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Triply Orthogonal Line Congruences with Common Middle Surface

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Abstract. Let S be a non parabolic line congruence in E^3, whose middle surface $P(u,v)$ is different from its middle envelope $M(u,v)$. We prove that there exist two line congruences S', S'' orthogonal to S and to each other with common middle surface $P(u,v)$ iff S is isotropic or the straight lines of S', S'' are directed by the tangent vectors of the spherical image of the S-principal ruled surfaces of S, in case S is not isotropic. Then, studying the properties of a triplet S, S', S'', we find a new geometric interpretation for the curvature of S.

Key Words: Orthogonal line congruences, middle surface, middle envelope, curvature of a line congruence

MSC 2000: 53A25

1. Introduction

In a three-dimensional Euclidean space E^3 let S, S' be two line congruences, whose straight lines correspond one-to-one. S, S' are called *orthogonal* iff their corresponding straight lines are orthogonal to each other. N. K. Stephanidis [4], G. Stamou [3] and the authors [2] studied orthogonal line congruences with common middle surface. The present paper expands this study, focusing on triplets of orthogonal line congruences with common middle surface. An example of such a triplet is obtained by considering the normal line congruence S of a minimal surface $P(u,v)$. In this case the middle surface $P(u,v)$ of S coincides with its middle envelope and there exist exactly two line congruences S', S'' orthogonal to S and to each other, which have $P(u,v)$ as middle surface [4, p. 324]. The straight lines of S' and S'' are tangent to the asymptotic lines of $P(u,v)$. Since we discuss extensively about S, S', S'' in [2], we shall exclude the above triplet from our study. Thus, we assume that *S is not the normal line congruence of a minimal surface, that is, the middle envelope of S is different from its middle surface.*

Firstly, we examine when there exist triply orthogonal line congruences S, S', S'' sharing the same middle surface and then, we find various properties connecting invariants of S, S',

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S''. Among them appears a new geometric interpretation for the curvature of the given line congruence S.

Suppose S is a line congruence in E^3, defined on a simply connected domain G in the (u,v)-plane by

$$\overline{x}(u,v,t) = \overline{OP} + t\,\overline{e}_3, \quad -\infty < t < +\infty, \tag{1.1}$$

where $\overline{OP} = P(u,v)$ is the position vector for the surface of reference and $\overline{e}_3(u,v)$ is the unit vector in the direction of the straight lines of S.

Let $\mathcal{D} = \{\overline{e}_i(u,v) \,|\, i = 1,2,3\}$ be an orthonormal, positively oriented moving frame of S and $\overline{OM} = M(u,v)$ be the middle envelope of S.

We assume that S satisfies the following conditions:

(a) The functions $P(u,v)$, $M(u,v)$ and $\overline{e}_i(u,v)$, $i = 1,2,3$, are of class C^4 throughout G.

(b) The spherical representation of S is one-to-one.

(c) The middle envelope $M(u,v)$ is a regular surface having no parabolic or umbilical points.

(d) There is a one-to-one mapping between the points of the middle surface and the points of the middle envelope.

Referring to the moving frame $\mathcal{D}$, we may write

$$dP = \sum_{i=1}^{3} \sigma_i\,\overline{e}_i, \tag{1.2}$$

$$d\overline{e}_j = \sum_{i=1}^{3} \omega_{ji}\,\overline{e}_i, \quad \omega_{ij} + \omega_{ji} = 0, \quad i,j = 1,2,3, \tag{1.3}$$

where σ_i, ω_{ij} are linear differential forms for $i,j = 1,2,3$. We denote by "$\wedge$" the exterior product of two differential forms. According to condition (b) the differential forms ω_{31}, ω_{32} are linearly independent, i.e.,

$$\omega_{31} \wedge \omega_{32} \neq 0. \tag{1.4}$$

Thus, for the exterior derivatives $d\omega_{31}$, $d\omega_{32}$ of the differential forms ω_{31}, ω_{32} we may put

$$d\omega_{31} = q\,\omega_{31} \wedge \omega_{32}, \quad d\omega_{32} = \widetilde{q}\,\omega_{32} \wedge \omega_{31}, \tag{1.5}$$

where $q, \widetilde{q}$ are functions of u and v defined on G. Then it is well-known [4, p. 319] that

$$\omega_{12} = q\,\omega_{31} - \widetilde{q}\,\omega_{32}. \tag{1.6}$$

The surface of reference $\overline{OP} = P(u,v)$ is the middle surface of S iff [4, p. 319]

$$\omega_{31} \wedge \sigma_2 + \sigma_1 \wedge \omega_{32} = 0. \tag{1.7}$$

From now on, we assume that $\overline{OP} = P(u,v)$ is the middle surface of S. There exist functions l, m, n of u and v defined on G, such that

$$\sigma_1 = -m\omega_{31} - n\omega_{32}, \quad \sigma_2 = l\omega_{31} + m\omega_{32}. \tag{1.8}$$

