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Equioptic Curves of Conic Sections

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Abstract. Given two plane curves c_1 and c_2 we call the set of points from which c_1 and c_2 are seen under equal angle the *equioptic curve*. We give some basic results concerning equioptic curves in general. Then we pay our attention to the seemingly simple case of equioptic curves of conic sections. Mainly we are interested in upper bounds of the algebraic degree of these curves. Some examples illustrate the results.

Key Words: equioptic curves, isoptic curve, orthoptic curve

MSC 2000: 51N35

1. Introduction

To any given plane curve c the locus i of points where c is seen under a given fixed angle ϕ is called the *isoptic curve* of c. Curves appearing as isoptic curves are well studied, see for example [5] and [8] and the references given there. The papers [10, 11] deal with curves having a circle or an ellipse for an isoptic curve.

The name isoptic curve was suggested first by TAYLOR in [7]. The kinematic generation of isoptic and orthoptic curves is also studied there. The locus of points where a tangent of c_1 intersects a tangent of c_2 at a certain angle ϕ is considered as a generalization of the classical notion of isoptic curves. These investigations also deal with the general forms of pedal curves. Especially the isoptics of concentric cycloids are studied in [9].

Due to the algebraic nature of its definition it is clear that the isoptic of an algebraic curve is also algebraic. The Plücker characteristics of the isoptic of a curve c can be expressed in terms of the characteristics of c, cf. [4, 7].

Isoptic curves of conic sections have been studied in [3] and [6]. It turned out that the isoptic curves of ellipses or hyperbolae, i.e., conic sections with center having an equation of the form

$$\alpha x^2 + \beta y^2 = 1 \tag{1}$$

are quartic curves given by

$$i: \ (\alpha + \beta - \alpha\beta(x^2 + y^2))^2 \sin^2\phi - 4\alpha\beta(\alpha x^2 + \beta y^2 - 1)\cos\phi^2 = 0. \tag{2}$$

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where ϕ is the desired optic angle.

It is not mentioned in the literature — though very easy to verify — that the isoptic curves of parabolae with equation

$$2py = x^2 \tag{3}$$

are hyperbolae given by

$$(4x^2 + (p - 2y)^2)\cos^2\phi - (p + 2y)^2 = 0. \tag{4}$$

Fig. 1 displays some isoptic curves of the three affine types of conic sections.

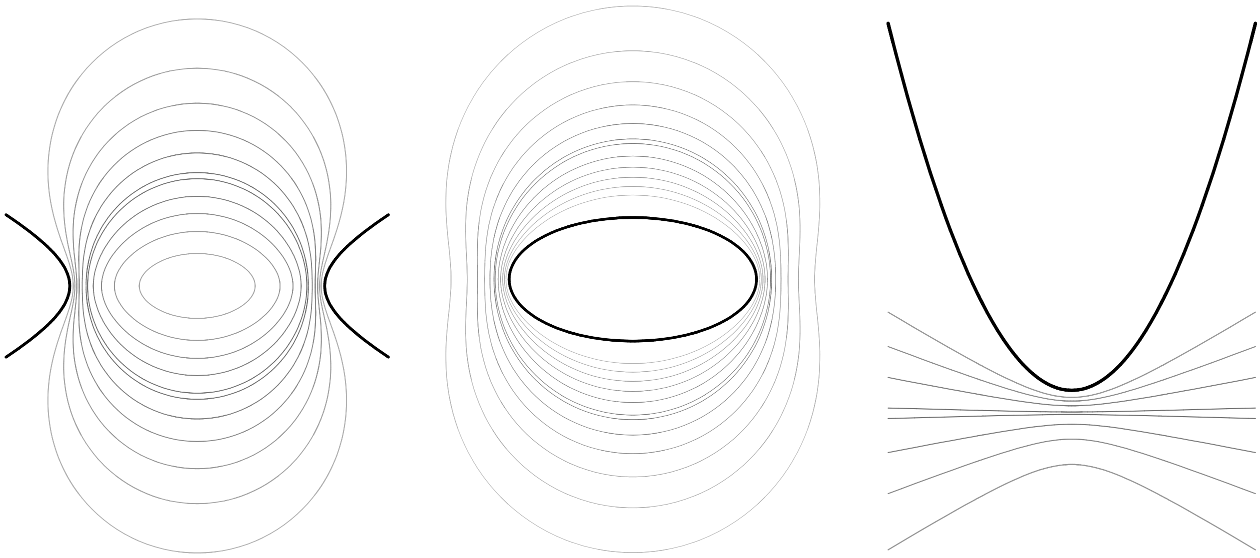

Figure 1: Isoptics of a hyperbola, an ellipse, and a parabola

Special cases of so called *orthoptics*, which are the isoptics for $\phi = \frac{\pi}{2}$ are well known in any case: If c is an ellipse with a and b for the lengths of its major and minor axis, then the orthoptic curve is a concentric circle with radius $\sqrt{a^2 + b^2}$. In case of a hyperbola c we find a circle concentric with c with radius $\sqrt{a^2 - b^2}$ provided that $a > b$. The orthoptic curve of a parabola is its directrix. This is clear when inserting $\phi = \frac{\pi}{2}$ into the respective equations of isoptics.[1]

Further it is worth to be noted that any quartic curve that appears as the isoptic curve of a conic section c is the isoptic curve of a further conic section $c' \neq c$ at the same time, see [5].

The isoptics of conic sections are *spiric curves*, see [5, 8], which can be obtained as planar intersections of a torus. Naturally these curves are bicircular like the torus, i.e., the curves have double points at the absolute points (of Euclidean geometry) whereas the surface has the absolute conic (of Euclidean geometry) for a double curve, see [5].

So far we have collected some facts on isoptic curves and the interested reader may ask himself: What has all this to do with equioptic curves? Consider a point X on the equioptic curve $e(c_1, c_2)$ of two different plane curves c_1 and c_2.[2] The generic point $X \in e$ is the common

[1] It is clear from Eq. (2) that the orthoptic of a conic section with center is also a quartic curve. To be more precise it is a circle of multiplicity 2. Similiarly the orthoptic of a parabola is a repeated line.

[2] At first we do not restrict the huge class of plane curves. We only assume that they have tangents at any point, i.e., from the differential geometric point of view we assume that the curves c_1 and c_2 are of class C^1.

