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Reflected Surfaces and Their Directing Cones

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Abstract. This article is devoted to the surface which arises after reflecting parallel rays along a planar section of a reflecting surface. The resulting "reflected rays surface" enables to decompose the congruence of the reflected rays into a one-parametric set of reflected rays surfaces. Three director curves of the reflected rays surfaces are determined: The given planar section, along which the reflected rays surface is constructed, a double line, along which pairs of reflected rays are intersecting, and a directing cone. By the use of the directing cone as the third director curve of the reflected rays surface the law of reflection is realized. There is a connection between the direction of incident rays, the directing cone of the normal surface and the directing cone of the reflected rays surface. Proposed algorithms based on the directing cone give us the possibility to describe the reflected rays surfaces along planar sections of particular reflecting surfaces analytically and synthetically.

Key Words: ruled surface, directing cone, line congruence, reflected rays surface, normals surface

MSC 2000: 51N05

1. Introduction

Reflections play a central role in many applications of solar energy utilization, in illumination by natural and artificial light and in architecture [4, 5, 6]. Properties of the reflection in spheres, paraboloids of revolution, parabolic and circular cylinders are well investigated in the case when the direction of incident rays is parallel to the axis of these surfaces. Otherwise the reflected picture of rays changes and requires additional investigation. Also the reflecting properties of many other families of surfaces need to be investigated.

The reflection in any surface along a planar section, when the normals of the planar section coincide with the surface normals, is a planar task. In all other cases it is necessary to consider the spatial situation, which is more complicated. Our strategy is to consider the congruence of reflected rays as a one-parametric set of reflected rays surfaces [8]. Here the

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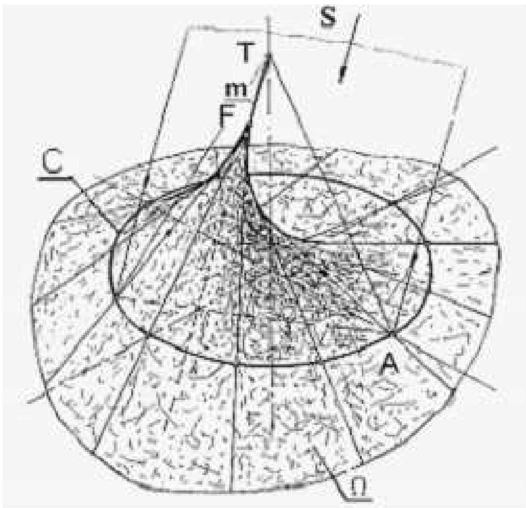

Figure 1: Reflected rays surface

term *"reflected rays surface"* stands for the ruled surface formed by incident parallel rays after reflection along a planar section of the reflecting surface.

The reflected rays surface with a source of rays at infinity is a ruled surface with three directing lines. In [1] it was proved, that such directing lines are

- the planar section **C** along which the reflected rays are constructed;
- a double line **m**, along which the rays are pairwise meeting;
- a directing cone **T** of the reflected rays surface.

An axonometric view of the reflected rays surface reveals concentrations of the reflected rays. In Fig. 1 the reflected rays surface $\mathbf{\Omega}$ along a circle **C** is displayed. In the case of a canal surface or a surface of revolution the normal surface along **C** can be assumed as a right cone with apex T. The given incident rays are parallel to the direction S.

2. Directing cone of the reflected rays surface

The surface $\mathbf{\Omega}$ of the reflected rays along the planar section **C** of a reflecting surface as displayed in Fig. 1 is a ruled surface with a straight double line **m** and the section **C** as second directrix. In addition, the law of reflection must be fulfilled: the angle of incidence equals that after reflection. To carry out this law, it is possible to use the cylinder of incident rays that passes through the section **C** and the normal surface of the reflecting surface along **C**.

It is known, that tangent planes to a second order surface along a planar section $\mathbf{C}^2$ envelope a second order cone. In articles [7, 2, 3] the surfaces of normals along planar sections of second order surfaces are investigated.

Using a directing cone as the third directrix of the reflected rays surface $\mathbf{\Omega}$ we realize the law of reflection. There is a connection between the direction S of incident rays, the directing cone $\mathbf{\Psi}^n$ of the normal surface and the directing cone $\mathbf{T}^n$ of the reflected rays surface. For the construction of the reflected rays surface it is necessary to know the construction of these two directing cones:

The form of the directing cone $\mathbf{T}^n$ depends on the form of the directing cone $\mathbf{\Psi}^n$ of the normal surface along the planar section $\mathbf{C}^n$. For the second order reflecting surfaces it is possible to separate the following three types of directing cones $\mathbf{\Psi}^n$ of the normal surfaces:

1. Elliptic cone (i.e., quadratic, non-degenerate and not a cone of revolution).
2. Right cone, i.e., cone of revolution.
3. Flat cone, i.e., all surface normals are parallel to a plane $\boldsymbol{\beta}$.

In the sequel we confine ourselves to these three types of cones $\mathbf{\Psi}^n$ and study the corresponding reflected rays surfaces.

