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Rose Surfaces and their Visualizations

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Abstract. In this paper we construct a new class of algebraic surfaces in three-dimensional Euclidean space that are generated by roses. We derive their parametric and implicit equations, investigate their singularities and visualize them with the program *Mathematica*.

Key Words: Rose curve, Rose surface, Singular point

MSC 2000: 51N20, 51M15, 14J25, 14J17

1. Introduction

In [2], by using an $(n+2)$-degree inversion defined in [1], we elaborated the pedal surfaces of special first order line congruences. The directing lines of these congruences are roses given by the polar equation $r = \cos n\varphi$, where n is an odd positive integer. The cases with special positions of the pole appeared to be very interesting and led us to explore a new construction of surfaces where the generating curve was a rose with a finite number of petals. The resulting surfaces had various attractive shapes, a small number of high singularities and were convenient for algebraic treatment and visualization in the program *Mathematica*. Some special examples of these surfaces are given in [9] and another attempt to generalize roses is given in [10].

2. Roses

Roses or *rhodonea* curves $R(n,d)$, treated here, can be expressed by the following polar equation:

$$r = \cos \frac{n}{d}\,\varphi, \tag{1}$$

where $\frac{n}{d}$ is a positive rational number in reduced form, i.e. $GCD(n,d) = 1$.

According to [13] and [7], these curves are particular trochoids: epitrochoids for $n > d$ and hypotrochoids for $n < d$. They are also a special type of cyclic-harmonic curves (foliate cyclic-harmonic curves), [8], [5].

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If $n \cdot d$ is *odd*, the curves close at polar angles $d \cdot \pi$ and have n petals. They are algebraic curves of the order $n + d$, with an n-fold singularity in the origin and with $\frac{1}{2}n(d-1)$ double points. If $n \cdot d$ is *even*, the curves close at polar angles $2d \cdot \pi$ and have $2n$ petals. They are algebraic curves of the order $2(n + d)$, with a $2n$-fold singularity in the origin and with $2n(d-1)$ double points [6, pp. 358–369], [11], [12] (see Table 1).

Table 1: Properties of $R(n, d)$

$n \cdot d$	*order*	*multiplicity of the point O*	*number of double points*	*period*	*number of petals*
odd	$n + d$	n	$\frac{1}{2}n(d-1)$	$d \cdot \pi$	n
even	$2(n + d)$	$2n$	$2n(d-1)$	$2d \cdot \pi$	$2n$

According to [6] we can derive the following implicit equation of $R(n, d)$:

$$\Big(\sum_{k=0}^{\lfloor d/2 \rfloor} \sum_{j=0}^{k} (-1)^{k+j} \binom{d}{2k} \binom{k}{j} (x^2 + y^2)^{\frac{n+d}{2} - k + j}\Big)^s - \Big(\sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n}{2i} x^{n-2i} y^{2i}\Big)^s = 0, \qquad (2)$$

where $s = 1$ if $n \cdot d$ is odd and $s = 2$ if $n \cdot d$ is even. According to [4, p. 251][1], the tangent lines at the origin are given by the following equations:

– if $n \cdot d$ is odd

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k} = 0, \qquad (3)$$

– if n is even

$$\Big(\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}\Big)^2 = 0, \qquad (4)$$

– if d is even ($\lfloor \frac{d}{2} \rfloor = \frac{d}{2}$)

$$(x^2 + y^2)^n - \Big(\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}\Big)^2 = 0. \qquad (5)$$

3. Rose Surfaces

Definition 1. Let $P(0, 0, p)$ be any point on the axis z and let $R(n, d)$ be a rose given by Eq. (1) in the plane $z = 0$. A *rose surface* $\mathcal{R}(n, d, p)$ is the system of circles c_i which lie in the planes ζ through the axis z and have diameters $\overline{PR_i}$, where $R_i \neq O$ are the intersection points of the rose $R(n, d)$ and the plane ζ (see Fig. 4).

[1] See the quotation that follows in the proof of Theorem 1.