The curvature k, the mean curvature h and the limit distance $2z$ of S are given by the formulae

$$k = l\,n - m^2, \quad 2h = l + n, \tag{1.9}$$

$$2z = \sqrt{(l-n)^2 + 4m^2} = 2\sqrt{h^2 - k}. \tag{1.10}$$

Considering $\overline{e}_3(u,v)$ as the unit normal vector of $M(u,v)$ and $\mathcal{D}$ as the moving frame on $M(u,v)$, there exist linear differential forms ρ, σ such that

$$dM = \rho\,\overline{e}_1 + \sigma\,\overline{e}_2\,. \tag{1.11}$$

Moreover, a middle plane of S is tangent to the middle envelope. Hence, there are functions $a = a(u,v)$, $b = b(u,v)$, $(u,v) \in G$, such that

$$\overline{OP} = \overline{OM} + a\,\overline{e}_1 + b\,\overline{e}_2\,. \tag{1.12}$$

Let us now denote the Pfaffian derivatives with respect to the forms ω_{31}, ω_{32} by ∇_i, $i = 1,2$, and the principal radii of $M(u,v)$ by r_i, $i = 1,2$. The functions $a = a(u,v)$, $b = b(u,v)$ satisfy the condition [4, p. 321]

$$\nabla_1 a + \nabla_2 b - \widetilde{q}a - qb = r_1 + r_2 \tag{1.13}$$

and the relation [4, p. 320]

$$\sigma_3 = -a\,\omega_{31} - b\,\omega_{32} \tag{1.14}$$

is valid.

We assume that *the middle envelope $M(u,v)$ of S is different from its middle surface $P(u,v)$*. At every point of $P(u,v)$ we consider a positively oriented orthonormal frame $\mathcal{D}' = \{\overline{e}'_i(u,v) \,|\, i = 1,2,3\}$, such that

$$\overline{e}'_1 = \overline{e}_3\,, \tag{1.15}$$

$$\overline{e}'_2 = \sin\varphi\,\overline{e}_1 - \cos\varphi\,\overline{e}_2, \tag{1.16}$$

$$\overline{e}'_3 = \cos\varphi\,\overline{e}_1 + \sin\varphi\,\overline{e}_2, \tag{1.17}$$

where $\varphi = \varphi(u,v)$ is the oriented angle between $\overline{e}_1(u,v)$ and $\overline{e}'_3(u,v)$. Each line congruence $S'(\varphi)$, whose straight lines are directed by the unit vector $\overline{e}'_3(u,v)$, is obviously orthogonal to S. In addition, it is well-known [4, p. 322] that in a neighbourhood of each point $(u_0,v_0) \in G$, there are infinitely many line congruences, which are orthogonal to S and have the same middle surface $P(u,v)$. All these congruences are defined by the differentiable functions $\varphi(u,v)$ that satisfy the equation

$$b\,\nabla_1\varphi - a\,\nabla_2\varphi - m\cos 2\varphi + \frac{l-n}{2}\sin 2\varphi + \widetilde{q}a + qb = 0 \tag{1.18}$$

or equivalently

$$a\Gamma + b\Delta - m\cos 2\varphi + \frac{l-n}{2}\sin 2\varphi = 0, \tag{1.19}$$

where

$$\Gamma = \widetilde{q} - \nabla_2\varphi, \quad \Delta = q + \nabla_1\varphi. \tag{1.20}$$

Suppose S, $S'(\varphi)$ are two orthogonal line congruences with the same middle surface $P(u,v)$. We denote the elements of $S'(\varphi)$ by the accentuated symbols of the corresponding elements of S. Thus, referring to the moving frame $\mathcal{D}'$, similarly to the formulae (1.2), (1.3), we may write

$$dP = \sum_{i=1}^{3}\sigma'_i\,\overline{e}'_i, \tag{1.21}$$

$$d\,\overline{e}'_j = \sum_{i=1}^{3}\omega'_{ji}\,\overline{e}'_i, \quad \omega'_{ij} + \omega'_{ji} = 0, \quad i,j = 1,2,3. \tag{1.22}$$

We know [4, p. 321] that

$$\sigma_1' = \sigma_3, \quad \sigma_2' = \sin\varphi\,\sigma_1 - \cos\varphi\,\sigma_2, \quad \sigma_3' = \cos\varphi\,\sigma_1 + \sin\varphi\,\sigma_2, \tag{1.23}$$

$$\omega_{12}' = \sin\varphi\,\omega_{31} - \cos\varphi\,\omega_{32}, \tag{1.24}$$

$$\omega_{31}' = -\cos\varphi\,\omega_{31} - \sin\varphi\,\omega_{32}, \tag{1.25}$$

$$\omega_{32}' = -\omega_{12} - d\varphi = -\Delta\omega_{31} + \Gamma\omega_{32}. \tag{1.26}$$

Besides [2, p. 125]

$$\omega_{31}' \wedge \omega_{32}' = D\,\omega_{31} \wedge \omega_{32}, \tag{1.27}$$

where

$$D = -\left(\Gamma\cos\varphi + \Delta\sin\varphi\right). \tag{1.28}$$

The linear differential forms ω_{31}', ω_{32}' are linearly independent iff $D \neq 0\ \forall (u,v) \in G$.

