Dilation-Induced Perspectivities among Triangles

Clark Kimberling¹, Peter J. C. Moses²

¹Dept. of Mathematics, University of Evansville
1800 Lincoln Avenue, Evansville, IN 47722, USA
email: ck6@evansville.edu

²Moparmatic Co., 1154 Evesham Road, Astwood Bank
Redditch, Worcestershire B96 6DT, England
email: mows@mopar.freeserve.co.uk

Abstract. In the plane of a triangle $ABC$, let $DEF$ be a triangle and $U$ a point. Except for special cases, there are at most two nontrivial $U$-dilations of $DEF$ that are perspective to $ABC$. Of particular interest are configurations in which $DEF$ is perspective to $ABC$, as when $DEF$ is a cevian or anticevian triangle. Midcevian and mid-isotomic triangles are introduced. Many loci associated with perspectivities among these triangles are $Z$ cubics.

Key Words: perspective triangles, cevian triangles, anticevian triangles, midcevian triangles, mid-isotomic triangles

MSC 2000: 51M04

1. Introduction

In the plane of a fixed triangle $ABC$, we shall use the notation $DEF$ for a triangle and $U$ and $X$ for points. Certain assumptions and conventions will remain in effect throughout:

(A1) $A, B, C, D, E, F, U$ are distinct points;

(A2) coordinates are homogeneous barycentric coordinates (henceforth simply barycentrics) relative to triangle $ABC$;

(A3) the phrase “triangles $D_tE_tF_t$ and $ABC$ are perspective with perspector $U$” means that the lines $AD_t$, $BE_t$, $CF_t$ concur in $U$.

Regarding (A3), five other possible perspectivities, such as the concurrence of lines $AE_t$, $BF_t$, $CD_t$, are not considered; accordingly, it is helpful to think of $D_t$, $E_t$, $F_t$ as a labeled triangle; viz., $D_t$ is the $A$-vertex, $E_t$ the $B$-vertex, and $F_t$ the $C$-vertex. For example, if $DEF$ is the cevian triangle of a point $X = x : y : z$, then

$$D = 0 : y : z, \quad E = x : 0 : z, \quad F = x : y : 0;$$
if \(DEF\) is the anticevian triangle of \(X\), then
\[
D = -x : y : z, \quad E = x : -y : z, \quad F = x : y : -z.
\]

A point \(X = x : y : z\) is a \textit{finite point} if \(x + y + z \neq 0\), and a \textit{regular point} if \(xyz \neq 0\); i.e., if \(X\) does not lie a sideline \(BC, CA,\) or \(AB\) of \(ABC\). These and other introductory features of triangle geometry are presented in [6] and [11].

The notation \(\sum\) will always mean a cyclic sum; e.g., \(\sum a = a + b + c\); and \(\sum p(vq\beta - wr\gamma)a^2\) means
\[
p(vq\beta - wr\gamma)a^2 + q(wr\gamma - up\alpha)\beta^2 + r(up\alpha - vq\beta)\gamma^2.
\]

Notation of the form \(X_i\) refers to triangle centers as indexed in [7]. The first four \(X_i\) all appear in the sequel and are given by
\[
X_1 = a : b : c = \text{incenter} \quad X_2 = 1 : 1 : 1 = \text{centroid} \quad X_3 = a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2) = \text{circumcenter} \quad X_4 = 1/(b^2 + c^2 - a^2) : 1/(c^2 + a^2 - b^2) : 1/(a^2 + b^2 - c^2) = \text{orthocenter}.
\]

As a final introductory note, we mention that this article is a sequel to [10].

2. Dilation

For given \(U = u : v : w\) and \(X = x : y : z\), the point \(D_t\) given by
\[
D_t = tkx + (1-t)hu : tky + (1-t)hv : tkz + (1-t)hw,
\]
where
\[
h = x + y + z \quad \text{and} \quad k = u + v + w,
\]
is the \textit{t-dilation} of \(X\) from \(U\). Note that \(D_0 = U\) and \(D_1 = X\). For numerical values of \(a, b, c\), the parameter \(t\) in (1) is merely a real number, but when \(a, b, c\) are regarded as indeterminates or variables, the parameter \(t\) is a 0-degree homogeneous symmetric function in \(a, b, c\); that is, for \(q\) an indeterminate, we have
\[
t(qa, qb, qc) = t(a, b, c); \quad t(a, b, c) = t(b, c, a) = t(a, c, b).
\]

Taking \(D\) to be successively the points \(D, E, F\), we obtain from (1) points \(D_t, E_t, F_t\). The triangle \(D_tE_tF_t\) is a \textit{U-dilation triangle of} \(DEF\), as typified by Fig. 1. Note that \(D_tE_tF_t\) is homothetic to \(DEF\) with homothetic ratio \(|UD_t|/|UD| = t\).