2.1. Elliptic cone as directing cone of normals

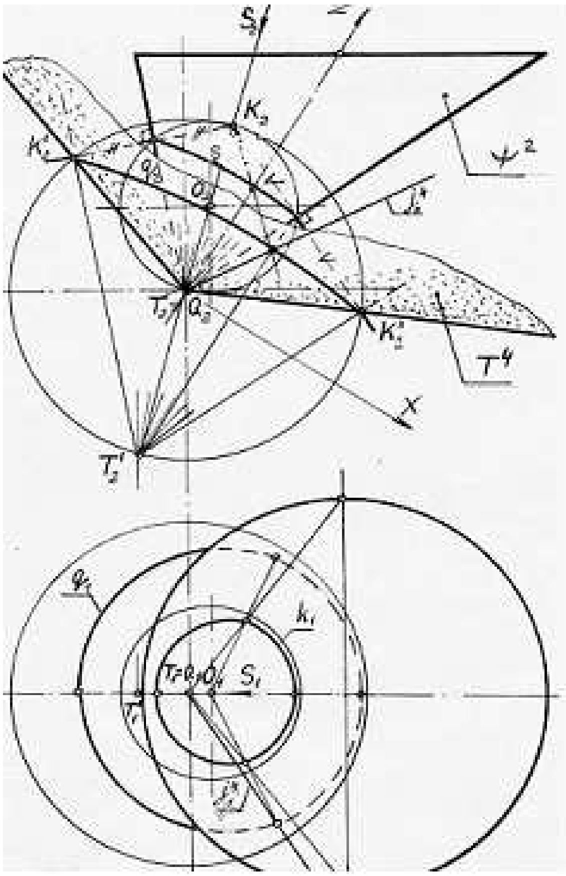

Figure 2: Directing cone $\mathbf{T}^4$ of the reflected rays surface with the elliptic cone $\mathbf{\Psi}^2$ as directing cone of normals

Theorem 1. When the directing cone of normals along a planar section of the reflecting surface is elliptic, then the directing cone of the reflected rays surface $\mathbf{\Omega}$ is of order four.

Proof: Let's consider the directing cone $\mathbf{T}^n$ of the reflected rays surface $\mathbf{\Omega}$ along a planar section $\mathbf{C}^2$ of any surface in the case, that the directing cone $\mathbf{\Psi}^2$ of the normal surface is elliptic (Fig. 2). The direction of the incident rays is given by line S passing through the apex T of the cone $\mathbf{\Psi}^2$.

The generators of the directing cone of the reflected rays surface $\mathbf{\Omega}$ are symmetric to the incident ray S with respect to the generators of the directing cone of normals $\mathbf{\Psi}^2$ according to the law of reflection.

To construct the directing cone of the reflected rays surface $\mathbf{\Omega}$ lets take a point K on the ray S and construct the mirror of K after reflection in any generator of the directing cone of normals. The feet of perpendiculars drawn from K onto lines through T are placed on the sphere $\mathbf{Q}$ with diameter KT. The feet of perpendiculars onto generators of the directing cone of normals belong to a curve $\mathbf{s}$ which is the line of intersection between the sphere $\mathbf{Q}$ and the cone $\mathbf{T}^2$. Thus in general a curve $\mathbf{s}$ of the fourth order is obtained.[1]

Points symmetrical to K with respect to generators of the directing cone of normals lie on a curve $\mathbf{q}$. It is obtained from $\mathbf{s}$ by a dilation with center K and factor 2. The order of the curve $\mathbf{q}$ is also four.

The generators of the directing cone of the reflected rays surface $\mathbf{\Omega}$ pass through the apex T of the cone $\mathbf{\Psi}^2$ and meet the curve $\mathbf{q}$ of fourth order. The above mentioned dilation maps T onto a point T' and the cone $\mathbf{\Psi}^2$ onto a cone $\mathbf{\Psi}^{2\prime}$ which is translatory congruent to $\mathbf{\Psi}^2$ and has the apex T'. The sphere $\mathbf{Q}$ is transformed into a sphere $\mathbf{Q}'$ with center T and radius KT. Hence curve $\mathbf{q}$ is the intersection of the cone $\mathbf{\Psi}^{2\prime}$ with the sphere $\mathbf{Q}'$.

The directing cone $\mathbf{T}^2$ of the reflected rays surface $\mathbf{\Omega}$ connecting the apex T with the fourth order curve $\mathbf{q}$ is in general of order 4 — except in the case mentioned in Footnote 1 which results in a right cone. □

We summarize: The directing cone of the reflected rays surface $\mathbf{\Omega}$ is constructed by the following algorithm:

1. On the incident ray S passing through the apex T of the cone $\mathbf{\Psi}^2$ of normals specify any point K and draw the sphere $\mathbf{Q}'$ with center T and radius KT.
2. The directing cone $\mathbf{\Psi}^2$ of normals is translated into the cone with apex T' which is the mirror of K with respect to T.
3. Curve $\mathbf{q}$ is the line of intersection between the sphere $\mathbf{Q}'$ and the translated cone.
4. The directing cone of the reflected rays surface $\mathbf{\Omega}$ has the apex T and passes through the curve $\mathbf{q}$.