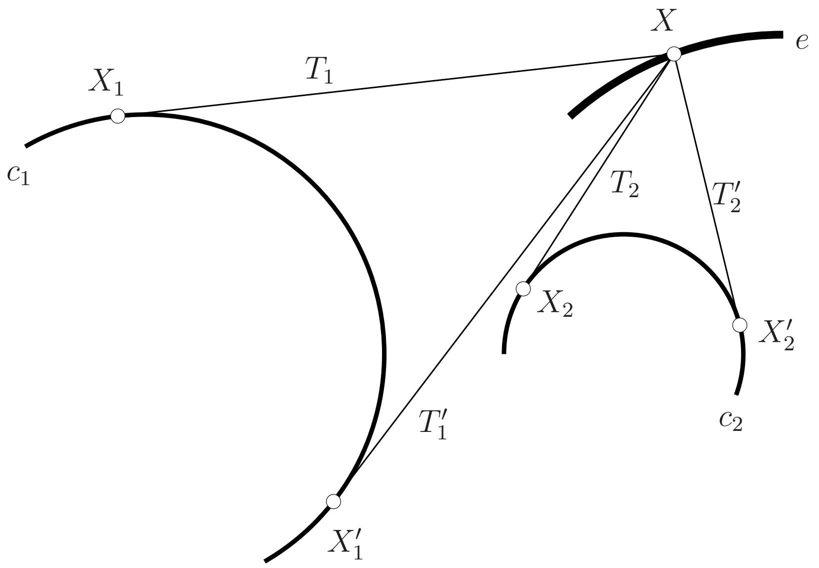

Figure 2: Two curves c_i and a point X on the equioptic curve e, the tangents T_i, T_i' and the corresponding contact points X_i, X_i'

point of two tangents of c_1 enclosing a certain angle, say ϕ. So X is a point of the isoptic curve $i_1(\phi)$ of c_1 to the angle ϕ. Since X is also a common point of two tangents of c_2 enclosing the same angle ϕ, it is a point of the isoptic curve $i_2(\phi)$ of c_2, cf. Fig. 2. Hence any point of the equioptic curve $e(c_1, c_2)$ of curves c_1 and c_2 is the intersection of two isoptic curves $i_1(\phi)$ and $i_2(\phi)$, respectively, for a certain value of ϕ. So we have

$$e(c_1, c_2) = \{i_1(\phi) \cap i_2(\phi) : \ \phi \in [0, 2\pi[\}. \tag{5}$$

It will turn out that the curve e has real branches even if ϕ is not real and $|\cos\phi| > 1$. Basically, these branches consist of interior points of curves (especially in the case of conic sections).

The aim of this paper is to give some basic results on equioptic curves. We pay our attention to algebraic curves, especially to conic sections. Further we want to show the algebraic way to find the equations of equioptics of a pair of algebraic curves. For that purpose we describe curves in terms of Cartesian coordinates in Euclidean plane $\mathbb{R}^2$. Whenever necessary we use the projective closure and the complex extension of $\mathbb{R}^2$.

The paper is organized as follows: First we collect some facts on equioptic curves of algebraic curves in general in Section 2 and then we focus on conic sections and their equioptics in Section 3. Unfortunately pairings of different affine types of conic sections need separate treatment. Section 4 is devoted to the study of existence and the counting of equioptic points of three given conic sections. Finally we conclude in Section 5 and address some open problems.

2. General remarks on equioptics

Let c_1 and c_2 be two algebraic curves of respective degrees d_1 and d_2. The equation of either curve shall be given in implicit form by a polynomial $F_i(x, y)$ of degree d_i. We try to find an upper bound for the algebraic degree of the equioptic $e(c_1, c_2)$. For this purpose we write down the system of (algebraic) equations determining the equioptic.

Any point $X = [x, y]^{\mathrm{T}}$ on $e(c_1, c_2)$ is the locus of concurrency of two tangents T_1, T_1' of c_1 and two further tangents T_2, T_2' of c_2. We assign the coordinates $X_i = [x_i, y_i]^{\mathrm{T}}$, $X_i' = [\xi_i, \eta_i]^{\mathrm{T}}$ to the contact points of the tangents T_i and T_i' with the respective curves c_i, cf. Fig. 2. The contact points have to fulfil

$$\begin{aligned} F_1(x_1, y_1) = 0, \quad F_1(\xi_1, \eta_1) = 0, \\ F_2(x_2, y_2) = 0, \quad F_2(\xi_2, \eta_2) = 0. \end{aligned} \tag{6}$$

Now we introduce the abbreviations $g_i := \operatorname{grad} F_i(X_i)$ and $g_i' := \operatorname{grad} F_i(X_i')$. The tangents T_i and T_i' have to pass through X which gives further relations between coordinates of contact points and the point X on e:

$$\begin{aligned} \langle g_1, X - X_i \rangle = 0, \quad \langle g_1', X - X_i' \rangle = 0, \\ \langle g_2, X - X_i \rangle = 0, \quad \langle g_2', X - X_i' \rangle = 0. \end{aligned} \tag{7}$$

Finally the condition on the tangents T_1 and T_1' to enclose the same angle as T_2 and T_2' is given by

$$\langle g_1, g_1' \rangle \cdot \|g_2\|^2 \cdot \|g_2'\|^2 = \langle g_2, g_2' \rangle \cdot \|g_1\|^2 \cdot \|g_1'\|^2. \tag{8}$$

Eqs. (6) and (7) together with Eq. (8) are nine equations in ten unknowns X_1, Y_1, ξ_1, η_1, X_2, Y_2, ξ_2, η_2, x, y. In order to determine an equation of $e(c_1, c_2)$ one has to eliminate all but x and y from these equations.

Table 1 shows the degrees of the nine equations with respect to the unknowns.

Table 1: Degrees of Eqs. (6), (7), (8).

Eq.	x_1, y_1	ξ_1, η_1	x_2, y_2	ξ_2, η_2	x, y
(6.1)	d_1	0	0	0	0
(6.2)	0	d_1	0	0	0
(6.3)	0	0	d_2	0	0
(6.4)	0	0	0	d_2	0
(7.1)	$d_1 - 1$	0	0	0	1
(7.2)	0	$d_1 - 1$	0	0	1
(7.3)	0	0	$d_2 - 1$	0	1
(7.4)	0	0	0	$d_2 - 1$	1
(8)	$2(d_1 - 1)$	$2(d_1 - 1)$	$2(d_2 - 1)$	$2(d_2 - 1)$	0

One can easily see from Table 1 that the system of algebraic equations (6), (7), and (8) is solved by successive elimination of variables. In a first cycle we eliminate X_1, ξ_1, X_2, and ξ_2 by computing the resultants

$$\begin{aligned} \operatorname{Res}((6.1), (7.1), X_1), \quad \operatorname{Res}((6.2), (7.2), \xi_1), \\ \operatorname{Res}((6.3), (7.3), X_2), \quad \operatorname{Res}((6.4), (7.4), \xi_2), \end{aligned} \tag{9}$$

and

$$\operatorname{Res}(\operatorname{Res}(\operatorname{Res}(\operatorname{Res}((8), (6.4), \xi_2), (6.3), X_2), (6.2), \xi_1), (6,1), X_1). \tag{10}$$

Table 2: The degrees of the resultants given in Eq. (9) and Eq. (10).