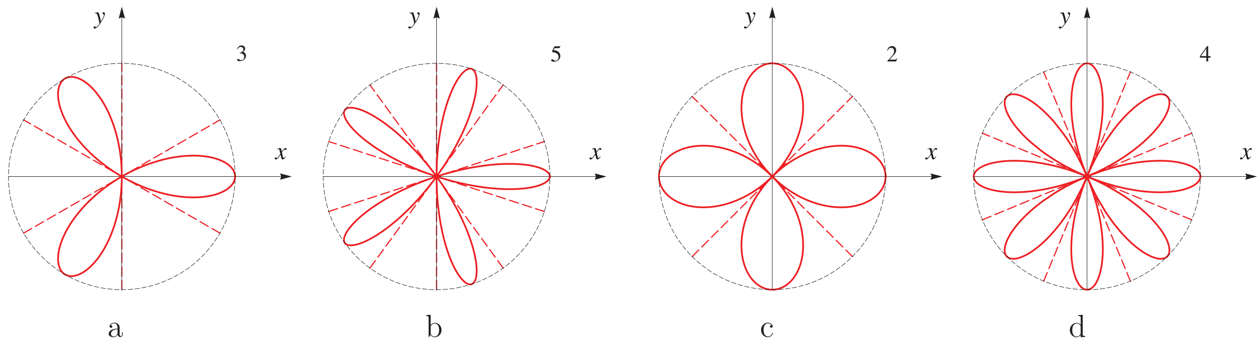


Figure 1: If n is odd, the rose $R(n, 1)$ is an n-petalled curve with n tangent lines at the origin (Figs. a and b). If n is even, the rose $R(n, 1)$ is an $2n$-petalled curve with n double tangent lines at the origin (Figs. c and d).

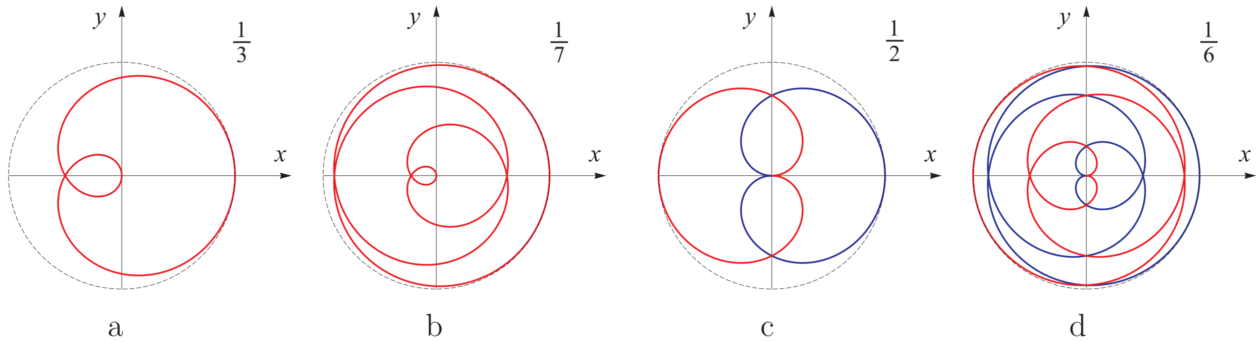


Figure 2: If d is odd, the rose $R(1, d)$ has only one petal (Figs. a and b). If d is even, the rose $R(1, d)$ has two petals (Figs. c and d).

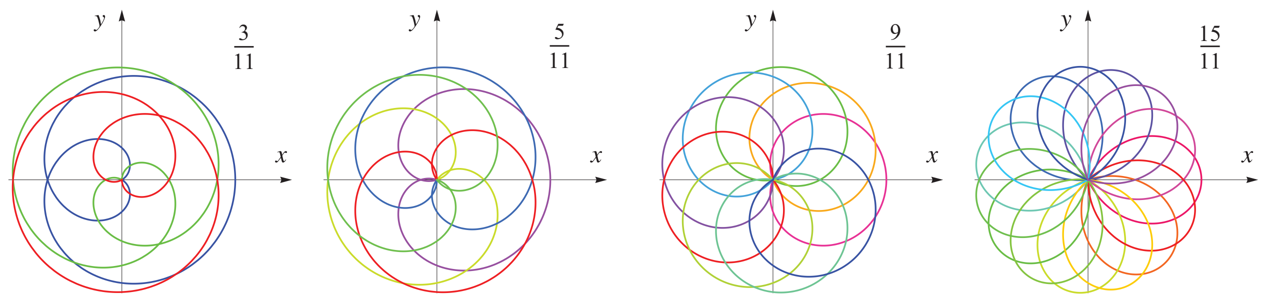


Figure 3: Four roses with petals in different colors.