Hereafter we assume $D \neq 0$, i.e.,

$$\Gamma\cos\varphi + \Delta\sin\varphi \neq 0 \quad \forall (u,v) \in G. \tag{1.29}$$

Then, there exist functions l', m', n', a', b' of u and v, defined on G, such that

$$\sigma_1' = -m'\omega_{31}' - n'\omega_{32}', \tag{1.30}$$

$$\sigma_2' = l'\omega_{31}' + m'\omega_{32}', \tag{1.31}$$

$$\sigma_3' = -a'\omega_{31}' - b'\omega_{32}'. \tag{1.32}$$

From the preceding relations, taking into account (1.8), (1.14), (1.19), (1.23)–(1.26) and (1.28), we derived [2, p. 126]

$$m' = \frac{1}{D}\left(a\Gamma + b\Delta\right) = \frac{1}{D}\left(m\cos 2\varphi - \frac{l-n}{2}\sin 2\varphi\right), \tag{1.33}$$

$$l' = -\frac{1}{D}\left[(m\Gamma + n\Delta)\sin\varphi + (l\Gamma + m\Delta)\cos\varphi\right], \tag{1.34}$$

$$n' = \frac{1}{D}\left(a\sin\varphi - b\cos\varphi\right), \tag{1.35}$$

$$a' = \frac{1}{D}\left[(m\Gamma + n\Delta)\cos\varphi - (l\Gamma + m\Delta)\sin\varphi\right], \tag{1.36}$$

$$b' = -\frac{1}{D}\left(l\sin^2\varphi + n\cos^2\varphi - m\sin 2\varphi\right). \tag{1.37}$$

In the following sections we assume that the given line congruence S *is not parabolic* ($k \neq 0$).

2. The existence

We consider a line congruence S whose *middle surface* $P(u,v)$ *is different from its middle envelope* $M(u,v)$. Then, in a neighbourhood of each point $(u_0,v_0)\in G$, there is always a line congruence $S'(\varphi)$ orthogonal to S, which has also $P(u,v)$ as middle surface, where $\varphi(u,v)$ is a solution of the differential equation (1.18). The aim of this section is to answer the question, when there exist a third line congruence $S'(\psi)$, which is orthogonal to S, $S'(\varphi)$ and has a common middle surface with them.

Such a line congruence $S'(\psi)$ corresponds to the function $\psi=\varphi+\frac{\pi}{2}$ (or $\psi=\varphi+\frac{3\pi}{2}$), which satisfies (1.18). That is,

$$b\,\nabla_1\psi - a\,\nabla_2\psi - m\cos 2\psi + \frac{l-n}{2}\sin 2\psi + \widetilde{q}a + qb = 0 \tag{2.1}$$

or equivalently

$$b\,\nabla_1\varphi - a\,\nabla_2\varphi + m\cos 2\varphi - \frac{l-n}{2}\sin 2\varphi + \widetilde{q}a + qb = 0. \tag{2.2}$$

Since the straight lines of $S'(\psi)$ for $\psi=\varphi+\frac{\pi}{2}$ and $\psi=\varphi+\frac{3\pi}{2}$ have the same direction, we continue to have only the function $\psi=\varphi+\frac{\pi}{2}$. By subtracting (1.18) from (2.2), we find

$$2m\cos 2\varphi - (l-n)\sin 2\varphi = 0. \tag{2.3}$$

Relations (2.2) and (2.3) imply

$$b\,\nabla_1\varphi - a\,\nabla_2\varphi + \widetilde{q}a + qb = 0 \tag{2.4}$$

or, taking into account (1.20),

$$a\Gamma + b\Delta = 0. \tag{2.5}$$

Conversely, we suppose that (2.3) is valid. Then, by virtue of (1.18), we obtain (2.4) and as a consequence (2.2) holds. Thus, we deduce:

Proposition 2.1. *Let S, $S'(\varphi)$ be orthogonal line congruences with the same middle surface $P(u,v)$. A line congruence $S'(\varphi+\frac{\pi}{2})$ is orthogonal to S, $S'(\varphi)$ and has $P(u,v)$ as middle surface iff (2.3) or equivalently (2.4) is valid.*

Proposition 2.2. *Given a line congruence S with middle surface $P(u,v)$ and the line congruences $S'(\varphi)$, $S'(\varphi+\frac{\pi}{2})$ orthogonal to S and to each other. If two of the following propositions*

(i) $P(u,v)$ *is the middle surface of* $S'(\varphi)$,

(ii) $P(u,v)$ *is the middle surface of* $S'(\varphi+\frac{\pi}{2})$,

(iii) $2m\cos 2\varphi-(l-n)\sin 2\varphi=0$

hold true, then the third one is also valid.

(A) Let us, firstly, consider that S is isotropic, i.e.,

$$l-n=0 \quad\text{and}\quad m=0 \quad \forall(u,v)\in G. \tag{2.6}$$

By virtue of (2.6), the equation (1.18) reduces to (2.4). Evidently, if $\varphi(u,v)$ satisfies (2.4), then every function $\varphi(u,v)+c$, where $c=\text{const.}$, also satisfies (2.4). Putting $c=\frac{\pi}{2}$, we obtain the following:

Proposition 2.3. *If S is an isotropic line congruence with middle surface $P(u,v)$, there are infinitely many pairs of line congruences $S'(\varphi)$, $S'(\varphi+\frac{\pi}{2})$ orthogonal to S and to each other, which have $P(u,v)$ as middle surface.*

In this case, we also have:

Proposition 2.4. *Suppose S, $S'(\varphi)$, $S'(\varphi+\frac{\pi}{2})$ are triply orthogonal line congruences. Then, the following conditions are equivalent:*

(i) $P(u,v)$ *is the middle surface of* $S'(\varphi)$.