Eq.	y_1	η_1	y_2	η_2	x, y
(9.1)	$d_1(d_1-1)$	0	0	0	$d_1(d_1-1)$
(9.2)	0	$d_1(d_1-1)$	0	0	$d_1(d_1-1)$
(9.3)	0	0	$d_1(d_1-1)$	0	$d_2(d_2-1)$
(9.4)	0	0	0	$d_1(d_1-1)$	$d_2(d_2-1)$
(10)	$2d_1(d_1-1)$	$2d_1(d_1-1)$	$2d_2(d_2-1)$	$2d_1(d_1-1)$	0

Table 2 collects the degrees of the resultants in the remaining unknowns Y_1, η_1, Y_2, η_2, and x, y.

The entries of the last column are actually $\max(d_i(d_i-1), d_i) = d_i(d_i-1)$, since in general $d_i \neq 0$.

In order to obtain an upper bound for the degree of the equioptic curve, we compute the final resultant of resultants

$$\operatorname{Res}\left(\operatorname{Res}\left(\operatorname{Res}\left(\operatorname{Res}\left((9.4), (10), \eta_2\right), (9.3), Y_2\right), (9.2), \eta_1\right), (9.1), Y_1\right). \tag{11}$$

which is of degree $d_1^2 d_2^2 (d_1-1)^2 (d_2-1)^2$. Hence we have:

Theorem 2.1. *Let c_1 and c_2 be two algebraic curves of degree d_1 and d_2, respectively. The degree of the equioptic curve $e(c_1, c_2)$ is at most*

$$d_1^2 d_2^2 (d_1-1)^2 (d_2-1)^2.$$

Remark.

1. There are several reasons why the actual degree of the equioptic curve can be lower. The degree of resultants computed in (9), (10), and (11) can be lower as expected, for example see [1, 2]. Sometimes resultants can factor and the corresponding components may not be essential. An example will show up in Sec. (3.1) when we deal with equioptics of circles.
2. The geometric definition of the equioptic curve somehow differs from the algebraic definition. The algebraic formulation of the geometric properties of the equioptic curves by means of Eqs. (6), (7), and (8) is not flawless. Especially, after squaring the angle criterion in order to get (8) it expresses the fact that $\cos^2 \angle(T_1, T_1') = \cos^2 \angle(T_2, T_2')$. This implies that either $\angle(T_1, T_1') = \angle(T_2, T_2')$ or $\angle(T_1, T_1') = \pi - \angle(T_2, T_2')$. Consequently the algebraically defined equioptic curve contains branches where the tangents fulfil the desired condition, though the curves c_1 and c_2 are not actually seen under the same angle.
   This phenomenon will not occur when we compute the equioptic of conic sections. These curves will be obtained by intersecting isoptics to equal angles.
3. In general the curves defined by the resultant (11) contain the equioptics defined by (5) and curves which can be called *quasi-equioptics*, i.e., the locus of points where one curve is seen under the angle ϕ and the other curve is seen under $\pi - \phi$.
4. Furthermore we observe that parasitic branches of the equioptic curve $e(c_1, c_2)$ may occur. These are the sets of real points where no real tangents of either curve may pass through. At these points $|\cos \phi| > 1$ and the angle enclosed by tangents T_i and T_i' is imaginary.

We observe the following properties of equioptic curves:

Corollary 2.1. *Let c_1 and c_2 be two (algebraic) plane curves. Denote the intersection points by S_i with and the common tangents by L_j.*

1. *The equioptic curve $e(c_1, c_2)$ passes through the common points S_i of c_1 and c_2.*
2. *The equioptic curve $e(c_1, c_2)$ contains the intersection points $L_{ij} := L_i \cap L_j$ of common tangents of c_1 and c_2 if both curves are locally in the same halfplanes of both tangents L_i and L_j, respectively.*

Proof:

1. At a point $S_i \in \{c_1 \cap c_2\}$ the curve c_1 as well as c_2 are seen under the angle of 180°.
2. Let $L_{ij} = L_i \cap L_j$ be one intersection point of i-th and j-th common tangent of c_1 and c_2. At the point L_{ij} the tangent L_i plays the role of T_1 and L_j that of T_1', say. Further L_i also plays the role of T_2 and L_j that of T_2'. Consequently $\angle(T_1, T_1') = \angle(T_2, T_2')$. □

3. Equioptic curves of conic sections

In this section we focus on equioptics of conic sections. We are not only interested in the degree of such curves. We are also looking for special pairings of conic section. The singularities of equioptics will also be payed attention to.

According to Eq. (5) the equioptics of two conic sections c_1 and c_2 can be found by writing down the equations of the respective isoptics $i_1(\phi)$ and $i_2(\phi)$. Then we eliminate ϕ, i.e., we intersect any pair of isoptics to the same angle ϕ. From the algebraic degrees of the isoptics we can conclude the following:

Theorem 3.1. *Let c_1 and c_2 be two conic sections given by irreducible quadratic equations. Then the algebraic degree of the equioptic curve $e(c_1, c_2)$ of c_1 and c_2 is bounded by the following values:*

1. *The algebraic degree of the equioptic of two conic sections with center is at most* 6.
2. *The algebraic degree of the equioptic of a conic section with center and a parabola is at most* 6.
3. *The algebraic degree of the equioptic of two parabolae is at most* 4.

Proof: A conic section of any affine type can be transformed into any conic section of the same affine type by applying an affine mapping. Since we are interested in certain relations on angles measured between tangents we restrict ourselves to equiform mappings. The coordinate representation of an equiform mapping in the Euclidean plane reads

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}, \tag{12}$$

which is a Euclidean motion if and only if $a^2 + b^2 = 1$. So the only degrees of freedom when mapping one conic section with center to another one are the ratio $\alpha : \beta$ (cf. Eq. (1)), and the two coordinates of the center.

1. The equioptic of two conic sections with center is obtained by eliminating ϕ from two equations of the form (2). Practically this can be done be letting $\cos^2 \phi = K$. The algebraic degrees are not harmed when applying any affine (indeed projective) transformation in order to find the isoptic of conic sections in more general positions. According

to Bézout's theorem we can expect that any two isoptics $i_1(\phi)$ and $i_2(\phi)$ to the same angle ϕ have $\deg i_1 \cdot \deg i_2 = 4 \cdot 4 = 16$ points in common. Note that this is also true for the orthoptics since these are circles with multiplicity two. Surprisingly we observe that the resultant of i_1 and i_2 with respect to K is a polynomial of degree 6 in the unknowns x and y, respectively.