2.2. Right cone as directing cone of normals

Theorem 2. When the directing cone of normals along a planar section of the reflecting surface is a cone $\mathbf{\Psi}^2$ of revolution, then the directing cone of the reflected rays surface $\mathbf{\Omega}$ has order four.

Proof: Let's consider the directing cone $\mathbf{T}^n$ of the reflected rays surface $\mathbf{\Omega}$ along a planar section of any surface (surface of rotation, canal surface) for the case, when the directing cone of the normal surface is a right cone $\mathbf{\Psi}^2$ (Fig. 3).

In this case the directing cone of the reflected rays surface $\mathbf{\Omega}$ is constructed with the following algorithm:

1. The unit sphere $\mathbf{Q}$ with the center at point T is constructed.
2. The right cone of normals $\mathbf{\Psi}^2$ is translated in direction of the incident rays into a cone with the apex T' on the sphere $\mathbf{Q}$.
3. The curve $\mathbf{q}$ of intersection between the sphere $\mathbf{Q}$ and the translated cone is obtained.
4. The directing cone of the reflected rays surface $\mathbf{\Omega}$ has the apex T and passes through curve $\mathbf{q}$.

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[1] The only exception shows up when the sections of $\mathbf{\Psi}^2$ perpendicular to the ray S are circles, because then $\mathbf{s}$ is a circle on $\mathbf{Q}$. This is related to the stereographic projection of $\mathbf{Q}$ with center T.

We control this analytically: The equation of a cone of revolution with the apex at the origin and the generator $z = -\tan\alpha \cdot x$ is

$$\frac{x^2}{1} + \frac{y^2}{1} - \frac{z^2}{\tan^2\alpha} = 0\,. \tag{1}$$

The translated cone with apex at the point T' can be represented as

$$\frac{(x+m)^2}{1} + \frac{y^2}{1} - \frac{(z+p)^2}{k^2} = 0 \tag{2}$$

with $k = -\tan\alpha$. The equation of the unit sphere is

$$x^2 + y^2 + z^2 = 1\,. \tag{3}$$

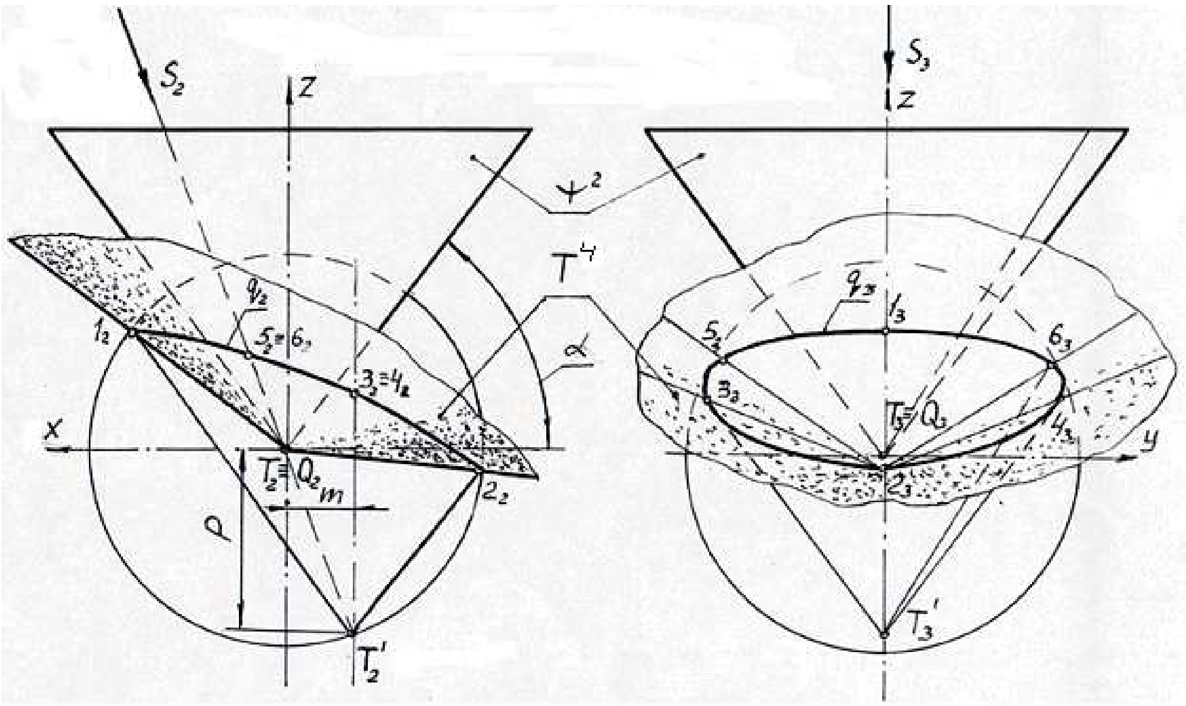

Figure 3: Directing cone $\mathbf{T}^4$ of the reflected rays surface with the cone $\mathbf{\Psi}^2$ of revolution as directing cone of the normal surface

The common solution of (2) and (3) will give the equation of a spatial curve $\mathbf{q}$ of order four. Projected on the plane ZOX it is the curve

$$z^2(k^2+1) + 2zp - 2kmx - k^2 - k^2m^2 + p^2 = 0\,. \tag{4}$$

This is a parabola. The projection onto the plane ZOY gives an ellipse. The same holds for the plane XOY. Projection from the center T gives a cone of order 4 — like in Theorem 1. $\square$

2.3. The directing cone of normals is flat

Theorem 3. When the directing cone of the normal surface is flat (Fig. 4) the directing cone of the reflected rays surface $\mathbf{\Omega}$ is a cone $\mathbf{T}^2$ of revolution.