(ii) $P(u,v)$ *is the middle surface of* $S'(\varphi+\frac{\pi}{2})$.

(iii) $b\,\nabla_1\varphi - a\,\nabla_2\varphi + \widetilde{q}a + qb = 0$.

(B) Now, we assume that S *is a nonisotropic* line congruence and $P(u,v)$ is its middle surface. Without loss of generality, we consider that the S-principal ruled surfaces of S are the parameter surfaces defined by $\omega_{31}=0$, $\omega_{32}=0$. That happens iff

$$m \equiv 0. \tag{2.7}$$

Then, $\overline{e}_1$, $\overline{e}_2$ are the tangent vectors to the spherical images of the S-principal ruled surfaces of S. On account of (2.7), the equation (2.3) takes the form

$$(l-n)\sin 2\varphi = 0. \tag{2.8}$$

Since S is nonisotropic, $l-n \neq 0$ for all $(u,v) \in G$. Hence the relation (2.8) is valid iff $\varphi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. From now on, *we shall keep only the values* $\varphi=0$, $\varphi=\frac{\pi}{2}$, because the straight lines of $S'(0)$ and $S'(\pi)$ (resp. $S'(\frac{\pi}{2})$ and $S'(\frac{3\pi}{2})$) have the same direction. Moreover, in view of (1.17), the straight lines of $S'(0)$ and $S'(\frac{\pi}{2})$ are directed by the vectors $\overline{e}_1$ and $\overline{e}_2$, respectively. Therefore we derive:

Proposition 2.5. *Let S be a nonisotropic line congruence with middle surface $P(u,v)$. There exist exactly two line congruences $S'(0)$, $S'(\frac{\pi}{2})$ orthogonal to S and to each other, which have $P(u,v)$ as middle surface. The straight lines of $S'(0)$, $S'(\frac{\pi}{2})$ are directed by the tangent vectors to the spherical images of the S-principal ruled surfaces of S.*

In this case, inserting consecutively the values $\varphi=0$ and $\varphi=\frac{\pi}{2}$ into (2.4), we find

$$\widetilde{q}a + qb = 0, \tag{2.9}$$

which leads to

Proposition 2.6. *Suppose $P(u,v)$ is the middle surface of a nonisotropic line congruence S. For the triply orthogonal line congruences S, $S'(0)$, $S'(\frac{\pi}{2})$ the following conditions are equivalent.*

(i) $P(u,v)$ *is the middle surface of* $S'(0)$,

(ii) $P(u,v)$ *is the middle surface of* $S'(\frac{\pi}{2})$,

(iii) $\widetilde{q}a + qb = 0$.

Summing up the conclusions of (A) and (B), we may answer the question of the existence that we posed in the beginning.

Proposition 2.7. *Let S be a nonparabolic line congruence whose middle surface $P(u,v)$ is different from its middle envelope $M(u,v)$. Then,*

if (A), S is isotropic, there exist infinitely many triplets of mutually orthogonal line congruences S, S', S'' with common middle surface $P(u,v)$, while

if (B), S is not isotropic, there exists exactly one such triplet S, S', S''. In this case the straight lines of S', S'' are directed by the tangent vectors to the spherical images of the S-principal ruled surfaces of S.

3. Case with S isotropic

In this section we assume that:

(A) *S is isotropic and S, S', S'' are triply orthogonal line congruences with common middle surface $P(u,v)$.*

S' and S'' are defined by the functions φ and $\varphi + \frac{\pi}{2}$ respectively, which satisfy (2.4). The straight lines of S' and S'', via the relation (1.17), are directed by the vectors

$$\overline{e}'_3 = \cos\varphi\,\overline{e}_1 + \sin\varphi\,\overline{e}_2 \quad \text{and} \quad \overline{e}''_3 = -\sin\varphi\,\overline{e}_1 + \cos\varphi\,\overline{e}_2\,,$$

respectively. Moreover, according to the conclusion of the Remark 2.1 of [2, p. 127]

$$\overline{e}'_3 \neq \pm\frac{\overline{MP}}{|\overline{MP}|}, \quad \overline{e}''_3 \neq \pm\frac{\overline{MP}}{|\overline{MP}|}. \tag{3.1}$$

For the line congruence S' the relations (1.15)–(1.17) and (1.20)–(1.29) are valid. In addition, taking into account (2.6) the formulae (1.33)–(1.37) for S' become

$$m' = 0, \tag{3.2}$$

$$l' = l, \tag{3.3}$$

$$n' = \frac{1}{D}(a\sin\varphi - b\cos\varphi), \tag{3.4}$$

$$a' = \frac{l}{D}(\Delta\cos\varphi - \Gamma\sin\varphi), \tag{3.5}$$

$$b' = -\frac{l}{D}\,. \tag{3.6}$$

On the other hand, substituting the value $\varphi + \frac{\pi}{2}$ into relations (1.15)–(1.17) and (1.25)–(1.28) and denoting the elements of S'' by double accentuated symbols, we obtain

$$\overline{e}''_1 = \overline{e}_3\,, \quad \overline{e}''_2 = \cos\varphi\,\overline{e}_1 + \sin\varphi\,\overline{e}_2\,, \quad \overline{e}''_3 = -\sin\varphi\,\overline{e}_1 + \cos\varphi\,\overline{e}_2, \tag{3.7}$$