The reduction of the degree is caused by the following facts: Since both absolute points of Euclidean geometry are double points on either isoptic $i_j(\phi)$ (for any ϕ) exactly 8 of the common points coincide with the absolute points. Further, the ideal line splits off with multiplicity 2 from the equation of the equioptic. (This can be seen by computing the homogeneous equation of the equioptic from the homogeneous equations of the isoptics.) Note that from any real ideal point a conic section with center can be seen under the angle of $0°$.

2. Any parabola can be obtained from the parabola $c_1\colon\ 2py - x^2 = 0$ by a suitable equiform transform of the above given kind. Apply the transform to the parabola as well as to its isoptic hyperbola given in (4). Then eliminate ϕ from both, the equations of the isoptic i_1 of the parabola c_1 and from the isoptic i_2 of the conic section c_2 with center given in (2). This obviously results in a polynomial whose degree is at most 6 in x and y.
3. For a pair of parabolae we go a similar way and end up with intersecting two families of hyperbolae comprising the set of equioptic curves of the two parabolae c_1 and c_2. So any two isoptics to the same angle intersect in four points (algebraically counted). □

Note that the results on degrees of equioptic curves of conic sections from Theorem 3.1 undershoot the upper bound given in Theorem 2.1 by far. According to Theorem 2.1 the degree of $e(c_1, c_2)$ could reach most 16, provided that c_1 and c_2 are conic sections, i.e., algebraic curves of degree $d_1 = d_2 = 2$.

Remark. The computation of the equioptic curve $e(c_1, c_2)$ of two conic sections c_1 and c_2 can be performed by eliminating ϕ from the equations of the respective isoptic curves. For variable ϕ the curves $i_1(\phi)$ and $i_2(\phi)$ can be viewed as level sets of two functions defined on the common plane of either c_i. In this sense the equioptic curve $e(c_1, c_2)$ is the orthogonal projection of the intersection of two graph surfaces to the plane of either c_i.

As a consequence of Corollary 2.1 we have:

Corollary 3.1. *Assume c_1 and c_2 are two conic sections. Let the common points be denoted by S_i and the common tangents may be labelled by L_i.*

1. *The four intersection points of c_1 and c_2 belong to the equioptic curve.*
2. *The six intersection points $L_{ij} = L_i \cap L_j$ of the four common tangents of the conic sections c_1 and c_2 are contained in their equioptic curve $e(c_1, c_2)$ if both curves are locally in the same halfspace of both tangents L_i and L_j, respectively.*

Figure 4 illustrates the contents of Corollary 3.1.

As mentioned earlier in this paper the orthoptic curve of an ellipse or hyperbola is a circle. The orthoptic of a parabola is its directrix. This allows to count the number of *orthoptic points*, i.e., points from which either curve can be seen under right angles:

Theorem 3.2. *For two arbitrarily given conic sections c_1 and c_2 the number o of points where both curves can be seen under right angles is at most 2. This bound is sharp except the case, when c_1 and c_2 are parabolae.*

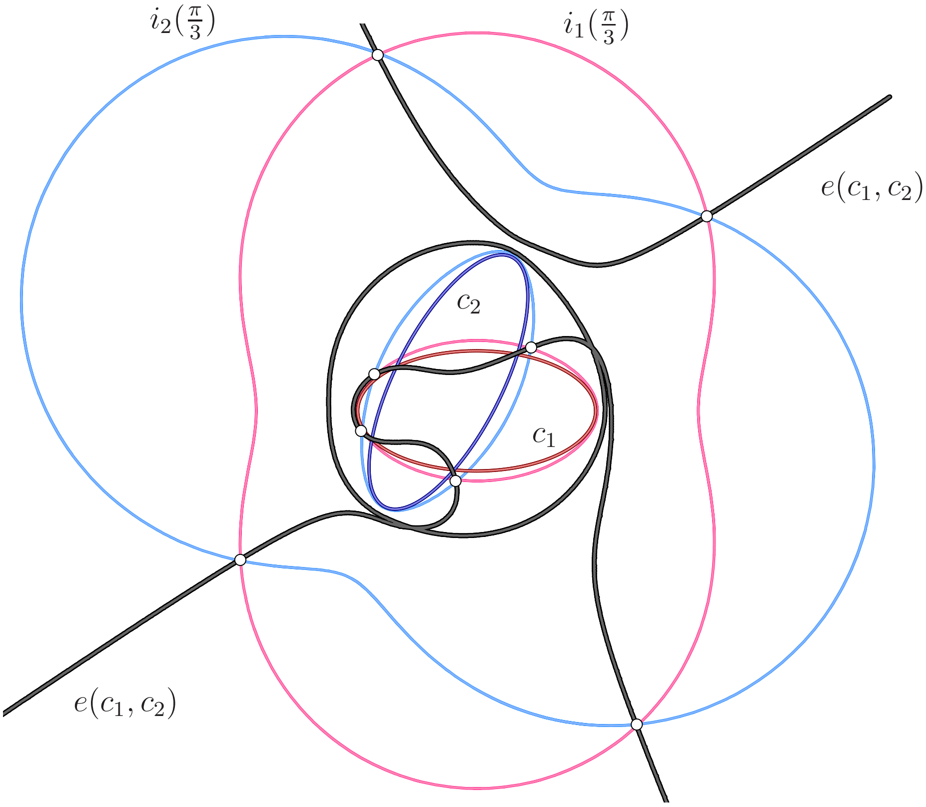

Figure 3: Points of the equioptic curve $e(c_1, c_2)$ of two ellipses c_1 and c_2 can be found as the intersection of their isoptic curves $i_1(\phi)$ and $i_2(\phi)$

Proof: In the case of a conic section with center the orthoptic is a circle, where as in the case of a parabola it is the directrix. In order to clarify the number of orthoptic points one has to discuss the possible intersections of circle and circle, or circle and line, or line and line. □

The shape of the equioptic curve of two confocal conics is regulated by:

Theorem 3.3. *Let (c_1, c_2) be confocal conic sections such that c_1 is chosen from one family and c_2 is chosen from the other family.*

1. *The equioptic curve of a pair (c_1, c_2) of confocal conic sections (with center) is the union of the two-fold ideal line, two further pairs of conjugate complex lines, and a circle containing the four common points of c_1 and c_2.*
2. *The equioptic curve of a pair (c_1, c_2) of confocal parabolae is the union of a straight line connecting the common points of c_1 and c_2, the ideal line and the pair of isotropic lines through the common focus of c_1 and c_2.*