Proof: According to the proof of Theorem 1 we apply the following algorithm:

1. On the incident ray S passing through the apex T of the directing cone of normals, which in this case is located in a plane $\boldsymbol{\beta}$, a point K is chosen and furthermore the sphere $\mathbf{Q}$ with center T and radius TK is constructed.
2. The circle $\mathbf{s}$ is the line of intersection between the sphere $\mathbf{Q}$ and the plane $\boldsymbol{\beta}$.
3. The dilation with center K and factor 2 transforms the circle $\mathbf{s}$ into the circle $\mathbf{q}$.
4. The directing cone $\mathbf{T}^2$ of the reflected rays surface $\mathbf{\Omega}$ has the apex T and passes through the curve $\mathbf{q}$. This gives a right cone, as stated.

The equation of the directing cone $\mathbf{T}^2$ can be obtained as follows: The direction of incident rays is given by $S(m,n,p)$. As the direction S is parallel to the plane ZOY, we have $n=0$, and the equation of a generator of the cone $\mathbf{T}^2$ (compare Fig. 4) is

$$z = \frac{p}{m} \cdot x \,. \tag{5}$$

The cone is generated by rotation of the generator (5). Instead of x we substitute $\sqrt{x^2+y^2}$ and obtain

$$z^2 = \frac{p^2}{m^2}\left(x^2+y^2\right). \tag{6}$$

After transformations the equation of a directing cone of the reflected rays surface is

$$\frac{x^2}{m^2} + \frac{y^2}{m^2} - \frac{z^2}{p^2} = 0 \,. \tag{7}$$

For $z=-p$ we obtain the cone basis

$$x^2+y^2 = m^2. \tag{8}$$

The described directing cone formalizes the law of reflection and it is one of three directrices of the reflected rays surface along a planar section of the reflecting surface. The algorithms for obtaining the directing cones of the reflected rays along planar sections of reflecting surfaces are thus presented for three cases of normal surfaces. □

3. Reflected rays surface along planar sections with a flat directing cone of normals

Let's consider concentrators of solar installations with cylinders of revolution and canal surfaces. For the investigation of the flow density of the reflected solar rays (line congruence) it is necessary to know properties of the reflected rays surfaces along planar sections of the reflecting surface that have a flat directing cone of normals.

For a reflecting surface $\mathbf{\Phi}^2$ as the cylinder of revolution or canal surfaces the offered algorithm is appropriate. All normals along the planar sections are parallel to a plane, and by Theorem 3 the directing cone of the reflected rays surface is a cone of revolution.

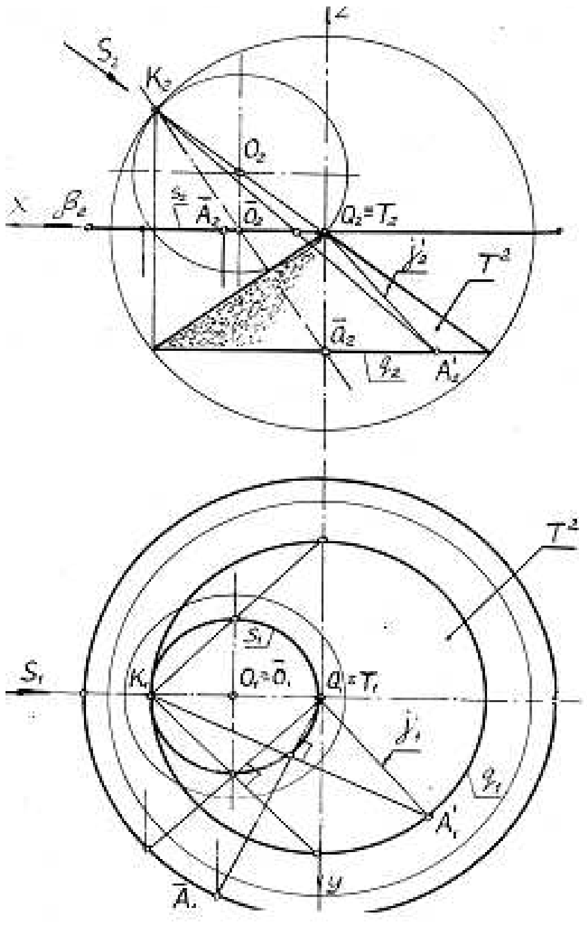

Figure 4: Directing cone $\mathbf{T}^n$ of the reflected rays surface with flat directing cone of normals