$$\omega''_{31} = \sin\varphi\,\omega_{31} - \cos\varphi\,\omega_{32}\,, \quad \omega''_{32} = -\Delta\omega_{31} + \Gamma\omega_{32}\,, \tag{3.8}$$

$$\omega''_{31} \wedge \omega''_{32} = D^*\omega_{31} \wedge \omega_{32}, \tag{3.9}$$

where

$$D^* := \Gamma\sin\varphi - \Delta\cos\varphi. \tag{3.10}$$

We assume $D^* \neq 0\ \forall (u,v) \in G$. That makes ω''_{31}, ω''_{32} linearly independent. Then, in view of (1.28), (2.6), (3.10) the relations (1.33)–(1.37) for S'' take the form

$$m'' = 0, \tag{3.11}$$

$$l'' = l, \tag{3.12}$$

$$n'' = \frac{1}{D^*}(b \sin\varphi + a\cos\varphi), \tag{3.13}$$

$$a'' = \frac{lD}{D^*}, \tag{3.14}$$

$$b'' = -\frac{l}{D^*}. \tag{3.15}$$

Evidently, from (3.3) and (3.12) we have

$$l'' = l' = l. \tag{3.16}$$

Besides, from the equations (3.4) and (3.13), using (1.28), (3.10) and the assertion *(iii)* of Proposition 2.4, which is equivalent to

$$a\Gamma + b\Delta = 0, \tag{3.17}$$

we find

$$n'' = n'. \tag{3.18}$$

We shall prove the following

Proposition 3.1. *Given (A), the relations*

$$k' = k'', \quad h' = h'', \quad 2z' = 2z'' \tag{3.19}$$

hold true, where k', k'' are the curvatures, h', h'' the mean curvatures and $2z'$, $2z''$ the limit distances of the line congruences S', S'', respectively.

Proof: Applying the formulae (1.9), (1.10) to S', S'', it follows

$$k' = l'n' - m'^2, \quad 2h' = l' + n', \tag{3.20}$$

$$2z' = 2\sqrt{h'^2 - k'}, \tag{3.21}$$

and

$$k'' = l''n'' - m''^2, \quad 2h'' = l'' + n'', \tag{3.22}$$

$$2z'' = 2\sqrt{h''^2 - k''}, \tag{3.23}$$

respectively. From (3.20)–(3.23), making use of (3.2), (3.11), (3.16) and (3.18), we immediately derive the relations (3.19). □

Since S is isotropic, the relations

$$z = 0, \quad k = l^2, \quad h = l \tag{3.24}$$

are valid and, by means of (3.20), (3.22) and (3.16), Proposition 3.1 implies

Corollary 3.1. *If one of S', S'' is normal, then*

(i) *the other one is also normal and*

(ii) $k' = k'' = -k$.

Corollary 3.2. *If one of S', S'' is isotropic, then*

(i) *the other one is isotropic too,*

(ii) $k' = k'' = k$ *and*

(iii) $h' = h'' = h$.

Moreover, according to Proposition 2.2 of [2, p. 128] S', S'' are not parabolic. Thus, a direct consequence of Proposition 3.1 is the following

Corollary 3.3. *Both of S', S'' are simultaneously elliptic or hyperbolic line congruences.*

Let P be an arbitrary point on the middle surface $P(u, v)$ and g', g'' the straight lines of S', S'', respectively, that pass through P. The lines g', g'' are perpendicular and lie on the middle plane of S at P.

We assume firstly that S', S'' *are hyperbolic:* For the angle $2\omega'$ (resp. $2\omega''$) between the focal planes of S' (resp. S''), which contains g' (resp. g''), the formula

$$\cos 2\omega' = \frac{h'}{\sqrt{h'^2 - k'}} \quad (\text{resp. } \cos 2\omega'' = \frac{h''}{\sqrt{h''^2 - k''}})$$

holds true [1, p. 154]. Thus, via Proposition 3.1, it follows

Corollary 3.4. *The angles $2\omega'$, $2\omega''$ between the corresponding focal planes of S' and S'' are equal.*

We denote by F'_i, F''_i, $i = 1, 2$, the focal points of g', g'' respectively. According to Proposition 3.1 the focal distances of S', S'' are equal. Hence

Corollary 3.5. *If (A) is valid and S', S'' are hyperbolic line congruences, the focal points F'_i, F''_i, $i = 1, 2$, define a square, whose center is the common midpoint P. The length of its diagonals is equal to the common focal distance $2\sqrt{-k'}$ and its area is $-2k'$.*

Further, taking into account Corollary 3.1, we get

Corollary 3.6. *If one of the line congruences S', S'' is normal, the length of each diagonal of the square $F'_1F''_1F'_2F''_2$ equals $2\sqrt{k}$ and its area is $2k$, where k is the curvature of S.*

Next, we assume that S', S'' *are nonisotropic:* Similarly, using Proposition 3.1, for the limit points Z'_i, Z''_i, $i = 1, 2$, of g', g'' respectively, we deduce:

Corollary 3.7. *If (A) is valid and S', S'' are nonisotropic line congruences, the limit points Z'_i, Z''_i, $i = 1, 2$, are the vertices of a square with the common midpoint P at the center. The length of its diagonals is $2z'$ and its area equals to $2z'^2$.*

Let us now discuss the middle envelopes $\overline{OM'} = M'(u, v)$ and $\overline{OM''} = M''(u, v)$ of S' and S'' respectively. Applying the formula (1.12), we have for S' and S''

$$\overline{OM'} = \overline{OP} - a'\overline{e}'_1 - b'\overline{e}'_2, \tag{3.25}$$

$$\overline{OM''} = \overline{OP} - a''\overline{e}''_1 - b''\overline{e}''_2. \tag{3.26}$$

We insert the right-hand sides of (1.15), (1.16), (3.5), (3.6) (resp. (3.7), (3.14), (3.15)) into (3.25) (resp. (3.26)) and, using (3.10), it turns out

$$\overline{M'P} = \frac{l}{D}(-\sin\varphi\, \overline{e}_1 + \cos\varphi\, \overline{e}_2 - D^*\overline{e}_3), \tag{3.27}$$

$$\overline{M''P} = \frac{l}{D^*}(-\cos\varphi\,\overline{e}_1 - \sin\varphi\,\overline{e}_2 + D\overline{e}_3). \tag{3.28}$$

Obviously, $\overline{M'P} \neq \overline{0}$, $\overline{M''P} \neq \overline{0}$.

Proposition 3.2. *Given (A), the following properties are valid:*

(i) $k = -\langle \overline{M'P}, \overline{M''P} \rangle$, $k' = -\langle \overline{MP}, \overline{M'P} \rangle$, $k'' = -\langle \overline{MP}, \overline{M''P} \rangle$.

(ii) $\overline{M'M''}$, $\overline{MP}$ *are orthogonal.*

(iii) The points P, M, M', M'' *are coplanar.*

Proof: (i) Since S is isotropic and S', S'' are orthogonal to S with $P(u,v)$ as middle surface, it is an immediate consequence of the Proposition 2.1 of [2, p. 128]

$$k' = -\langle \overline{MP}, \overline{M'P} \rangle, \quad k'' = -\langle \overline{MP}, \overline{M''P} \rangle. \tag{3.29}$$

In addition, using (3.27), (3.28) and the second relation of (3.24) we obtain

$$k = -\langle \overline{M'P}, \overline{M''P} \rangle. \tag{3.30}$$

(ii) From Proposition 3.1 we know that $k' = k''$, which, by virtue of (3.29), can be written

$$\langle \overline{MP}, \overline{M''P} - \overline{M'P} \rangle = 0 \tag{3.31}$$

or equivalently

$$\langle \overline{MP}, \overline{M'M''} \rangle = 0. \tag{3.32}$$

Hence the property *(ii)* holds true.

(iii) Taking into account the relations (3.27), (3.28),

$$\overline{MP} = a\overline{e}_1 + b\overline{e}_2, \quad a^2 + b^2 \neq 0, \tag{3.33}$$

and applying (1.28), (3.10), we find

$$\det\left(\overline{MP}, \overline{M'P}, \overline{M''P}\right) = -\frac{l^2}{DD^*}(a\Gamma + b\Delta). \tag{3.34}$$

Then, in view of (3.17), it follows

$$\det\left(\overline{MP}, \overline{M'P}, \overline{M''P}\right) = 0, \tag{3.35}$$

which proves *(iii)*. □

The following is evident because of (3.30):

Corollary 3.8. *If (A) holds, then*

$$\left|\overline{M'M''}\right|^2 = \left|\overline{M'P}\right|^2 + \left|\overline{M''P}\right|^2 + 2k.$$

Besides, if one of S', S'' is also isotropic, via the relations $k = k' = k''$ (Corollary 3.2) and the assertions *(i)*, *(ii)* of Proposition 3.2, we derive

$$\langle \overline{MP}, \overline{M'M''} \rangle = 0, \quad \langle \overline{M'P}, \overline{MM''} \rangle = 0, \quad \langle \overline{M''P}, \overline{MM'} \rangle = 0.$$

Hence we obtain

Corollary 3.9. *If (A) is valid and one of* S', S'' *is isotropic, then* P *is the orthocenter of the triangle* $MM'M''$.

4. Case with S nonisotropic

Hereafter, we suppose that

(B) *S is a nonisotropic line congruence and S', S'' are two line congruences, whose straight lines are directed by the tangent vectors to the spherical images of the S-principal ruled surfaces of S.*

According to Proposition 2.7, the line congruences S, S', S'' are triply orthogonal and have the same middle surface $P(u, v)$.