Proof:

1. Assume c_1 and c_2 are given by an equation of the form (1). Let $\alpha = 1/a^2$, $\beta = 1/b^2$ and $a > b$ for c_1. Without loss of generality we can assume that $a > b$. Further $\alpha' = 1/(c^2 - b^2)$ and $\beta' = -1/(a^2 - c^2)$ guarantee that c_1 and c_2 span a confocal family. Write down the isoptics of either conic sections in homogeneous coordinates and eliminate the angle (parameter). This yields

$$x_0^2(c^2x_0^2 - x_1^2 - x_1^2) \cdot \big((a^2-b^2)^2x_0^4 - 2(a^2-b^2)x_0^2(x_1^2-x_2^2)) + (x_1^2+x_2^2)^2\big) = 0. \qquad (13)$$

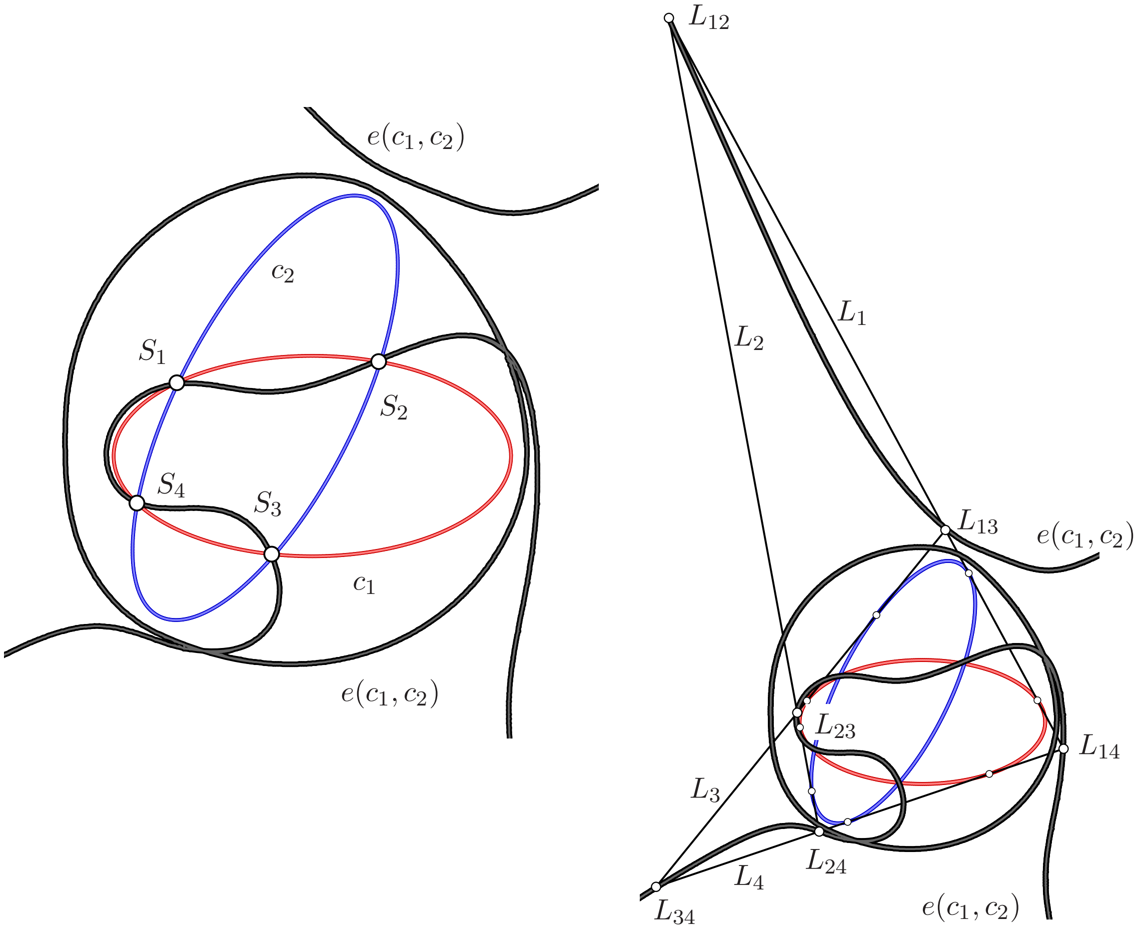

Figure 4: Left: The common points of c_1 and c_2 belong to $e(c_1, c_2)$. Right: The points L_{ij} of intersection of common tangents L_i and L_j of c_1 and c_2 are also contained in $e(c_1, c_2)$

The first factor corresponds to the two-fold ideal line. The second factor is the equation of a circle centered at $[0,0]^{\mathrm{T}}$ with radius c carrying real points if $c^2 > 0$. In this case c_1 and c_2 have the four real points

$$\frac{1}{\sqrt{a^2-b^2}}\left[\pm\sqrt{a(c^2-b^2)}, \pm\sqrt{b(a^2-c^2)}\right]^{\mathrm{T}} \tag{14}$$

in common, which are located on the circle. This holds true even if $c^2 < 0$. Note that the tangents to this circle are bisectors of the angles enclosed by c_1 and c_2 at their common points.
The third factor splits into the equations of the four isotropic lines

$$x \pm \sqrt{a^2-b^2} \pm iy = 0 \tag{15}$$

through c_1's common foci.

2. Assume the parabolae are given by $c_1 :\ 2y - \frac{x^2}{a^2} + a^2 = 0$ and $c_2 :\ 2y + \frac{x^2}{b^2} - b^2 = 0$. The equioptic curve has the homogeneous equation

$$x_0(x_1^2 + x_2^2)((a^2-b^2)x_0 + 2x_2) = 0. \tag{16}$$

The first factor is the equation of the ideal line and the second factor corresponds to the isotropic lines through the common focus $[0,0]^{\mathrm{T}}$. The last factor is the equation of a real line which carries the intersection points $[\pm ab, \frac{1}{2}(a^2-b^2)]^{\mathrm{T}}$ of c_1 and c_2. $\square$