3. General result

What dilations of a triangle \(DEF\) from a point \(U\) are perspective to \(ABC\)? We shall see in Theorem 1 that much can be said even if the triangle \(DEF\) is not assumed to be a member of any particular class of triangles, such as cevian or anticevian. Those cases are considered in Sections 5 and 6.
Figure 1: $D_tE_tF_t$ is a $U$-dilation triangle of $DEF$

**Theorem 1.** Suppose that $A, B, C, D, E, F, U$ are distinct finite points, and let $n$ be the number of $U$-dilation triangles $D_tE_tF_t$ that are perspective to $ABC$. If $n$ is finite then $n \leq 2$.

More precisely, there exist conics $C_0, C_1, C_2$ such that

- (B1) if $U \notin C_2$, then $n \leq 2$;
- (B2) if $U \in C_2$ and $U \notin C_1$, then $n = 1$;
- (B3) if $U \in C_1 \cap C_2$ and $U \notin C_0$, then $n = 0$.

**Proof:** Write $U = u : v : w$, and, to facilitate application of a computer algebra system, assign symbols as follows:

$$D = d_1 : d_2 : d_3 \quad E = d_4 : d_5 : d_6 \quad F = d_7 : d_8 : d_9$$
$$D_t = u_1 : u_2 : u_3 \quad E_t = u_4 : u_5 : u_6 \quad F_t = u_7 : u_8 : u_9.$$ 

The line $AD_t$ is given by an equation $l_1\alpha + l_2\beta + l_3\gamma = 0$, where $\alpha : \beta : \gamma$ is a variable point. We represent the coefficients for this line and two others by

$$AD_t = l_1 : l_2 : l_3 \quad BE_t = l_4 : l_5 : l_6 \quad CF_t = l_7 : l_8 : l_9.$$ 

Let

$$h = u + v + w \quad k_1 = d_1 + d_2 + d_3 \quad k_2 = d_4 + d_5 + d_6 \quad k_3 = d_7 + d_8 + d_9,$$

so that the vertices of $\triangle D_tE_tF_t$ are given by

$$u_1 = td_1h + (1 - t)uk_1 \quad u_2 = td_3h + (1 - t)vk_1 \quad u_3 = td_3h + (1 - t)wk_1$$
$$u_4 = td_4h + (1 - t)uk_2 \quad u_5 = td_5h + (1 - t)vk_2 \quad u_6 = td_6h + (1 - t)wk_2$$
$$u_7 = td_7h + (1 - t)uk_3 \quad u_8 = td_9h + (1 - t)vk_3 \quad u_9 = td_9h + (1 - t)wk_3,$$

and coefficients for lines $AD_t, BE_t, CF_t$ by

$$l_1 = 0 \quad l_2 = -u_3 \quad l_3 = u_2$$
$$l_4 = u_6 \quad l_5 = 0 \quad l_6 = -u_4$$
$$l_7 = -u_8 \quad l_8 = u_7 \quad l_9 = 0.$$
The determinant
\[ \delta = \begin{vmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{vmatrix} \]
has the form
\[ t(u + v + w) \left[ r_0(u, v, w) + tr_1(u, v, w) + t^2r_2(u, v, w) \right], \]
where \( r_i(u, v, w) \) has degree 2 in \( u, v, w \), for \( i = 0, 1, 2 \). Let \( C_i \) be the conic \( r_i(\alpha, \beta, \gamma) = 0 \) for \( i = 0, 1, 2 \). The hypothesis that \( U \) is a finite point implies that \( t(u + v + w) \neq 0 \), so that the equation \( \delta = 0 \) is equivalent to
\[ r_0(u, v, w) + tr_1(u, v, w) + t^2r_2(u, v, w) = 0, \quad (2) \]
which is equivalent to the concurrence of the lines \( AD_t, BE_t, CF_t \), which is equivalent to the conditions (B1)–(B3). Specifically, if \( U \notin C_2 \), then (2) has at most two solutions \( t \), so that \( n \leq 2 \). If \( U \in C_2 \) and \( U \notin C_1 \), then (2) linear, so that \( n = 1 \). The only remaining possibility is (B3). (Of course, if \( n > 2 \), then \( D_tE_tF_t \) is perspective to \( ABC \) for all \( t \), contrary to the hypothesis of the theorem.)