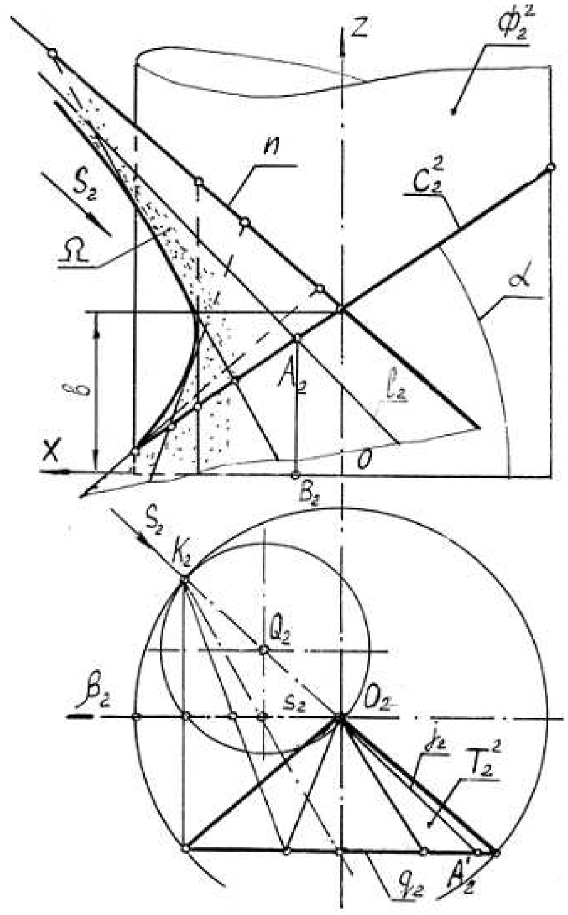

Figure 5: The reflected rays surface $\boldsymbol{\Omega}$ along the planar section $\mathbf{C}^2$ of a cylinder of revolution

In Fig. 5 the right cylinder $\boldsymbol{\Phi}^2$ as reflecting surface is represented. All normals of this reflecting surface $\boldsymbol{\Phi}^2$ are parallel to the plane $\boldsymbol{\beta}$ with center O. Any point K on a ray which passes through O determines the sphere of radius KO. The pedal points of the perpendiculars constructed from K to each ray of the flat pencil in $\boldsymbol{\beta}$ lie on a circle $\mathbf{s}$. This circle is the line of intersection between the sphere $\mathbf{Q}$ and the plane $\boldsymbol{\beta}$.

The directing curve $\mathbf{q}$ of the directing cone $\mathbf{T}^n$ of the reflected rays surface is obtained by the dilatation from the center K with factor 2.

Theorem 4. For a right cylinder or a canal surface as reflecting surface the curve $\mathbf{q}$ is a circle and the directing cone of the reflected rays surface along a planar section $\mathbf{C}^2$ is a right cone $\mathbf{T}^2$.

Let's describe the reflected rays surface along any planar section $\mathbf{C}^2$ of the right cylinder $\boldsymbol{\Phi}^2$ (Fig. 5): The director lines of this surface $\boldsymbol{\Omega}$ are a curve of the second order $\mathbf{C}^2$, a straight double-line $\mathbf{m}$ and directing cone $\mathbf{T}^2$.

The equation of a planar section $\mathbf{C}^2$ of the right cylinder $\boldsymbol{\Phi}^2$ with the axis OZ is

$$\left\{\begin{array}{l} x^2+y^2=R^2 \\ \dfrac{x}{a}+\dfrac{z}{b}=1\,. \end{array}\right. \tag{9}$$

The direction S of incident rays is

$$\frac{x}{m} = \frac{y}{l} = \frac{z}{p}\,. \tag{10}$$

The equation of the directrix $\mathbf{m}$ of the reflected rays surface is

$$\frac{x}{m} = \frac{z-b}{p}\,. \tag{11}$$

The direction S is specified parallel to the plane XOY ($l = 0$). Let the apex of the directing cone $\mathbf{T}^2$ coincide with the origin. One of its two contour generators coincides with the direction S. The equation of the directing cone $\mathbf{T}^2$ will be

$$\frac{x^2}{m^2} + \frac{y^2}{m^2} - \frac{z^2}{p^2} = 0\,. \tag{12}$$

The relationship between a point A on the planar section $\mathbf{C}^2$ of the reflecting cylinder and point A' on the base of the directing cone $\mathbf{T}^2$ is the following (Fig. 5):

From the point A the generator is drawn down to point B on the base circle of the cylinder. Points A and B lie on the same vertical, hence

$$\frac{x}{x_B} = \frac{y}{y_B}\,; \quad x_B^2 + y_B^2 = R^2. \tag{13}$$

As a result of substitutions and transformations the equation of the ray OB will be

$$y = kx \quad \text{with} \quad k = \frac{y_B}{\sqrt{R^2 - y_B^2}}\,. \tag{14}$$