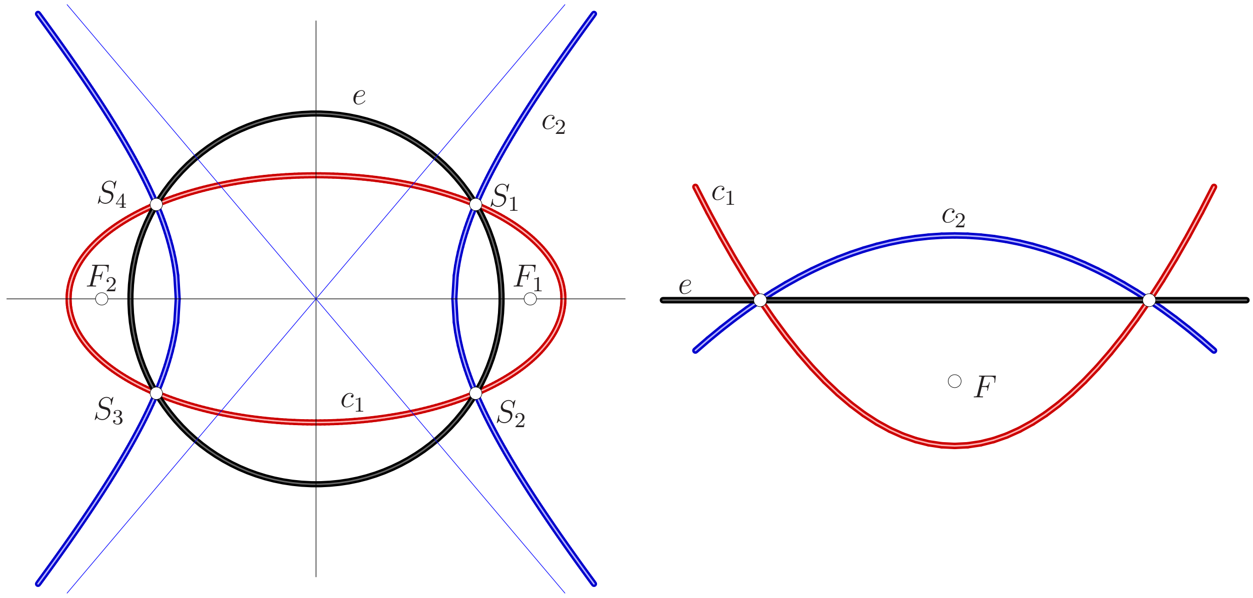

Figure 5: Equioptic curves of pairs of confocal conic sections

The equioptic curve $e(c_1, c_2)$ of two conic sections c_1 and c_2 carries singularities. It is possible to find some of them immediately:

Theorem 3.4.

1. *The absolute points of Euclidean geometry are singular points on* $e(c_1, c_2)$.
2. *The intersection points of the orthoptics of either conic section are singular points on* $e(c_1, c_2)$.

Proof:

1. Any isoptic curve of a conic section has double points at the absolute points. As the points of the equioptic appear as the intersection of two isoptics to the same angle both curves share their singularities and so the family of intersection points also contains these points.
2. The isoptic curves to ellipses and hyperbolae for the angle $\frac{\pi}{2}$ are circles actually having multiplicity 2 as is clearly seen from (2). The orthopic of a parabola is its directrix which has also multiplicity 2 which follows from (4) by substituting $\phi = \frac{\pi}{2}$. Obviously their common points have at least multiplicity 2 and therefore they are singular. $\square$

Figure 6 shows the generic case of singular points on an equioptic curve of two conic sections. Further singularities can occur if the conic sections are in higher order contact which is also illustrated in Fig. 6 at hand of an osculating pair.

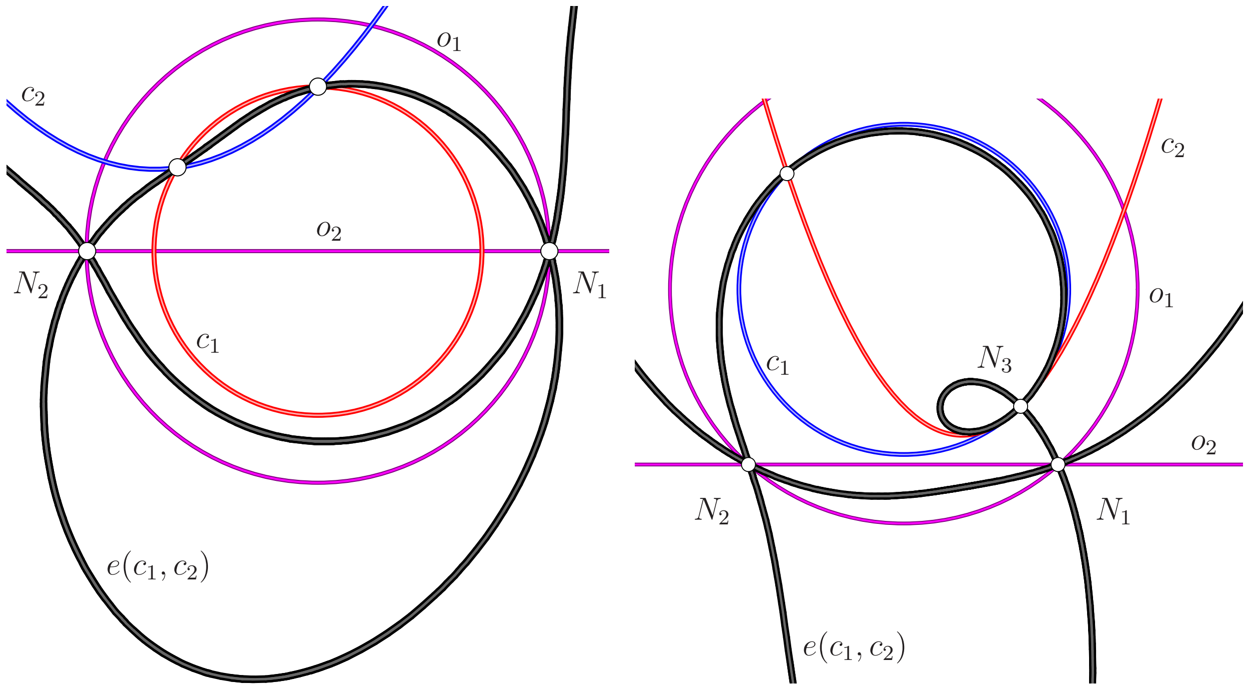

Figure 6: Singular points N_i on an equioptic curve $e(c_1, c_2)$ of two conic sections. Left: Two conics in general position. Right: The equioptic of a parabola c_2 with one of its osculating circles c_1 has a further node at the point of osculation.

3.1. Equioptic of two circles

A circle c with center $M = [m, n]^{\mathrm{T}}$ and radius R shall be given in terms of Cartesian coordinates as

$$c: \ (x - m)^2 + (y - n)^2 - R^2 = 0. \tag{17}$$

The isoptic $i(\phi)$ is again a circle, centered at M, with radius $R \cdot \operatorname{cosec} \frac{\phi}{2}$ and thus described by the equation

$$i(\phi): \ (1 - K)((x - m)^2 + (y - n)^2) - 2^2 R = 0, \tag{18}$$

where $K := \cos\phi$. It is worth to be noted that (18) is linear in K. This will have much influence on the degree of the equioptic curve of two circles.