The conic sections given by \( r_i(\alpha, \beta, \gamma) = 0 \) can be written out by substituting \((\alpha, \beta, \gamma)\) for \((u, v, w)\) and using the following conveniences. Let
\[
\begin{align*}
c_0 &= h^2(d_2d_6d_7 - d_3d_4d_8) \\
c_1 &= h(u(d_2d_6k_3 - d_3d_4k_2) + v(d_6d_7k_1 - d_3d_4k_3) + w(d_2d_7k_2 - d_3d_8k_1)) \\
c_2 &= uvk_1(d_7k_2 - d_4k_3) + wuk_2(d_2k_3 - d_8k_1) + uvk_3(d_8k_1 - d_4k_2) \\
Q &= t^2c_0 + t(1 - t)c_1 + (1 - t)^2c_2.
\end{align*}
\]
Then \( r_0(u, v, w) = c_2, \ r_1(u, v, w) = c_1 - 2c_2, \ r_2(u, v, w) = c_2 - c_1 + c_0, \) and (2) is equivalent to \( Q = 0 \). Consequently, if \( c_0 = 0 \), then \( d_2d_6d_7 = d_3d_4d_8 \) or \( h = 0 \); either the triangles \( ABC \) and \( DEF \) are perspective, so that \( n \) is infinite, contrary to the hypothesis of Theorem 1, or else, if \( h = 0 \), then \( U \) lies on the line at infinity, but this, too, is contrary to the hypothesis. Therefore \( c_0 \neq 0 \). If \( c_1 = 0 \), then \( U \) lies on the line
\[
(d_2d_6k_3 - d_3d_8k_2)\alpha + (d_6d_7k_1 - d_3d_4k_3)\beta + (d_2d_7k_2 - d_4d_8k_1)\gamma = 0,
\]
and if \( c_2 = 0 \), then \( U \) lies on the conic
\[
k_1(d_7k_2 - d_4k_3)\beta \gamma + k_2(d_2k_3 - d_8k_1)\gamma \alpha + k_3(d_6k_1 - d_3k_2)\alpha \beta = 0,
\]
which passes through the vertices \( A, B, C \). If all the barycentrics in Theorem 1 are normalized (so that each point \( x : y : z \) satisfies \( x + y + z = 1 \), then the expressions simplify considerably because \( h = k_1 = k_2 = k_3 = 1 \).

4. Perspective images of \( ABC \)

Suppose that \( DEF \) is perspective to \( ABC \), and let \( X = x : y : z \) denote the perspector, so that
\[
D = t_1 : y : z, \quad E = x : t_2 : z, \quad F = x : y : t_3,
\]
for some triple \((t_1, t_2, t_3)\). If \(a, b, c\) are numerical, then so are \(t_1, t_2, t_3\), whereas if \(a, b, c\) are indeterminates or variables, then \(t_1, t_2, t_3\) are functions of \((a, b, c)\) having the same degree of homogeneity as \(x, y, z\). We assume that \(t_1 \neq x, t_2 \neq y, t_3 \neq z\) and refer to \(DEF\) an \(X\)-perspective image of \(ABC\). Much studied examples of such \(DEF\) are cevian, anticevian, circumcevian, and circum-anticevian triangles. In the sequel, we shall also present examples which we call midcevian and mid-isotomic triangles.

**Theorem 2.** Suppose that \(DEF\) is an \(X\)-perspective image of \(ABC\) and that \(U\) is a finite point other than \(X\). Let \(D_tE_tF_t\) be the \(t\)-dilation of \(DEF\) from \(U\). Let

\[
M = \sum vw \left[ xt_1(z-y) + yt_2(z+y) - zt_3(z+y) - xt_1(t_2 + t_3) + x(z^2 - y^2) \right] ;
\]

\[
N = \sum [ u^2( yz(t_3 - t_2 - z + y) + vw x(yt_3 - zt_2 - t_1 t_3 + xy - xz)] .
\]

If \(MN \neq 0\), then \(D_tE_tF_t\) is perspective to \(ABC\) if and only if \(t \in \{0, 1, -M/N\}\). If \(t = -M/N\), the perspector is the point \(P(X, U) = p : q : r\) given by

\[
p = \frac{y - z - (t_2 - t_3)}{vz(x+y) - wy(x+z) - (wy t_2 - vz t_3)};
\]

\[
q = \frac{z - x - (t_3 - t_1)}{wx(y+z) - uz(y+x) - (uz t_3 - wx t_1)};
\]

\[
r = \frac{x - y - (t_1 - t_2)}{uy(z+x) - vx(z+y) - (vx t_1 - uy t_2)}.
\]

**Proof:** Referring to the proof of Theorem 1, we find that (2) has a nontrivial linear factor \(M + Nt\):

\[
\delta = t(1-t) (u + v + w) (M + Nt),
\]

Putting \(t = -M/N\), we find the perspector

\[
l_5 l_9 - l_3 l_8 : l_6 l_7 - l_4 l_9 : l_4 l_8 - l_5 l_7
\]

to be, after simplifications, as asserted.