3.1. Parametric equations of $\mathcal{R}(n, d, p)$

Let φ be the angle between the planes $\zeta(\varphi)$ and $y = 0$. The parametric equations of the circle c with the diameter $\overline{PR}$ in the plane $\zeta(\varphi)$ are the following:

$$\begin{aligned} r &= \frac{1}{2}\left(\cos\frac{n}{d}\varphi + \sqrt{p^2 + \cos^2\frac{n}{d}\varphi}\,\sin\theta\right), \\ z &= \frac{1}{2}\left(p + \sqrt{p^2 + \cos^2\frac{n}{d}\varphi}\,\cos\theta\right), \end{aligned} \qquad \theta \in [0, 2\pi), \tag{6}$$

where $\varphi \in [0, d \cdot \pi)$ if $n \cdot d$ is odd, and $\varphi \in [0, 2d \cdot \pi)$ if $n \cdot d$ is even.

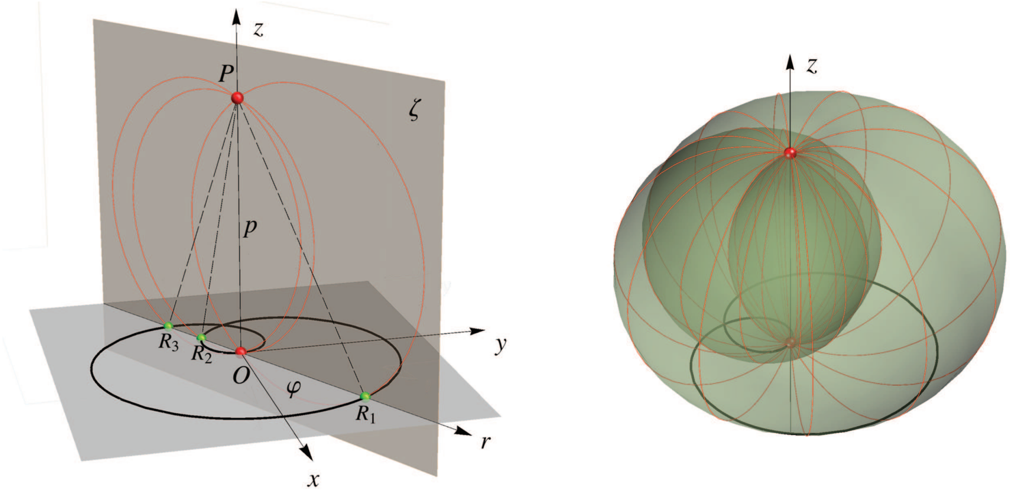


Figure 4: If $n \cdot d$ is odd or even, the number of points $R_i \in \zeta$ is d or $2d$, respectively.

Therefore, the parametric equations of the rose surface $\mathcal{R}(n, d, p)$ are the following:

$$\begin{aligned} x &= \frac{1}{2}\cos\varphi(\cos\frac{n}{d}\,\varphi + \sqrt{p^2+\cos^2\frac{n}{d}\,\varphi}\,\sin\theta) \\ y &= \frac{1}{2}\sin\varphi(\cos\frac{n}{d}\,\varphi + \sqrt{p^2+\cos^2\frac{n}{d}\,\varphi}\,\sin\theta) \\ z &= \frac{1}{2}(p + \sqrt{p^2+\cos^2\frac{n}{d}\,\varphi}\,\cos\theta), \end{aligned} \tag{7}$$

where $(\varphi,\theta) \in [0, d\cdot\pi) \times [0, 2\pi)$ if $n \cdot d$ is odd, and $(\varphi,\theta)) \in [0, 2d\cdot\pi) \times [0, 2\pi)$ if $n\cdot d$ is even. For *Mathematica* the equations (7) allow visualizations of the surfaces $\mathcal{R}(n, d, p)$, see [3].