Now we assume that we are given two circles c_1 and c_2. Without loss of generality we can assume that c_1 is centered at $[0, 0]^{\mathrm{T}}$ and has radius $R \in \mathbb{R} \setminus \{0\}$. The circle c_2 shall be centered at $[d, 0]^{\mathrm{T}}$ with $d \neq 0$ and its radius shall be $r \in \mathbb{R} \setminus \{0\}$.

We are writing down the isoptics $i_1(\phi)$ and $i_2(\phi)$ of c_1 and c_2, respectively, in terms of homogenous coordinates by letting $x = x_1 x_0^{-1}$ and $y = x_2 x_0^{-1}$ and eliminate the angle ϕ. This yields a homogenous equation of the equioptic curve $e(c_1, c_2)$ as

$$e(c_1, c_2): \ x_0^2((r^2 - R^2)(x_1^2 + x_2^2) + 2dR^2 x_0 x_1 - d^2 R^2 x_0^2) = 0. \tag{19}$$

This leads to the following result:

Theorem 3.5. *Let c_1 and c_2 be two circles with radii R and r and the distance d between their centers. The equioptic curve $e(c_1, c_2)$ of c_1 and c_2 is*

1. *the union of a circle containing the two centers of similitude of c_1 and c_2, respectively, and the ideal line with multiplicity 2, if $d \neq 0$ and $r \neq R$,*

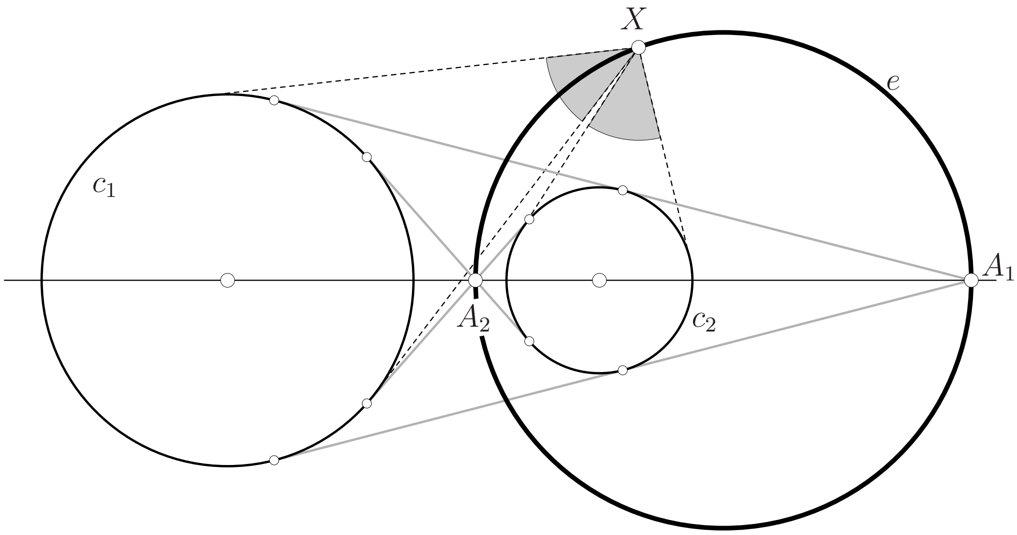

Figure 7: Equioptic circle of two circles passing through the centers of similitude A_i

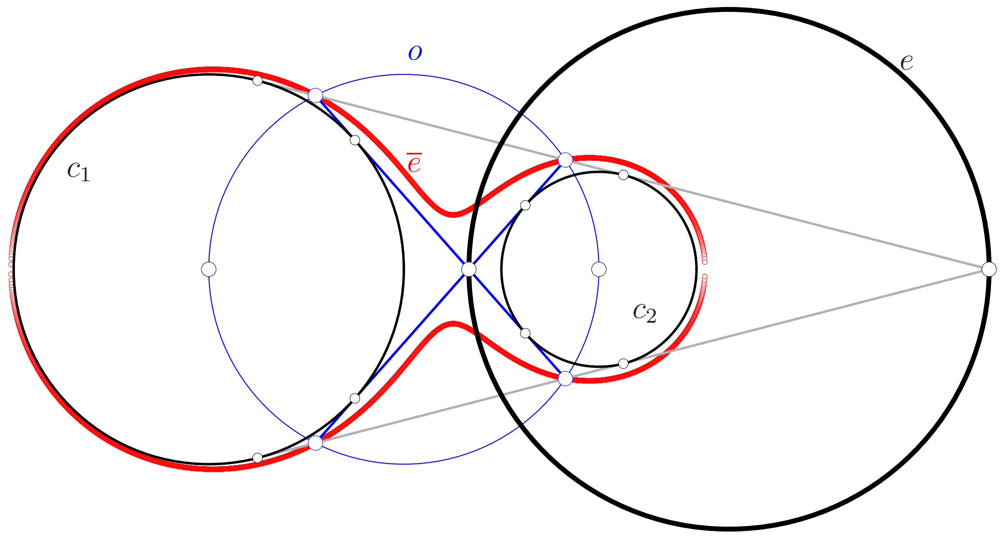

Figure 8: The quasi-equioptic curve $\overline{e}$ of two circles is passing through the four intersection points of the common tangents of c_1 and c_2 but not through the centers of similitude

2. *the union of the bisector of the centers (with multiplicity* 1*) and the three-fold ideal line if c_1 and c_2 are congruent and*
3. *the union of the two-fold ideal line and a the pair of isotropic lines through the common center if c_1 and c_2 are concentric.*

Proof:

1. The centers of similitude of c_1 and c_2 are given by

$$S_1 = \left[\frac{dR}{R+r}, 0\right]^{\mathrm{T}} \quad \text{and} \quad S_2 = \left[\frac{dR}{R-r}, 0\right]^{\mathrm{T}}$$

and they annihilate the second factor of Eq. (19).

2. Insert $R = r$ into (19) and note that $d \neq 0$.
3. Insert $d = 0$ into (19) and note that $R \neq r$. □

Remark. 1. In Fig. 7 we observe that four intersection points of the common tangents of the two circles c_1 and c_2 are not located on the equioptic circle $e(c_1, c_2)$. From these four points one circle is seen under an angle of ϕ whereas the other curve is seen under the angle $\pi - \phi$. So these points do not belong to the geometrically defined equioptic curve.
There is also a reason why these points are not located on the algebraically determined equioptic curve: Earlier in this paper we have noticed that the equations of the isoptic curves $i_1(\phi)$ of c_i the value K shows up only linear. In the case of the isoptic of an ellipse or hyperbola K appear only in second powers, i.e., $\cos\phi$ apears only in squares. Therefore these equations give the equations for the isoptics to an angle ϕ and the angle $\pi - \phi$. This is not the case for the isoptics of circles.
Figure 8 shows the locus $\overline{e}$ of points where c_1 is seen under the angle ϕ and c_2 is seen under the angle $\pi - \phi$. We call this curve the *quasi-equioptic* of c_1 and c_2, respectively.