In Fig 2, \(DEF\) is an \(X\)-perspective image of \(ABC\), as in Theorem 2. As \(D_t\) traces line \(DU\), the \(U\)-dilation \(D_tE_tF_t\) takes three positions for which it is perspective to \(ABC\) — that is, for which \(def\) is a single point. Of the three, two are trivial (when \(d = e = f = X\) and when \(d = e = f = U\)). The existence of the other position is established by Theorem 2. As \(D_t\) traces line \(DU\), each of the points \(d, e, f\) traces a conic (not shown), and the three conics meet in the three perspectors.

It will be convenient in the sequel to refer to \(D_tE_tF_t\), for \(t = -M/N \neq 0\), as the special triangle, special in the sense that it is the only triangle that is both homothetic to \(DEF\) and also perspective to \(ABC\).

We turn now to certain loci related to special triangles. A subclass of these loci are “self-isogonal circumcubics” — elegantly described in a context of linear algebra by H.S.M. Coxeter [1] — “self-isogonal” in the sense that if \(J\) is any regular point on such a cubic \(Z\), then the isogonal conjugate of \(J\) is on \(Z\) also; and “circum-” because \(Z\) passes through the vertices \(A, B, C\). The full class, in which “self-isogonal” is replaced by “self-\(P\)-isoconjugate,” consists of cubics each the locus of a point \(\alpha : \beta : \gamma\) satisfying an equation of the form

\[
\sum p(\nu q \beta - w r \gamma) \alpha^2 = 0
\]
for some points $P = p : q : r$ and $U = u : v : w$. This equation is used to define the cubic $Z(U, P)$ in [8] and [9], where the coordinates are homogeneous trilinears, and the same class of cubics is defined when the coordinates are barycentrics, as will be the case in the sequel. Indeed, when working with barycentrics, it is common to use the $pK$ classification scheme, presented in [5]. It will be helpful to be able to cross back and forth between the two kinds of classification. First suppose $Z(U, P)$ is given, where $U$ and $P$ are in trilinears. Then

$$Z(U, P) = pK(P, U'),$$

where $U'$ is the trilinear product given by

$$U' = X_2 \cdot X = bcu : cav : abw.$$

For the reverse, we start with $pK(P, U)$ where $U$ and $P$ are in barycentrics. Then

$$pK(P, U) = Z(U, P'),$$

where $P'$ is the barycentric product given by

$$P' = X_{31} \ast P = a^3p : b^3q : c^3r.$$  

For example, the Thomson cubic is $Z(X_{30}, X_1) = pK(X_6, X_{30})$; the Lucas cubic is $Z(X_{69}, X_{31}) = pK(X_2, X_{69})$. In [9] and elsewhere, hundreds of $Z$ cubics are discussed in families and individually.

**Theorem 3.** Suppose that $DEF$ is an $X$-perspective image of $ABC$ and that $U$ is a finite point other than $X$ such that $MN \neq 0$ (as in Theorem 2). Let $D_tE_tF_t$ be the special triangle. The locus of a point $\alpha : \beta : \gamma$ whose cevian triangle $A'B'C'$ is perspective to $D_tE_tF_t$ is a $Z$ cubic. Likewise, the locus of a point $\alpha : \beta : \gamma$ whose anticevian triangle $A'B'C'$ is perspective to $D_tE_tF_t$ is a $Z$ cubic.

**Proof:** Let $u_1, u_2, \ldots, u_9$ be as in the proof of Theorem 1. Then the lines $A'D_t$, $B'E_t$, $C'F_t$ are given by the equations

$$h_{i+2}\alpha + h_{i+1}\beta + h_i\gamma = 0,$$
for \(i = 1, 4, 7\), respectively, where

\[
\begin{align*}
    h_1 &= \beta u_3 - \gamma u_2 & h_2 &= \gamma u_1 & h_3 &= -\beta u_1 \\
    h_4 &= -\gamma u_5 & h_5 &= \gamma u_4 - \alpha u_6 & h_6 &= \alpha u_5 \\
    h_7 &= \beta u_9 & h_8 &= -\alpha u_9 & h_9 &= \alpha u_8 - \beta u_7.
\end{align*}
\]

Let

\[
\delta = \begin{vmatrix}
    h_1 & h_2 & h_3 \\
    h_4 & h_5 & h_6 \\
    h_7 & h_8 & h_9
\end{vmatrix}.
\]

The three lines concur if and only if \(\delta = 0\). Since

\[
\delta = \left( u_3 u_4 u_8 - u_2 u_6 u_7 \right) \alpha \beta \gamma
\]

\[
+ \left( u_3 u_5 u_9 - u_3 u_6 u_8 \right) \beta - (u_2 u_5 u_9 - u_2 u_6 u_8) \gamma \alpha^2,
\]

\[
+ \left( u_1 u_4 u_9 - u_3 u_4 u_7 \right) \gamma - (u_3 u_6 u_7 - u_1 u_6 u_9) \beta^2,
\]

\[
+ \left( u_1 u_5 u_8 - u_2 u_4 u_8 \right) \alpha - (u_1 u_5 u_7 - u_2 u_4 u_7) \beta \gamma^2
\]

the locus, clearly cubic, is a \(Z\) cubic if and only if the first two terms cancel, and a computer calculation shows that we do indeed have \(u_3 u_4 u_8 = u_2 u_6 u_7\).