We consider again, without loss of generality, that the S-principal ruled surfaces of S are the parameter surfaces $\omega_{31} = 0$, $\omega_{32} = 0$, i.e., $m \equiv 0$. Then, the unit vectors in the direction of the straight lines of S', S'' are

$$\overline{e}'_3 = \overline{e}_1, \quad \overline{e}''_3 = \overline{e}_2 \tag{4.1}$$

and correspond to the values $\varphi = 0$ and $\varphi = \frac{\pi}{2}$ respectively (§2(B)). Here (3.1) is replaced by

$$\overline{e}_1 \neq \pm \frac{\overline{MP}}{|\overline{MP}|}, \quad \overline{e}_2 \neq \pm \frac{\overline{MP}}{|\overline{MP}|},$$

which, in view of (3.33), are equivalent to

$$b \neq 0 \text{ and } a \neq 0. \tag{4.2}$$

Substituting $\varphi = 0$ (resp. $\varphi = \frac{\pi}{2}$) into (1.16), (1.24)–(1.26) and using (1.6), (1.15), we obtain for S' (resp. S''):

$$\overline{e}'_1 = \overline{e}_3, \quad \overline{e}'_2 = -\overline{e}_2, \tag{4.3}$$

$$\omega'_{12} = -\omega_{32}, \quad \omega'_{31} = -\omega_{31}, \quad \omega'_{32} = -\omega_{12} = -q\omega_{31} + \widetilde{q}\omega_{32}, \tag{4.4}$$

$$\omega'_{31} \wedge \omega'_{32} = -\widetilde{q}\omega_{31} \wedge \omega_{32}, \tag{4.5}$$

(resp.

$$\overline{e}''_1 = \overline{e}_3, \quad \overline{e}''_2 = \overline{e}_1, \tag{4.6}$$

$$\omega''_{12} = \omega_{31}, \quad \omega''_{31} = -\omega_{32}, \quad \omega''_{32} = -\omega_{12} = -q\omega_{31} + \widetilde{q}\omega_{32}, \tag{4.7}$$

$$\omega''_{31} \wedge \omega''_{32} = -q\omega_{31} \wedge \omega_{32}). \tag{4.8}$$

The differential forms ω'_{31}, ω'_{32} (resp. ω''_{31}, ω''_{32}) are linearly independent iff

$$\widetilde{q} \neq 0 \text{ (resp. } q \neq 0) \quad \forall (u, v) \in G. \tag{4.9}$$

Note that for the constant values $\varphi = 0$ and $\varphi = \frac{\pi}{2}$, from (1.20) we get

$$\Gamma = \widetilde{q}, \quad \Delta = q\,. \tag{4.10}$$

Similarly, inserting $\varphi = 0$ (resp. $\varphi = \frac{\pi}{2}$) into (1.28), in view of (4.10), we obtain

$$D = -\widetilde{q} \text{ (resp. } D = -q\,). \tag{4.11}$$

Proposition 4.1. *Let k', h', $2z'$ and k'', h'', $2z''$ be the curvature, the mean curvature and the limit distance of S' and S'', respectively. If (B) holds true, then*

(i) $2z = 2|h' - h''|, \quad 2z' = 2|h'' - h|, \quad 2z'' = 2|h - h'|,$

(ii) $k'k'' = k(h' + h'' - h)^2,$

(iii) $z \leq z' + z''$. *The equality is valid iff* $h' \leq h \leq h''$ *or* $h'' \leq h \leq h'$.

Proof: The relations (1.33)–(1.35) for $\varphi = 0$ and, by means of (2.7), reduce to

$$m' = 0, \tag{4.12}$$

$$l' = l, \tag{4.13}$$

$$n' = \frac{b}{\widetilde{q}}. \tag{4.14}$$

Similarly, for $\varphi = \frac{\pi}{2}$ they become

$$m'' = 0, \tag{4.15}$$

$$l'' = n, \tag{4.16}$$

$$n'' = -\frac{a}{q}. \tag{4.17}$$

Taking into account (2.9), from (4.14) and (4.17) we find

$$n'' = n'. \tag{4.18}$$

We consider that for the corresponding invariants of S', S'' the formulae (3.20)–(3.23) are valid. Thus, applying (4.13), (4.16) and (4.18) to the second formulae of (3.20) and (3.22), we get

$$2h' = l + n', \quad 2h'' = n + n'. \tag{4.19}$$

Besides, making use of (4.12), (4.13), (4.15), (4.16) and (4.18), the first formula of (3.20) and (3.22) may be written

$$k' = l\, n', \quad k'' = n\, n'. \tag{4.20}$$

Finally, from (3.21), (3.23), via (4.19), (4.20), we obtain

$$2z' = |l - n'|, \quad 2z'' = |n - n'|. \tag{4.21}$$

(i) To prove the first relation of *(i)*, it suffices to use the first formula of (1.10), (2.7) and (4.19). Similarly, by virtue of (4.19) and the second equation of (1.9), we derive from (4.21) the other two formulae of *(i)*.

(ii) On account of (2.7), the first equation of (1.9) becomes

$$k = l\, n. \tag{4.22}$$

Thus, from (4.20), (4.22) it follows

$$k'k'' = kn'^2. \tag{4.23}$$

By using now (4.19) and the second of the formulae (1.9), we deduce

$$(h' + h'' - h)^2 = n'^2 \tag{4.24}$$

and consequently

$$k'k'' = k(h' + h'' - h)^2. \tag{4.25}$$

(iii) Because of (4.21), we have

$$2z = |l - n| = |l - n' - n + n'| \leq |l - n'| + |n - n'| = 2z' + 2z''.$$

Suppose now that

$$z = z' + z''. \tag{4.26}$$

According to *(i)*, the relation (4.26) takes the form

$$|h' - h''| = |h'' - h| + |h - h'|. \tag{4.27}$$

That happens iff $(h'' - h)(h - h') \geq 0$, i.e.,

$$h' \leq h \leq h'' \text{ or } h'' \leq h \leq h'. \tag{4.28}$$

Conversely, if (4.28) is valid, the equality (4.26) obviously holds true. □

Remark 4.1: By a process similar to that in Proposition 4.1, it turns out that *the following relations are valid:*

(i) $kk' = k''(h + h' - h'')^2$, $kk'' = k'(h + h'' - h')^2$,

(ii) $z' \leq z'' + z$, $z'' \leq z + z'$.