2. From points of the ideal line the curves c_1 and c_2 can be seen at equal angles $\phi = 0$ since any two tangents from ideal points to any curve are parallel.
The five collinear points C_1, C_2 (centers of c_i), S_1, S_2 (respective centers of similitude), and E (center of the equioptic circle) can be arranged in quadruples in several ways and define cross ratios which are related by
$$\mathrm{cr}(E, C_1, S_1, S_2) \cdot \mathrm{cr}(E, C_2, S_1, S_2) = \mathrm{cr}(C_1, C_2, S_1, S_2) = -1.$$

3.2. Equioptics of conic sections with a circle

This short section is exclusively devoted to the computation of algebraic degrees:

Corollary 3.2. *The algebraic degree of the equioptic curve of a conic section and a circle is at most* 6.

Proof: At first we derive the equioptic curve $e(c_1, c_2)$ of a conic section c_2 with center and a circle c_1. Without loss of generality we can assume that c_2 is given by an equation of the form (1) and c_1 is centered at $[m, n]^{\mathrm{T}}$ and has radius R and is thus given by Eq. (17). Therefore the isoptics $i_1(\phi)$ of c_1 are given by (18) and the isoptics of c_2 have the equation (2). We eliminate ϕ from (18) and (2) by letting $K := \cos\phi$ and end up with an algebraic equation of degree 6 in the unknowns x and y.

The proof is almost the same for the equioptic of a parabola and a circle. □

Remark. We can use the projective closure of $\mathbb{R}^2$ and represent the isoptics appearing in the proof of Corollary 3.2 by their respective homogeneous equations. Eliminating K now results in a homogeneous polynomial of degree 8 which always factors into x_0^2 and a sextic form. Thus the ideal line (of course with multiplicity 2) is always a part of the equioptic of a circle and conic section with center. This is not the case for the equioptic of a circle and a parabola.

4. Equioptic points

We call a point $E(c_1, c_2, c_3)$ *equioptic point of three curves* c_1, c_2, and c_3 if there are two tangents, say T_i, T_i' of either curve c_i passing through E such that $\angle(T_1, T_1') = \angle(T_2, T_2') = \angle(T_3, T_3')$.

It is obvious that equioptic points appear as the intersections of equioptic curves. As a consequence of Theorem 3.1 we can give an upper bound for the number of equioptic points of three conic sections:

Theorem 4.1. *Assume that c_i with $i \in \{1,2,3\}$ are three conic sections. The number $v(c_1, c_2, c_3)$ of equioptic points of the three conic sections c_1, c_2, and c_3 is bounded by* 36.

Proof: Use Theorem 3.1 and apply Bézout's theorem. □

Remark. The value $v(c_1, c_2, c_3)$ drops if parabolae and circles are involved. The long-winded discussion of the number of equioptic points arbitrary triplets of conic sections (classified with respect to affine or even Euclidean properties) could be postponed to a forthcomming paper.

We can state:

Lemma 4.1. *Three generic conic sections do not have orthoptic points. In other words: In general there is no point from which three generic conic sections can be seen under the same angle.*

Proof: We cannot expect that the three circles appearing as orthoptic curves of three conic sections have common points (besides the absolute points of Euclidean geometry). □

For circles there is only the following result:

Lemma 4.2. *Three generic circles in arbitrary position with arbitrary radii have no proper equioptic point. They can only be seen from any ideal point under the angle of* $0°$.

Proof: Three circles in general position with not necessarily equal radii determine three circles as their orthoptic curves. These orthoptic circles are concentric with the given ones and naturally they are of course also in general position. Besides the absolute points of Euclidean geometry these three circles do not share any point. □

5. Conclusion

There are a lot of fine details to be studied. A matter of particular interest could be the exact number of equioptic points of the three arbitrary algebraic plane curves. The study of equioptic curves for special configurations of pairs of conic sections is far from beeing complete. For example one can classify pairs of conic sections from the viewpoint of Euclidean geometry and discuss the corresponding equioptic curves. There are only a few of the singularities detected so far. A huge amount of practical examples showed that in general there a no more singular points than the four discovered. More singularities on the equioptic curve appear if the conic sections are in higher order contact as shown in one example. But this needs a close inspection. What is the number and what are the types of singularities that can occur?

There is still much to do for algebraic curves of higher degree, i.e., for example of cubics, quartics, and so on. Under which circumstances do the degrees of equioptic curves drop? Are singular points on the given curves points of their equioptics? However, as long as the power of computers is not sufficient the algebraic (or computational) approach will shipwreck.

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References

[1] E. Brieskorn, H. Knörrer: *Ebene algebraische Kurven.* Birkhäuser, Basel-Boston-Stuttgart 1981.

[2] J.L. Coolidge: *A treatise on algebraic plane curves.* Dover Publications, New York 1959.

[3] G. Holzmüller: *Einführung in die Theorie der isogonalen Verwandtschaft.* B.G. Teubner, Leipzig 1882.

[4] A.T. Ljungh: *Über isoptische und orthoptische Kurven.* PhD thesis, Lund 1895.

[5] G. Loria: *Spezielle algebraische und transzendente ebene Kurven.* 1 & 2, B.G. Teubner, Leipzig-Berlin 1911.

[6] F.H. Siebeck: *Über eine Gattung von Curven vierten Grades, welche mit den elliptischen Funktionen zusammenhängen.* J. Reine Angew. Math. **57**, 359–370 (1860); **59**, 173–184 (1861).

[7] C. Taylor: *Note on a theory of orthoptic and isoptic loci.* Proc. R. Soc. London **XXXVII** (1884).

[8] H. Wieleitner: *Spezielle ebene Kurven.* Sammlung Schubert LVI, Göschen'sche Verlagshandlung, Leipzig 1908.

[9] W. Wunderlich: *Die isoptischen Kurven der Zykloiden.* Z. angew. Math. Mech. **17**, 56 (1937).

[10] W. Wunderlich: *Kurven mit isoptischem Kreis.* Aequat. math. **6**, 71–81 (1971).

[11] W. Wunderlich: *Kurven mit isoptischer Ellipse.* Monatsh. Math. **75**, 346–362 (1971).

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