The same method shows that starting with the anticevian triangle of \(\alpha : \beta : \gamma\) also yields a \(Z\) cubic. \(\square\)

5. Cevian triangles

The simplest perspective image of the reference triangle \(ABC\) is, for given \(X = x : y : z\), the cevian triangle of \(X\). Its vertices are the points

\[
AX \cap BC = 0 : y : z \quad BX \cap CA = x : 0 : z \quad CX \cap AB = x : y : 0.
\]

**Corollary 1.** Suppose \(X = x : y : z\) is a regular point and \(DEF\) is the cevian triangle of \(X\). Suppose \(U\) and \(D_t E_t F_t\) are as in Theorem 2. Let

\[
M = \sum u \left( w y - v z \right) x^2
\]

\[
N = \sum \left[ w v (y - z) + w^2 y - v^2 z \right] x^2.
\]

If \(MN \neq 0\), then \(D_t E_t F_t\) is perspective to \(ABC\) if and only if \(t \in \{0, 1, -M/N\}\). If \(t = -M/N\), then the perspector \(P\) is the point

\[
Y - z \\
\frac{v z (x + y) - w y (x + z)}{w x (y + z) - u z (y + x)} \\
\frac{z - x}{u y (z + x) - v x (z + y)}
\]

and the line \(UP\), given by

\[
\sum [v x (x + y) - w y (x + z)] \alpha = 0,
\]

passes through the point

\[
X^* = x(y + z) : y(z + x) : z(x + y).
\]
Proof: In the proof of Theorem 2, take \( t_1 = 0, t_2 = 0, t_3 = 0 \), leading to the solution \( t = -M/N \) of the equation \( M + Nt = 0 \). It is easy to check that \( UP \) is as asserted and that \( X^* \in UP \).

Regarding Fig. 3, starting with \( A, B, C, X, U \), the following are calculated as in Corollary 1: \( M, N, t = -M/N \), the special triangle \( D_rE_rF_r \), the perspector \( P \), and the line \( UP \). Then a variable point \( R \) is placed on line \( UP \), the corresponding \( t \) calculated, and the \( t \)-dilation of \( DEF \) from \( R \) is shown as \( D_rE_rF_r \). The main point of the figure is this: that the perspector \( P \) of \( D_rE_rF_r \) and \( ABC \) remains fixed as \( R \) varies on \( UP \).

A geometric rendering of Corollary 1 is that there is exactly one nontrivial \( D_rE_rF_r \) perspective to \( ABC \) if, for given \( U \), the point \( X \) lies on neither of the cubic curves

\[
\sum u (w\beta - v\gamma) \alpha^2 = 0 \quad \text{and} \quad \sum [vw(\beta - \gamma) + w^2\beta - v^2\gamma] \alpha^2 = 0,
\]

or equivalently, if, for given \( X \), the point \( U \) lies on neither of two conic curves

\[
\sum x (y^2 - z^2) \beta\gamma = 0 \quad \text{and} \quad \sum (y - z) (x^2\beta\gamma + yz\alpha^2) = 0.
\]

Example 1. If \( U \) in Corollary 1 is the centroid, \( 1 : 1 : 1 \), then \( t = -1/2 \), and the special triangle \( D_1E_1F_1 \) is especially simple:

\[
D_1 = y + z : z : y, \quad E_1 = z : z + x : x, \quad F_1 = y : x : x + y.
\]

We call \( D_1E_1F_1 \) the mid-isotomic triangle of \( X \). Note that the perspector of \( D_1E_1F_1 \) and \( ABC \) is the isotomic conjugate of \( X \).

In Section 7, we present a construction and properties of mid-isotomic triangles.

Example 2. \( X = X_4, U = X_1 \). Here, \( DEF \) is the orthic triangle, determined by

\[
x : y : z = \tan A : \tan B : \tan C,
\]

and \( U = a : b : c \). The perspector \( P \) is given by

\[
X_{72} = a (b + c) (b^2 + c^2 - a^2) : b (a + c) (a^2 - b^2 + c^2) : c (a + b) (a^2 + b^2 - c^2).
\]
Example 3. \( X = X_{63}, \ U = X_3 \). Here, we obtain \( P = a : b : c \), the incenter. The special triangle has vertices

\[
\begin{align*}
D_t &= f(a, b, c) : b : c \\
E_t &= a : f(b, c, a) : c \\
F_t &= a : b : f(c, a, b),
\end{align*}
\]

where

\[
f(a, b, c) = \frac{2a^2(b + c)(b^2 + c^2 - a^2)}{(a^2 - b^2 + c^2)(a^2 + b^2 - c^2)}.
\]