3.2. Implicit equations of $\mathcal{R}(n, d, p)$

From the identity $\cos d\frac{n}{d}\,\varphi = \cos n\varphi$ and the multiple angle formula $\cos n\varphi = \sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\binom{n}{2i}(\sin\varphi)^{2i}(\cos\varphi)^{n-2i}$ we obtain

$$\sum_{k=0}^{\lfloor d/2\rfloor}(-1)^k\binom{d}{2k}\left(\sin\frac{n}{d}\,\varphi\right)^{2k}\left(\cos\frac{n}{d}\,\varphi\right)^{d-2k} = \sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\binom{n}{2i}(\sin\varphi)^{2i}(\cos\varphi)^{n-2i}. \tag{8}$$

Since the implicite equation of the circle c in the plane $\zeta(\varphi)$ is

$$\left(r - \frac{\cos\frac{n}{d}\varphi}{2}\right)^2 + \left(z - \frac{p}{2}\right)^2 = \frac{1}{4}\,(\cos^2\frac{n}{d}\varphi + p^2), \tag{9}$$

by substituting $r = \sqrt{x^2+y^2}$ in (9), we obtain the following conditions for the points of $\mathcal{R}(n, d, p)$:

$$\cos\frac{n}{d}\,\varphi = \frac{x^2+y^2+z^2-p\cdot z}{\sqrt{x^2+y^2}}, \quad \sin\frac{n}{d}\,\varphi = \sqrt{1-\frac{(x^2+y^2+z^2-p\cdot z)^2}{x^2+y^2}}. \tag{10}$$

By substituting (10), $\cos\varphi = \dfrac{x}{\sqrt{x^2+y^2}}$, and $\sin\varphi = \dfrac{y}{\sqrt{x^2+y^2}}$ into (8), we obtain the following algebraic equations of $\mathcal{R}(n,d,p)$:

$$\begin{aligned}&(x^2+y^2)^{\frac{s(n-d)}{2}}\Big(\sum_{k=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{j=0}^{k}(-1)^{k+j}\binom{d}{2k}\binom{k}{j}(x^2+y^2+z^2-p\cdot z)^{d-2(k-j)}(x^2+y^2)^{k-j}\Big)^s\\ &=\Big(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}\Big)^s \qquad \dots \text{ for } n>d\,,\end{aligned} \tag{11}$$

$$\begin{aligned}&\Big(\sum_{k=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{j=0}^{k}(-1)^{k+j}\binom{d}{2k}\binom{k}{j}(x^2+y^2+z^2-p\cdot z)^{d-2(k-j)}(x^2+y^2)^{k-j}\Big)^s\\ &=(x^2+y^2)^{\frac{s(d-n)}{2}}\Big(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}\Big)^s \qquad \dots \text{ for } n<d\,,\end{aligned} \tag{12}$$

where $s=1$ if $n\cdot d$ is odd, and $s=2$ if $n\cdot d$ is even.

3.3. Properties of $\mathcal{R}(n,d,p)$

Theorem 1. *For rose surfaces $\mathcal{R}(n,d,p)$, the following table is valid:*

Table 2: Properties of $\mathcal{R}(n,d,p)$.

$n\cdot d$		*order*	*multiplicity of points O and P*	*multiplicity of the axis z*	*number of double circles in the planes ζ*
odd	$n>d$	$n+d$	n	$(n-d)$	$\frac{1}{2}n(d-1)$
even	$n>d$	$2(n+d)$	$2n$	$2(n-d)$	$n(2d-1)$
odd	$n<d$	$2d$	d	0	$\frac{1}{2}n(d-1)$
even	$n<d$	$4d$	$2d$	0	$n(2d-1)$
		A	**B**	**C**	**D**

Proof: Ad **A**: The order of an algebraic surface is equal to the degree of its algebraic equation. In Eqs. (11) and (12) the terms with the highest exponents (for $k=j$) are

$$\Big(2^{d-1}(x^2+y^2)^{\frac{n-d}{2}}(x^2+y^2+z^2)^d\Big)^s \quad \text{and}$$

$$\Big(2^{d-1}(x^2+y^2+z^2)^d\Big)^s-(x^2+y^2)^{\frac{s(d-n)}{2}}\Big(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}\Big)^s,$$

respectively.

Ad **B**: According to [4, p. 251]: If an n-th order surface in $\mathbb{E}^3$ which passes through the origin is given by the equation

$$F(x,z,y)=f_m(x,y,z)+f_{m+1}(x,y,z)+\cdots+f_n(x,y,z)=0,$$

where $f_k(x,y,z)$ $(1 \leq k \leq n)$ are homogeneous polynomials of degree k, then the tangent cone at the origin is given by the equation $f_m(x,y,z) = 0$.