The equation of the ray passing through a point K' and perpendicular to the straight line OB is

$$y = -\frac{1}{k}\,x + b\,. \tag{15}$$

And taking into account that the point K' has coordinates $x = R$ and $y = 0$, we obtain

$$y = -\frac{1}{k}\,x + \frac{R}{k}\,. \tag{16}$$

The coordinates of the point A' from the equation of the cylinder's basis circle are substituted in the equation (16)

$$x_{A'} = -\sqrt{R^2 - y_{A'}^2}\,, \tag{17}$$

and we obtain

$$y_{A'} = \frac{1}{k}\,\sqrt{R^2 - y_{A'}^2} + \frac{R}{k}\,. \tag{18}$$

After substitutions and transformations we end up with

$$y_{A'}^2\left(1 + \frac{1}{k}\right) - 2y_{A'}\,\frac{R}{k} = 0\,. \tag{19}$$

At the first solution $x = 0$ and $y = R$ the straight line meets the circle at the point K'. The second solution corresponds to the point A' and is

$$x_{A'} = \frac{R(1-k^2)}{1+k^2}\,, \quad y_{A'} = \frac{2Rk}{1+k^2} \quad \text{with} \quad k = \frac{y_B}{\sqrt{R^2 - y_B^2}}\,. \tag{20}$$

In a point A' the equation of the directing cone generating $\mathbf{T}^2$ is the following:

$$\frac{x}{x_{A'}} = \frac{y}{y_{A'}} = \frac{z}{z_{A'}} \quad \text{with} \quad z_{A'} = R\,\frac{m}{p}\,. \tag{21}$$

After substitutions and tranformations the equation of the directing cone $\mathbf{T}^2$ reads:

$$z = \frac{m \cdot (1+k^2)}{p \cdot (1-k^2)} \cdot x\,, \qquad z = \frac{m \cdot (1+k^2)}{2k \cdot p} \cdot y\,. \tag{22}$$

The generator of the reflected rays surface will pass through the point A and be parallel to the directing cone generator at the point A', and its equation will be:

$$z - z_A = a(x - x_A), \quad z - z_A = b(y - y_A) \quad \text{with} \quad a = \frac{m \cdot (1+k^2)}{p \cdot (1-k^2)}\,;\; b = \frac{m \cdot (1+k^2)}{2k \cdot p}\,. \tag{23}$$

If the reflecting surface is a sphere, torus or canal surface, the algorithm is applicable for the description of the reflected rays surface along a circle on any listed surface.

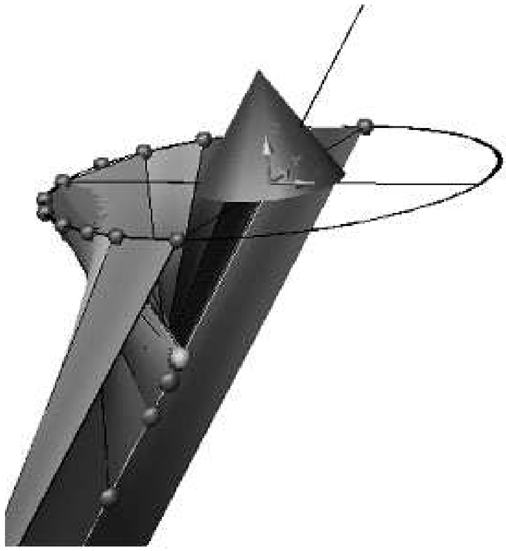

Figure 6: Visualization of the reflected rays surface $\mathbf{\Omega}$ along a meridian circle of a torus

4. Reflected rays surface along planar sections of a right cone

Let's consider the construction of the reflected rays surface $\mathbf{\Omega}$ along a planar section $\mathbf{C}^2$ of a right cone (Fig. 7). In this case the directing cone of normals is also a cone of revolution. The reflected rays surface $\mathbf{\Omega}$ along $\mathbf{C}^2$ is constructed by the following algorithm:

1. For determining the relationship between the point A on the section $\mathbf{C}^2$ and the point A' on the curve $\mathbf{q}$ the generator of the reflected rays surface $\mathbf{\Omega}$ is constructed through the point A and corresponding to the generator $\mathbf{l}$ on the cone of normals $\mathbf{\Psi}^2$.
2. The generator $\mathbf{l}'$ on the cone $\mathbf{T}'$ is parallel to the generator $\mathbf{l}$ of cone $\mathbf{\Psi}^2$, and the point A' of intersection between the generator $\mathbf{l}'$ and the curve $\mathbf{q}$ on the sphere $\mathbf{Q}$ is constructed.
3. Thus the generator TA' on the directing cone of the reflected rays surface $\mathbf{\Omega}$ is determined. The generator of the reflected rays surface $\mathbf{\Omega}$ passing through point A is parallel to TA'. This generator meets the curve $\mathbf{C}^2$ and is parallel to the directing cone $\mathbf{T}^4$.