Example 4. \( X = X_{69}, \ U \) and arbitrary finite point on the Euler line, other than \( X_3 \) (for which \( M = N = 0 \)). In this case, the perspector \( P \) is the orthocenter, \( X_4 \), so that the line \( PU \) is the Euler line. Representing the Euler line by a variable point \( U = U(\tau) \), given by

\[
u : v : w = a(bc + \tau a_1) : b(ca + \tau b_1) : c(ab + \tau c_1),
\]

where

\[
a_1 = a(b^2 + c^2 - a^2), \ b_1 = b(c^2 + a^2 - b^2), \ c_1 = c(a^2 + b^2 - c^2)
\]

(so that \( a_1 : b_1 : c_1 = \cos A : \cos B : \cos C \)), we find that the special triangle has vertices

\[
\begin{align*}
D_t &= f(a, b, c) : a^2 + b^2 - c^2 : a^2 - b^2 + c^2 \\
E_t &= b^2 - c^2 + a^2 : f(b, c, a) : b^2 + c^2 - a^2 \\
F_t &= c^2 + a^2 - b^2 : c^2 - a^2 + b^2 : f(c, a, b)
\end{align*}
\]

where

\[
f(a, b, c) = \frac{-4a^4(b^2\tau + c^2\tau - a^2\tau + 2b^2c^2)}{b^4\tau + c^4\tau - a^4\tau - 2b^2c^2\tau - 4a^2b^2c^2}.
\]

Example 5. \( X = X_7 \) (Gergonne point) and \( U = X_6 \) (symmedian point). Here, we obtain

\[
P = X_9 = a(b - a + c) : b(a - b + c) : c(a + b - c),
\]

and the special triangle has vertices

\[
\begin{align*}
D_t &= f(a, b, c) : b(a - b + c) : c(a + b - c) \\
E_t &= a(b + c - a) : f(b, c, a) : c(a + b - c) \\
F_t &= a(b + c - a) : b(c + a - b) : f(c, a, b)
\end{align*}
\]

where

\[
f(a, b, c) = \frac{a^2(2bc + 2ca + 2ab - a^2 - b^2 - c^2)}{b^2 + c^2 - ab - ac}.
\]

6. Anticevian triangles

Another simple perspective image of the reference triangle \( ABC \) is, for given \( X = x : y : z \), the anticevian triangle of \( X \), with vertices

\[
D = -x : y : z \quad E = x : -y : z \quad F = x : y : -z.
\]

Writing \( A' = AX \cap BC = 0 : y : z \), the point \( D \) is the \( \{A, A'\}\)-harmonic conjugate of \( X \), and likewise for \( E \) and \( F \).
Corollary 2. Suppose $X = x : y : z$ is a regular point and $DEF$ is the anticevian triangle of $X$. Suppose $U$ and $D_tE_tF_t$ are as in Theorem 2. Let
\[
M = \sum x(y - z)(x - y - z)vw;
N = \sum yz(y - z)u^2.
\]
If $MN \neq 0$, then $D_tE_tF_t$ is perspective to $ABC$ if and only if $t \in \{0, 1, -M/N\}$. If $t = -M/N$, then the perspector $P$ is the point
\[
y - z \\
wz(y - z + x) - vz(x + y - z) \\
z - x \\
x - y \\
wz(x - y + z) - uy(z + x - y),
\]
and the line $UP$ is given by
\[
\sum[vz(x + y - z) - wy(x + z - y)]\alpha = 0.
\]

Proof: In the proof of Theorem 2, take $t_1 = -x, t_2 = -y, t_3 = -z$, leading to the solution $t = -M/N$ of the equation $M + Nt = 0$. 

Example 6. As a complement to Example 1, if $U = 1 : 1 : 1$, then $t = -2$, and the special triangle $D_tE_tF_t$ is given by
\[
D_t = -x - y - z : x + y - z : x - y + z \\
E_t = y - z + x : -x - y - z : y + z - x \\
F_t = z + x - y : z - x + y : -x - y - z.
\]
Starting with $G = 1 : 1 : 1$ and $G_A = -1 : 1 : 1$, we can construct $D_t$ as follows: let
\[
P_1 = GX \cap BC = 0 : x - y : x - z \\
P_2 = P_1G_A \cap AX = -x : y : z \\
X^* = (\text{isotomic conjugate of } X) = 1/x : 1/y : 1/z \\
P_3 = AX^* \cap BC = 0 : z : y \\
D_t = P_2G \cap P_3G_A.
\]
The vertices $E_t$ and $F_t$ are likewise constructed. Note that $D_tE_tF_t$ is inscribed in the anti-complementary triangle, $G_AG_BG_C$.