Therefore, the tangent cones of $\mathcal{R}(n,d,p)$ given by Eqs. (11) and (12) at the points O and P are given by the following equations:

$$\begin{aligned}&\Big(\sum_{k=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{j=0}^{k}(-1)^{k+j}\binom{d}{2k}\binom{k}{j}(\mp p\cdot z)^{d-2(k-j)}(x^2+y^2)^{\frac{n-d+2(k-j)}{2}}\Big)^s\\&-\Big(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}\Big)^s=0\,,\end{aligned}\tag{13}$$

$$\begin{aligned}&\Big(\sum_{k=0}^{\lfloor\frac{d}{2}\rfloor}\sum_{j=0}^{k}(-1)^{k+j}\binom{d}{2k}\binom{k}{j}(\mp p\cdot z)^{d-2(k-j)}(x^2+y^2)^{k-j}\Big)^s\\&-\Big(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}(x^2+y^2)^{\frac{d-n}{2}}\Big)^s=0\,,\end{aligned}\tag{14}$$

respectively.

In these equations, $-p\cdot d$ corresponds with the point O, and $+p\cdot d$ with the point P as the origin.

Ad **C**: If $\mathcal{R}(n,d,p)$ is given by Eq. (11), any point $Z_0(0,0,z_0)$ lies on the surface and the tangent cone at Z_0, with the origin translated into Z_0, is given by the following equation:

$$(x^2+y^2)^{\frac{s(n-d)}{2}} = 0. \tag{15}$$

This equation represents the $\frac{s(n-d)}{2}$-fold pair of isotropic planes through the axis z.

If $\mathcal{R}(n,d,p)$ is given by Eq. (12), it is clear that a point $Z_0(0,0,z_0)$ on the axis z lies on $\mathcal{R}(n,d,p)$ iff $z_0^2 - p\cdot z_0 = 0$, i.e., $Z_0 = O$ or $Z_0 = P$.

Ad **D**: For $z=0$, Eq. (11) takes the form (2), while Eq. (12) also gives Eq. (2) but multiplied by $(x^2+y^2)^{\frac{s(d-n)}{2}}$. It means that the intersection of $\mathcal{R}(n,d,p)$ and the plane $z=0$ is $R(n,d)$ for $n>d$, while it splits on $R(n,d)$ and $\frac{s(d-n)}{2}$-fold isotropic lines through the origin for $n<d$. The circle c in the plane ζ is the double curve of $\mathcal{R}(n,d,p)$ iff the intersection point of ζ and $R(n,d)$ is a double point of $R(n,d)$. Thus, the number of double circles on $\mathcal{R}(n,d,p)$ is equal to the number of double points of $R(n,d)$ if $n\cdot d$ is odd. But, if $n\cdot d$ is even, other n double circles in the planes ζ exist on $\mathcal{R}(n,d,p)$. These circles lie in the planes through the double tangent lines of $R(n,d)$ at O and their diameters are $\overline{OP}$. If $O=P$, these circles degenerate into the pairs of isotropic lines. □

Corollary 1. *If $p=0$, the tangent cone of $\mathcal{R}(n,d,p)$ at $O=P$ splits into n or d planes.*

Proof: If $p=0$, Eqs. (13) and (14) take the following forms:

$$\left(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}\right)^s = 0, \tag{16}$$

