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# Two Approaches to Geometrography

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**Abstract.** Geometrography studies the complexity of ruler-and-compass constructions. According to our knowledge, so far the only remarkable work in this field is by É. Lemoine. We survey his method and present another method based on statistical modelling. We compare these methods by studying the complexity of certain constructions of a perpendicular to a line and a regular pentagon. A purpose of this paper is to show that geometrography has potential to enrich geometry and graphics education in school. The latter, statistical, method also provides tools for a more advanced analysis of error propagation through geometric transformation.

*Key Words:* geometric constructions, complexity, statistical modelling

*MSC 2010:* 51M15

## 1. Introduction

Ruler-and-compass constructions, which only less than a century ago constituted a core of mathematical education, are today almost fully ignored in teaching geometry. Indeed, nowadays even in upper secondary school, geometry education merely builds on students' intuitive and informal knowledge of their environment and space rather than on strict deductive reasoning. The focus is preferably on applications than on the logical structure of geometric knowledge. Also most ICT based dynamic geometry environments are designed to support teaching and learning problem solving skills related to, for example, measuring, rather than general deductive thinking. For a more detailed discussion, see, e.g., [15].

As an example of this global and rapid development, we mention that in Finland, the PISA winning country, only a few decades ago upper secondary school's matriculation examination regularly contained ruler-and-compass constructions. For example, in 1951 students were asked to draw a circle sector whose arc is 72° and area is equal to the area of a given circle.

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Today Finnish students are introduced to, if any, only a few trivial constructions like angle bisection etc.

There are several obvious reasons why ruler-and-compass constructions do not deserve so much emphasis in mathematics education today. On the other hand, there are also grounds why these constructions, in our opinion, warrant a little more attention than they nowadays are paid to. For example, studying the three classical unsolvable Greek construction problems still invokes even amateurs' interest toward mathematics in general. Also the value of these constructions to the fine arts is timelessly consistent. There are, however, also more modern and concrete possibilities to increase the relevance of ruler-and-compass constructions to mathematics education — and eventually for all mathematical sciences.

For instance, computer scientists have already noticed that [2, p. 85] "... the algorithmic problems occurring in geometric reasoning have also an enormous scientific appeal [to computer science]. In the past few years, geometric reasoning problems have provoked a whole spectrum of new algorithmic techniques." In other words, any suitable approach to study, for example, the complexity of algorithms in the context of geometry would arguably enrich mathematics teaching in school. Further, in school, statistics is often introduced only as an application of mathematics and the fact that both of these autonomous disciplines promote each other is not usually noticed. We believe that any relevant way to diversify students' knowledge in statistics is welcome, too. Last but not least, the consistent problem of mathematics education in universities is related to the training of prospective researchers: how to find reasonable ways to introduce students with the process of doing research in mathematics already at the level of basic studies? As some recent articles, e.g., [6] and [17] verify, it is still possible to achieve new results relying only on undergraduate mathematics. We see that participation into yielding such results is authentic and, hence, a motivating context for such activities. Therefore, we want to speak for surveying topics in the intersections of different areas of mathematical sciences – such as geometric constructions and statistics – since new, but only moderately challenging, observations are most probably found in these junctions.

We shall show that *geometrography*, i.e., studying the complexity of ruler-and-compass constructions, can answer the above demands. Especially, the new approach to geometrography to-be-introduced below provides in upper secondary school and in undergraduate mathematics education — particularly in mathematics teacher education — a natural setting to discuss many modern mathematical phenomena including algorithmic thinking and program design, in a classical geometric context. Indeed, this approach requires that geometric constructions are described precisely in an algorithmic form for computer software; the syntax and method of this description are yet simple and easy to learn. Further, the approach enables learners to study at a general level relations between abstract mathematical structures and their models, and how statistics can support mathematical reasoning and problem solving. Finally, it also provides a complementary approach to graphics education.

We shall first bring up the almost forgotten work of É. Lemoine [14] which seems to be so far the only comprehensive treatise in this field. We will survey his approach in Section 2. In Section 3, we will present an alternative, statistical, approach due to S. Mustonen [18]. Its main idea is that, in practice, the ruler and compass are never placed exactly right. The errors so arising are modelled statistically. The measure of the final error, defined in a certain reasonable way, is computed using simulation experiments. It turns out that the result expresses, besides the inaccuracy, also the complexity of the construction.

We will illustrate both approaches and compare them by studying first a construction of a perpendicular to a line in Section 4. Second, we will consider three constructions of a regular pentagon in Section 5. The results of this section motivate us to study correlations of certain complexity measures for the constructions of the regular pentagon. We will do it in Section 6.

The rise of CAD and the arithmetic of floating point numbers behind it have vitalized research on error propagation of geometric constructions in ICT environments, e.g., [11, 21, 23, 24, 25]. In Section 7, we will shortly compare Mustonen's approach and that of Hu and Wallner [11].