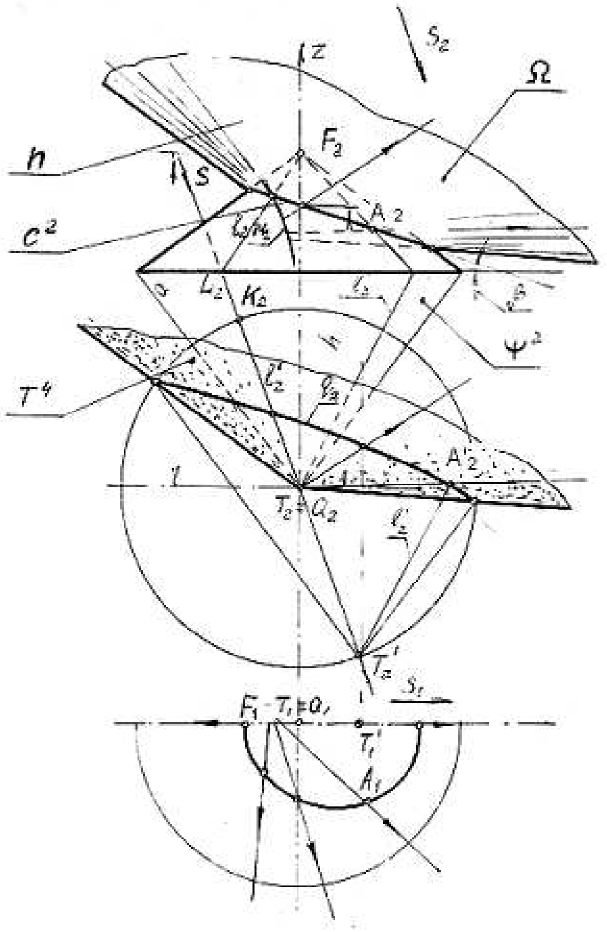

Figure 7: The reflected rays surface $\mathbf{\Omega}$ along a planar section $\mathbf{C}^2$ of the directing cone

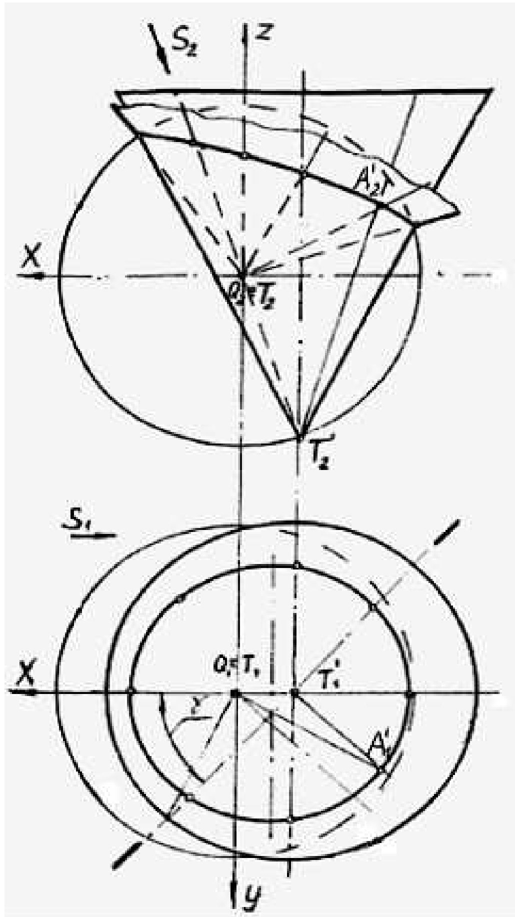

Figure 8: The relationship between the point A on the reflecting surface and the point A' on the directing cone of surface $\mathbf{\Omega}$.

To determine the relationship between the point A on the reflecting surface, a right cone, and the point A' on the directing of the reflected rays surface $\mathbf{\Omega}$ we can use as a parameter the angle γ of the generator $\mathbf{l}$ varying on the reflecting surface and simultaneously of the generator $\mathbf{l}'$ on the directing cone of normals (Fig. 8).

The generator $\mathbf{l}'$ belongs to the plane $y = \tan\gamma(x+m)$ which passes through the axis of the cone $\mathbf{T}'$

$$\frac{(x+m)^2}{1} + \frac{y^2}{1} - \frac{(z+p)^2}{k^2} = 0 \tag{24}$$

and cuts the cone along two generators

$$\begin{aligned}(x+m)\cdot k &= (z+p)\cdot\cos\gamma\,,\\ (x+m)\cdot k &= -(z+p)\cdot\cos\gamma\,.\end{aligned} \tag{25}$$

Lets consider one of the two generators

$$z = \frac{x+m)\cdot k}{\cos\gamma} - p \tag{26}$$

and put it into equation of the unit sphere

$$x^2 + y^2 + z^2 = 1 \tag{27}$$

and taking into account the equation of the plane. After transformation the coordinates of the point A' of intersection between the cone generator $\mathbf{l}'$ and the sphere are:

$$x_{A'} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad \text{with} \quad \begin{array}{l} A = \cos\gamma + \cos\gamma \cdot \tan^2\gamma + k \\ B = 2\cos\gamma \cdot \tan^2\gamma \cdot m + 2m \cdot k - 2p \cdot k \\ C = \cos\gamma \cdot m \cdot (\tan^2\gamma - m) + k \cdot (m - 2p) \end{array} \tag{28}$$

and $m^2 + p^2 = 1$. Then from Eq. (26)