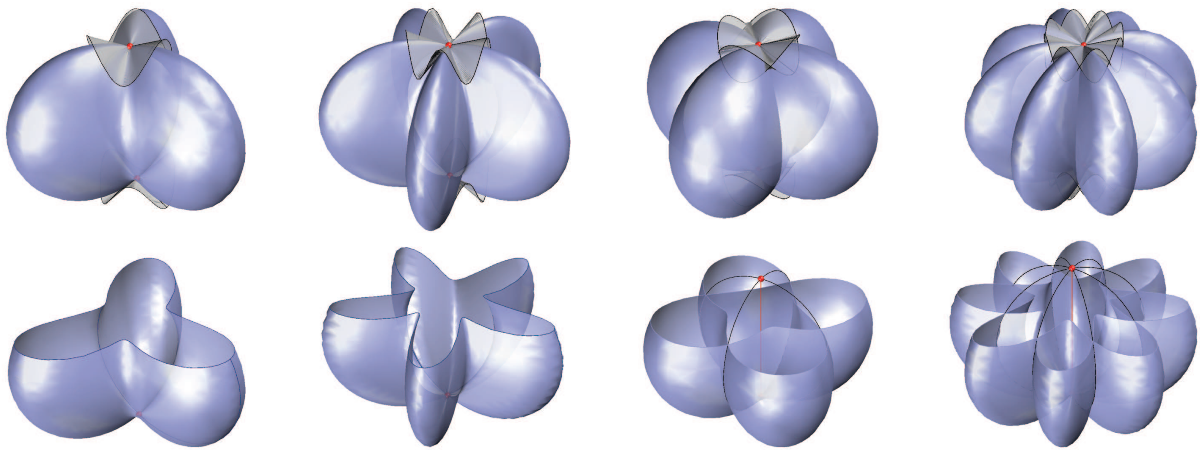
$$\left(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^i\binom{n}{2i}x^{n-2i}y^{2i}(x^2+y^2)^{\frac{d-n}{2}}\right)^s = 0, \tag{17}$$

respectively.

Since the polynomials in these equations are n-th (Eq. (16)) or d-th (Eq. (17)) degree homogeneous in x and y, therefore they can be reduced to linear and quadratic factors. These factors equal to 0 represent n or d planes (real or imaginary) through the axis z. □

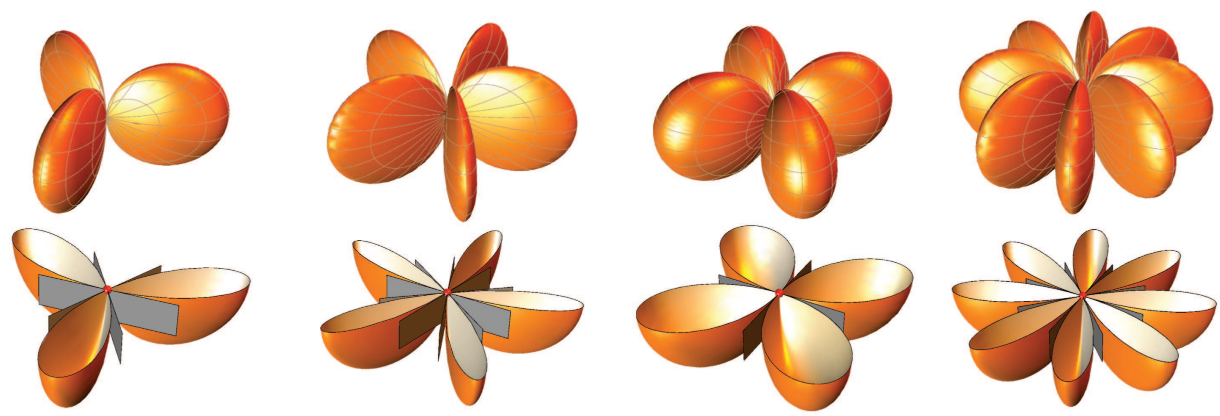
3.4. Visualizations of $\mathcal{R}(n, d, p)$

The following figures are computed and plotted by the software *Mathematica*.



$\mathcal{R}$(3,1,2) $\mathcal{R}$(5,1,2) $\mathcal{R}$(2,1,2) $\mathcal{R}$(4,1,2)

Figure 5: If $d = 1$ and $p \neq 0$, the tangent cones at the points O and P are proper cones.
If n is odd, there are no double circles on $\mathcal{R}(n, 1, p)$;
for even n, $2n$ double circles exist on $\mathcal{R}(n, 1, p)$.



$\mathcal{R}$(3,1,0) $\mathcal{R}$(5,1,0) $\mathcal{R}$(2,1,0) $\mathcal{R}$(4,1,0)

Figure 6: If $d = 1$ and $p = 0$, the tangent cones at O split into n planes.
If n is even, these planes are the double tangent planes of $\mathcal{R}(n, 1, 0)$.