The perspector of $D_tE_tF_t$ and $ABC$ is the point
\[
\frac{1}{y + z - x} : \frac{1}{z + x - y} : \frac{1}{x + y - z},
\]
which, for example, lies on the Euler line if $X = a^2 : b^2 : c^2 = X_6$.

The $Z$-cubic of Theorem 3, i.e., the locus of a point $\alpha : \beta : \gamma$ whose cevian triangle is perspective to $D_tE_tF_t$, is given by
\[
\sum x(y + z) [(x - y + z)\beta - (x + y - z)\gamma] \alpha^2 = 0.
\]
7. Mid-isotomic triangles

Mid-isotomic triangles $D_tE_tF_t$ are introduced in Example 1 as the special triangles when $DEF$ is a cevian triangle and $U = 1 : 1 : 1$. (Recall that the special triangle is the one homothetic to $DEF$ and perspective to $ABC$.) As these triangles may not have been considered as a class elsewhere, we present a construction (Fig. 4). Theorem 2 applies to these triangles, so that Theorem 3 also applies, and we give corresponding examples.

Figure 4: $D_1$ is the $A$-vertex of the midcevian triangle of $X$ and $D_2$ the $A$-vertex of the mid-isotomic triangle of $X$

Figure 4 shows the $A$-vertex, labeled $D_2$, of the mid-isotomic triangle of a point $X$. Its construction depends on that of the $A$-vertex, labeled $D_1$, of the midcevian triangle of $X$. Steps for constructing $D_1$ and $D_2$ follow:

$$
D = AX \cap BC
$$
$$
D_1 = \text{midpoint of segment } AD
$$
$$
D' = \text{reflection of } D \text{ in the midpoint of } BC
$$
$$
D_2 = \text{midpoint of segment } AD'
$$

Theorem 2 applies to mid-isotomic triangles; to see that this is so, start by writing the $A$-vertex as $D = y + z : z : y$. (That is, we write $DEF$ for the special triangle and will soon write $D_tE_tF_t$ for the special-of-special triangle.) Then $D = 1/y + 1/z : 1/y : 1/z$, and likewise for $E$ and $F$, so that $DEF$ is the $X^*$-perspective image of $ABC$, where $X^*$ is the isotomic conjugate of $X$. Now, for any $U \neq 1 : 1 : 1$, the special triangle $D_tE_tF_t$ as given by Theorem 2 has $A$-vertex

$$
D_t = x(y + z)[(u - v)(ux - wz)z - (u - w)(ux - vy)y]/(vy - wz)
$$
$$
: y(x - z)(v(y + z) - u(x + z)) : z(x - y)(w(y + z) - u(x + y)).
$$

By Theorem 2, $D_tE_tF_t$ is homothetic to the cevian triangle of $X^*$ and perspective to $ABC$. 
with perspector $P$ given by

$$p = \frac{x(y - z)}{v(x + y) - w(x + z)}$$
$$q = \frac{y(z - x)}{w(y + z) - u(y + x)}$$
$$r = \frac{z(x - y)}{u(z + x) - v(z + y)}.$$

We view the foregoing appearance of the mid-isotomic triangles in connection with Theorems 2 and 3 as sufficient reason for inquiring further into the properties of these triangles quite apart from the two theorems. Accordingly, continuing with the notation $DEF$ for the mid-isotomic triangle, suppose that $X = x : y : z$ is a finite regular point. The locus of a point $\alpha : \beta : \gamma$ whose cevian triangle $A'B'C'$ is perspective to $DEF$ is a $\mathcal{Z}$ cubic; to prove this, note that coefficients for the line $\alpha$ in [4]. It would be of interest to have a formula for a mapping $X$ indexed as $K_034$, $X_31$, $X_58$, $(y \beta - z \gamma) \alpha^2 + (z \gamma - x \alpha) \beta^2 + (x \alpha - y \beta) \gamma^2 = 0.$

From this product we see that if $1/x + 1/y + 1/z \neq 0$, then the expected cubic is the self-isotomic cubic given by

$$(y \beta - z \gamma) \alpha^2 + (z \gamma - x \alpha) \beta^2 + (x \alpha - y \beta) \gamma^2 = 0. \quad (3)$$

**Example 7.** If $X = X_{75} = bc : ca : ab$, then (3) is the Spieker perspector cubic $\mathcal{Z}(X_{75}, X_{31})$, indexed as $K_034$ in [4]. As $\alpha : \beta : \gamma$ traces $\mathcal{Z}(X_{75}, X_{31})$, the perspector $\alpha_1 : \beta_1 : \gamma_1$ of $\mathcal{Z}(X_{75}, X_{31})$ to $\mathcal{Z}(X_{2}, X_{58})$, for $i = 1, 2, 7, 8, 63, 75, 92, 347$, are mapped respectively to the points $X_i$ for $i = 1, 37, 226, 10, 9, 2, 281, 1214$, on $\mathcal{Z}(X_{2}, X_{58}).$

