\, 1)$, $B(-2,\, 3)$, $C(1,\, 4)$, $D(2,\, \frac{24275-31\sqrt{51937}}{6069}$, $E(1,\, 2)$, $k = 3$, and $\varepsilon = 0.01$.
- The third example (Fig. 3) has a convex hexagonal meridian with apices $A(-1,\, 1)$, $B(-2,\, 2)$, $C(-1,\, 3)$, $D(0,\, 6)$, $E(1,\, 3)$, $F(2,\, 2)$, $k = 2$, and $\varepsilon = 0.1$.
- The fourth example (Fig. 4) of a non-rigid toroidal surfaces is based on the convex hexagonal polygon with apices $D(0,\, \frac{167-\sqrt{6133}}{37})$, $E(1,\, 3)$, $F(2,\, 2)$, $A(-1,\, 1)$, $B(-2,\, 2)$, $C(-1,\, 3)$, $k = 2$, and $\varepsilon = 0.1$.

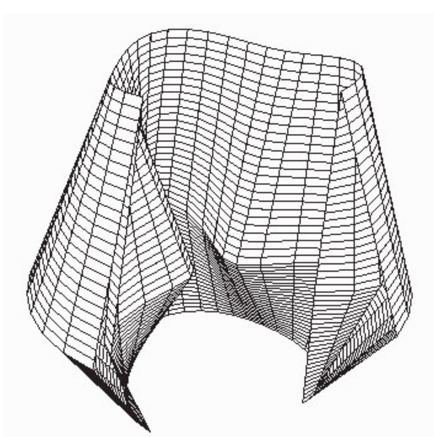

Figure 1:

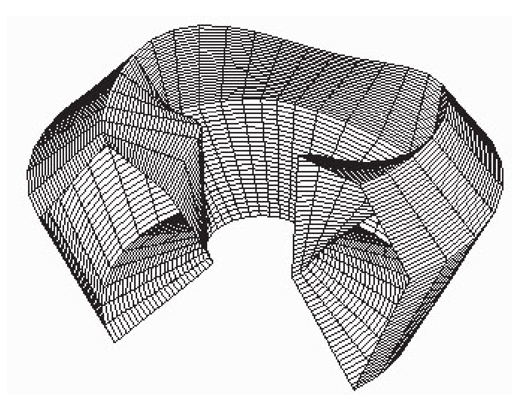

Figure 2:

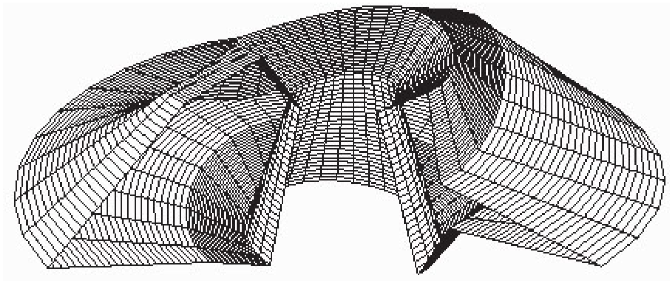

Figure 3:

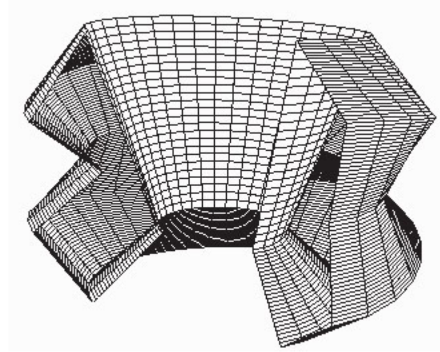

Figure 4:

6. Conclusions

In this paper the authors analyse a class of surfaces topologically equivalent to a torus. It is known that a circular torus is rigid. Based on theoretical considerations, the authors present a tool for examining the infinitesimal rigidity of a toroid generated by a polygonal meridian. In this way the family of surfaces that is non-rigid is enlarged, and the deformed surfaces are presented. The developed program *SurfBand* starts with nodes of a meridian, performs the analysis and displays the obtained toroid and its deformed shapes. Besides theoretical considerations based on Differential Geometry and Fourier Analysis, we use C++ and OpenGL to perform the underlying calculations and the 3D model visualization.

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