In addition, $z' = z'' + z$ *(resp.* $z'' = z + z'$*) iff* $h \leq h' \leq h''$ *or* $h'' \leq h' \leq h$ *(resp.* $h \leq h'' \leq h'$ *or* $h' \leq h'' \leq h$*).*

Since, according to (4.2), $b \neq 0$, from (4.23) and (4.14) we derive

$$(h' + h'' - h)^2 = \frac{b^2}{\widetilde{q}^2} > 0\,. \tag{4.29}$$

Thus, (4.25) leads to

Corollary 4.1. *If S is a hyperbolic line congruence, then one of S', S'' is also hyperbolic, while the other one is elliptic. If S is elliptic, then both of S', S'' are simultaneously elliptic or hyperbolic.*

We further assume that S is normal ($h = l + n \equiv 0$) but not the normal line congruence of a minimal surface $P(u, v)$ which has been excluded in §1. Using the relations (4.20), (1.10) together with claims (i) of Proposition 4.1, we conclude:

Corollary 4.2. *When S is a normal line congruence, the following properties hold true.*

(i) $k' = -k''$,

(ii) $z' = |h''|$, $z'' = |h'|$,

(iii) $k = -(h' - h'')^2$.

Again, we shall focus on the middle envelopes $\overline{OM'} = M'(u, v)$ and $\overline{OM''} = M''(u, v)$ of S' and S'', respectively. Taking into account (4.10), (4.11), for the values $\varphi = 0$ and $\varphi = \frac{\pi}{2}$, the relations (1.36), (1.37) turn into

$$a' = -\frac{nq}{\widetilde{q}}\,, \quad b' = \frac{n}{\widetilde{q}}\,, \tag{4.30}$$

$$a'' = \frac{l\widetilde{q}}{q}\,, \quad b'' = \frac{l}{q} \tag{4.31}$$

respectively. Then, by substituting (4.1), (4.3), (4.6), (4.30) and (4.31) into (3.25), (3.26), it follows

$$\overline{M'P} = -\frac{n}{\widetilde{q}}\,\overline{e}_2 - \frac{nq}{\widetilde{q}}\,\overline{e}_3, \tag{4.32}$$

$$\overline{M''P} = \frac{l}{q}\,\overline{e}_1 + \frac{l\widetilde{q}}{q}\,\overline{e}_3\,. \tag{4.33}$$

On account of (3.33), (4.32), (4.33) and using (4.14), (4.17), (4.20), (4.22) and (4.18), we deduce:

Proposition 4.2. *Let (B) hold true. Then, the formulae*

$$k = -\left\langle\overline{M'P},\,\overline{M''P}\right\rangle, \quad k' = -\left\langle\overline{MP},\,\overline{M''P}\right\rangle, \quad k'' = -\left\langle\overline{MP},\,\overline{M'P}\right\rangle \tag{4.34}$$

are valid.

An immediate consequence of the first equation of (4.34) is

Corollary 4.3. *The following relation holds true:*

$$|\overline{M'M''}|^2 = |\overline{M'P}|^2 + |\overline{M''P}|^2 + 2k \tag{4.35}$$

Thus, in case that S is hyperbolic (resp. elliptic) the angle $\widehat{M'PM''}$ is acute (resp. obtuce).

Moreover, by virtue of (3.33), (4.32), (4.33), we find

$$\det(\overline{MP},\,\overline{M'P},\,\overline{M''P}) = -\frac{l\,n}{q\widetilde{q}}(\widetilde{q}a + qb), \tag{4.36}$$

which, according to (2.9), vanishes. Therefore

Proposition 4.3. *The points P, M, M', M'' are coplanar.*

Remark 4.2: We see that Proposition 4.3, Corollary 4.3 and the first equation of Proposition 4.2 hold true for both cases (A) and (B), as well as for the case that S is the normal congruence of a minimal middle surface [2, p. 132]. Hence in any case a triplet of mutually orthogonal line congruences S, S', S'' with common middle surface has the above properties.

Notice that considering the relation (3.30), we may have *a new geometric interpretation for the curvature of each, nonparabolic, line congruence S*. In particular, if S is isotropic, whenever there are infinitely many pairs S', S'' consisting with S the preceding triplet S, S', S'' (§2, Proposition 2.3), the relation (3.30) does not depend on the function $\varphi(u,v)$. In other words, it is independent of the choice of the pair S', S''. Thus, we come to

Proposition 4.4. *Let S be a nonparabolic line congruence and $P(u,v)$ its middle surface. If S', S'' are line congruences sharing the same middle surface $P(u,v)$ and orthogonal to S and to each other, then for the curvature k of S the relation*

$$k = -\left\langle\overline{M'P},\,\overline{M''P}\right\rangle$$

is valid. Here $M'(u,v)$, $M''(u,v)$ stand for the middle envelopes of S', S'', respectively.

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