**Example 8.** For comparison with (3), for arbitrary finite regular $X$, the locus of $\alpha : \beta : \gamma$ whose anticevian triangle $A''B''C''$ is perspective to $DEF$ is also a $\mathcal{Z}$ cubic:

$$\sum [(x + y)(yz - zx + xy) \beta - (x + z)(yz + zx - xy) \gamma] \alpha^2 = 0. \quad (4)$$

Taking $X = X_{75} = bc : ca : ab$, we obtain in (4) the Spieker central cubic $\mathcal{Z}(X_{8}, X_{58})$, indexed as $K_033$ in [4]. As $\alpha : \beta : \gamma$ traces $\mathcal{Z}(X_{8}, X_{58})$, the perspector $\alpha_1 : \beta_1 : \gamma_1$ of $\mathcal{Z}(X_{8}, X_{58})$ traces $\mathcal{Z}(X_{2}, X_{58})$, already encountered in Example 7. Here, the points $X_i$ on $\mathcal{Z}(X_{8}, X_{58})$, for $i = 1, 4, 8, 10, 40, 65, 72$, are mapped respectively to the points $X_i$, for $i = 1, 281, 2, 37, 9, 226, 10$, on $\mathcal{Z}(X_{2}, X_{58}).$

8. Midcevian triangles

As indicated by the construction in Fig. 3, a class of triangles closely related to mid-isotomic triangles are a class we call midcevian triangles, defined by vertices of the form

$$D = y + z : y : z, \quad E = x : z + x : z, \quad F = x : y : x + y.$$
As $DEF$ is an $X$-perspective image of $ABC$, Theorem 2 applies. As usual, let $D_tE_tF_t$ denote the special triangle (homothetic to $DEF$ and perspective to $ABC$) determined by $X$ and a finite point $U \neq X$. The perspector of $D_tE_tF_t$ and $ABC$ is given by Theorem 2:

$$
\begin{align*}
p &= \frac{y - z}{vz(x + y) - wy(x + z)} \\
q &= \frac{z - x}{wx(y + z) - uz(y + x)} \\
r &= \frac{x - y}{uy(z + x) - vx(z + y)}.
\end{align*}
$$

Next, in the manner of Section 7, we consider perspectivities of the midcevian triangle $DEF$ with cevian and anticevian triangles. If $X = x : y : z$ is a regular point, then $DEF$ is perspective to the cevian triangle of a point $\Upsilon = \alpha : \beta : \gamma$ if and only if

$$
(x + y + z) \sum (\beta/y - \gamma/z)\alpha^2 = 0,
$$

which is to say that $\Upsilon$ is on the line at infinity or on the $Z$ cubic indicated. Finally, $DEF$ is perspective to the anticevian triangle of a point $\Upsilon$ if and only if $\Upsilon$ lies on the $Z$ cubic given by

$$
\sum [z(x + y) (x - y + z) \beta - y(x + z) (x + y - z) \gamma] \alpha^2 = 0.
$$

9. Concluding remarks

Here we summarize the foregoing results and mention problems for further study. Theorem 1 is very general in the sense the $DEF$ is an arbitrary triangle. In Theorem 2 and all the rest of the article except an example just below, $DEF$ is an $X$-perspective image of $ABC$. It is reasonable to expect that Theorem 1 has interesting implications for other other triangles $DEF$. Theorem 3 indicates that many loci associated with $D_tE_tF_t$ are $Z$ cubics. Subsequent examples in Sections 5–8 suggest that the locus of the perspector $P$ for $\alpha : \beta : \gamma$ on such a $Z$ cubic is again a $Z$ cubic, as in Examples 7 and 8. The associated mappings between the two cubics warrant further investigation.

We conclude with a choice of $DEF$ that is not a perspective image of $ABC$, namely the pedal triangle of the centroid. In the case, (2) has only one nonzero root, $t = -1$, and the triangle $D_tE_tF_t$ homothetic to $DEF$ and also perspective to $ABC$ (whereas $DEF$ is not perspective to $ABC$) is given by

$$
\begin{align*}
D_{-1} &= 4a^2 : a^2 - b^2 + c^2 : a^2 + b^2 - c^2 \\
E_{-1} &= b^2 + c^2 - a^2 : 4b^2 : b^2 - c^2 + a^2 \\
F_{-1} &= c^2 - a^2 + b^2 : c^2 + a^2 - b^2 : 4c^2.
\end{align*}
$$

References


Received November 30, 2008; final form March 10, 2010