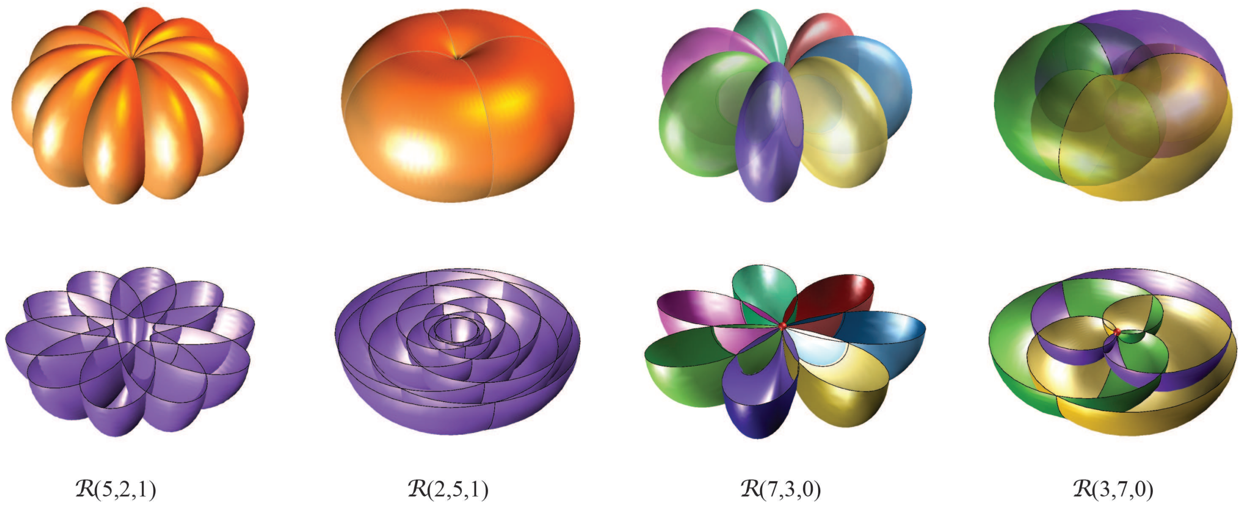


Figure 7: Four rose surfaces.

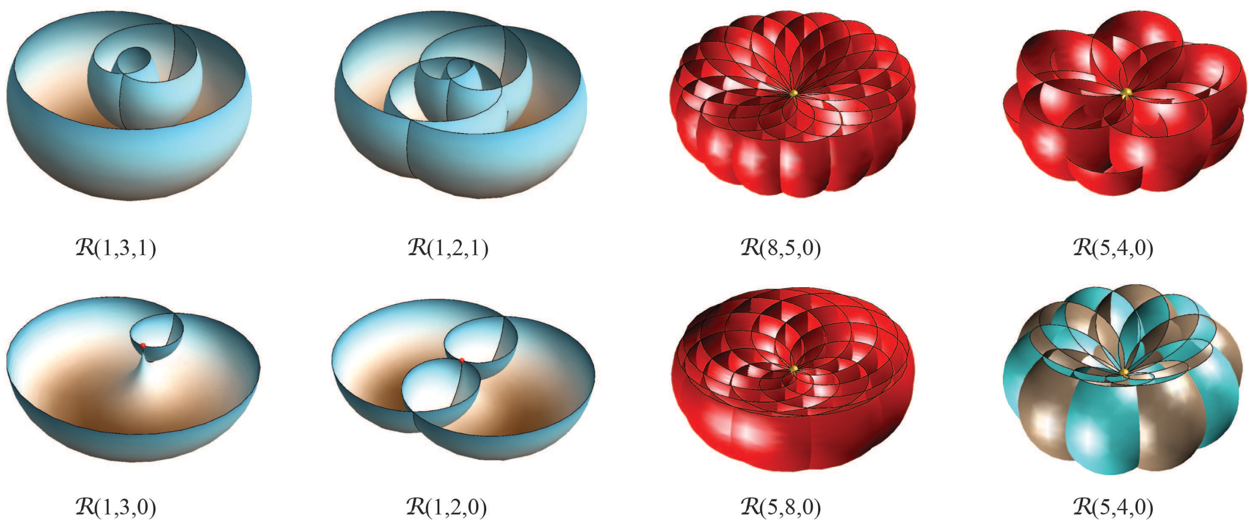


Figure 8: Seven rose surfaces.

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