$$z_{A'} = \frac{(x_{A'} + m) \cdot k}{\cos\gamma} - p\,. \tag{29}$$

The equation of the generator TA' of the directing cone $\mathbf{T}^2$ of the reflected rays that passes through the origin and the point A' is

$$x = \frac{x_{A'}}{z_{A'}} \cdot z\,. \tag{30}$$

The equation of the generator of the reflected rays surface $\mathbf{\Omega}$ that is parallel to the generator TA' of the directing cone $\mathbf{T}^2$ is

$$x - x_A = \frac{x_{A'}}{z_{A'}}(z - z_A). \tag{31}$$

Coordinates of the reflecting point A are computed by intersecting the section $\mathbf{C}^2$ with the plane with parameter γ. The equation of the reflecting cone is

$$\frac{x^2}{k^2} + \frac{y^2}{k^2} - \frac{(z-a)^2}{1} = 0 \quad \text{with} \quad k = \tan\alpha\,. \tag{32}$$

The equation of the axial plane is

$$y = \tan\gamma \cdot x\,; \tag{33}$$

that of the section plane is

$$z = x \cdot \tan\alpha + h. \tag{34}$$

The common solution of three last equations gives the coordinates of the reflecting point A.

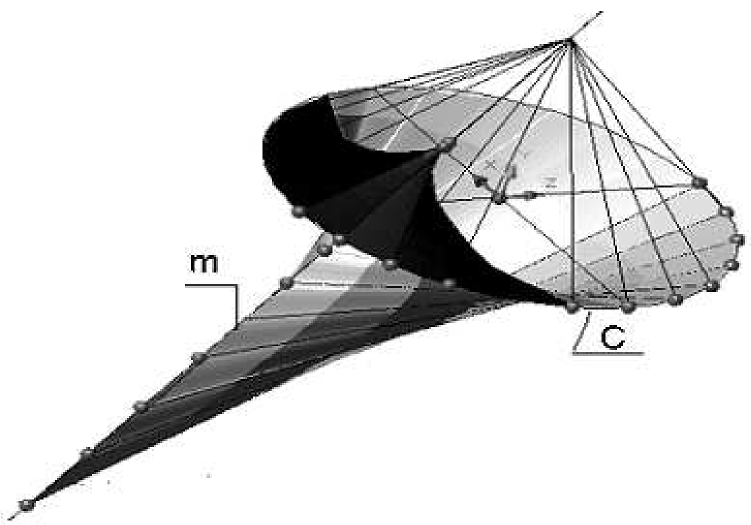

Figure 9: The reflected rays surface $\mathbf{\Omega}$ along a parallel circle $\mathbf{C}$ of any surface of revolution

5. Conclusion and future work

The proposed algorithms using the directing cone give us the possibility to describe the reflected rays surfaces along planar section of the reflecting surfaces analytically and synthetically. It was proved that the reflected rays surface along planar section of the reflecting surface is a surface with three directrices:

- $\mathbf{C}^n$, the section of the reflecting surface;
- $\mathbf{T}^n$, the directing cone;
- $\mathbf{m}$, a double curve.

Two main tasks for application of the reflection remain for future work:

- Determining zones with maximum concentration of the reflected. The search of a zone with greatest concentration of the rays, reflected by the surface, is of practical interest and can be used in designing solar installations for converting solar energy into thermal or electrical energy as well as for the design of reflecting surfaces of lamps. That is why the quasifocal line theory will be proposed.
- Determining zones where the reflected flow is absent at all. The ways of constructing *carst zones* where there are no reflected rays at all will be proposed.

References

[1] A.T. Dvoretsky, A. Kuryatov: *Installation on Discrete Stationary Solar Concentrator with Tubular Collectors for IPH Generation up to 130° C.* 8th Internat. Symposium on Solar Thermal Concentrating Technologies, Köln, Germany, 1996, vol. 2, pp. 911–916.

[2] A.T. Dvoretsky: *Geometrical models of solar energy reflected flows.* Proc. 10th ICGG, Kyiv 2002, vol. 1, pp. 222–225.

[3] A.T. Dvoretsky: *The Special Points of Reflected Rays Surfaces.* Applied Geometry and Engineering Graphics, Kyiv, **70**, 208–213 (2002).

[4] A.T. Dvoretsky: *Solar Thermal Concentrating Technologies in Installations with Stationary Concentrators.* The XIIth SolarPACES International Symposium on Concentrating Solar Power and Chemical Energy Systems, Seville, Spain, 2006.

[5] A.T. Dvoretsky: *Apparatus of Secondary Reflection.* Proc. 13th ICGG, Dresden, Germany, p. 81, 2008.

[6] G. Glaeser: *Reflections on Spheres and Cylinders of Revolution.* J. Geometry Graphics **3**, 121–139 (1999).

[7] O.L. Pidgorny: *Constructive Modelling of Bundles Chains and Congruences of Normals as Result of Reflections and Refractions.* Proc. 10th ICGG, Kyiv 2002, vol. 1, pp. 13–17.

[8] А.Л. Подгорный: Поверхности отраженных личей. Прикладная геометрия инженерная графика, Киев, “Будiвельник”, **20**, 13–16 (1975).

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