## 2. Lemoine's geometrography

A trivial way (see, e.g., [13]) to define the complexity of a ruler-and-compass construction is simply to count how many (straight) lines and circles must be drawn. So, if $l$ lines and $c$ circles are required, then the complexity is $l+c$. But these operations are geometric, not arithmetic, and as such incommensurable. Experience makes us to think that drawing a circle is more complicated than drawing a line, yet also the opposite can be argued, see [5]. Anyway, they are here considered equally complex and, if necessary elsewhere, this matter may be taken into account by defining also the "symbol" $lL+cC$ where $L$ and $C$ are indeterminates expressing relative complexities of drawing a line and a circle, respectively.

This definition, nevertheless, ignores the fact that single operations with the same instrument may have different complexities. For example, to draw a line through two given points is obviously more complicated than to do it through one given point. Further, to draw a circle with a given center and radius is clearly more complex than to do it with a given center only. Therefore a deeper analysis is needed and that has been done by Lemoine [14] (see also [3, 4, 7, 12, 20]). He distinguishes the following basic operations.

- L1. Place the ruler through a given point.
- L2. Draw a line.
- C1. Place one leg of the compass on a given point.
- C2. Place one leg of the compass on an indeterminate point of a given line.
- C3. Draw a circle.

If the numbers of these operations in a construction are respectively $l_1$, $l_2$, $c_1$, $c_2$, $c_3$, then the *complexity* of the construction is defined by $l_1+l_2+c_1+c_2+c_3$. Actually Lemoine calls it *simplicity* (and so do also the other references above) but we find "complexity" more appropriate. Since the summands are again incommensurable, it is more precise to define (as Lemoine does) the *symbol* of the construction by $l_1L_1+l_2L_2+c_1C_1+c_2C_2+c_3C_3$. Here the indeterminates $L_1$, $L_2$, $C_1$, $C_2$, $C_3$ express relative complexities of the respective operations. For brevity, we denote the symbol by the 5-tuple $(l_1,l_2,c_1,c_2,c_3)$.

These basic operations can be trivially partitioned into two sets according to the instrument but there is also another natural partition. In L1, C1 and C2 an instrument *is placed*, while in L2 and C3 it *is used*. We may plausibly think that the only factor effecting on accuracy is how exactly the instrument is placed; once it is done, then "arbitrarily thin" and (subject to the placing) "arbitrarily exact" lines and circles can be drawn. This leads us to define the *inexactness* of the construction by $l_1+c_1+c_2$. Actually Lemoine calls it *exactitude* (and so do also [4, 7, 12], while [3] and [20] do not mention this concept) but we find "inexactness" more appropriate. It turns out that in practice there is no essential difference between complexity and inexactness. We will discuss this in Section 6.

Lemoine's geometrography is not widely known today, neither did it get much attention even in his time. O'Connor and Robertson [20] write as follows.

> "He presented these results to the meeting of the Association Française pour l'Avancement des Sciences in 1888 at Oran in Algeria. One would have to say that these results were not thought to be particularly interesting by mathematicians at the meeting and there has been a similar lack of interest ever since.
> It is perhaps worth asking what is interesting in mathematics. Why are the results of Lemoine not found interesting? All I [Robertson] can add is that I agree with the mathematicians of the time who preferred a construction with a large number of easily understood steps to a shorter one with sophisticated, rather obscure, steps. Let me add that I do find Lemoine's results on symmedians of a triangle to be very interesting and beautiful!"

As a red herring, let it be told that the symmedian of a triangle is the reflected image of a median in the corresponding angle bisector. The symmedians intersect in the same point, called the *Lemoine point* (or the *Grebe point* or the *symmedian point*). According to Honsberger ([9, p. 53], see also [26]), this point is "one of the crown jewels of modern geometry". Further, Eves [4, p. 438] says that Lemoine's presentation of this point "can be claimed to have seriously started the modern study of the geometry of the triangle" .

We cite also Hudson [12, pp. 7–8] on Lemoine's geometrography. In the introduction of her book, she describes the section on geometrography as follows.

> "The idea of the last section of the same chapter is to make a numerical estimate of the length of a construction, by reckoning up all the different operations with ruler and compasses that it requires, so as to be able to say which is the shortest of different solutions of the same problem. This plan of 'giving marks' is little more than a pastime, and the scale of marking is very arbitrary; but Lemoine's book on Geometrography deserves to be better known, and some account of the matter is given here in the hope of introducing more English readers to his original work."

In our opinion, also Hudson's book [12] deserves to be known better.

## 3. Mustonen's geometrography

S. Mustonen's interest in the statistical accuracy of ruler-and-compass constructions originates from the circle-squaring problem. When comparing various approximate constructions, the degree of the approximation cannot be taken into account if Lemoine's measure is used, but Mustonen's model observes this effect reasonably as explained in [18, Section 4]. In this model, also the addition of incommensurable entities is avoided.

Thinking statistically, it is natural to assume that the actual point where the instrument is placed follows a two-dimensional normal distribution around the correct point.

In case of the intersection of two lines, the question arises how the distribution depends on the angle between them. The simplest answer is that the angle has no effect and hence the variance of the placement is a constant $\sigma^2$ in all directions. But if the angle is small, then one is tempted to think that the variation depends on the angle and, with a fixed angle, is greatest along the bisector and smallest in the orthogonal direction. For two such models, we refer to [18, Sections 5.1 and 5.2].

Hudson [12, pp. 109–111] also discusses the "ill-defined intersection" of two lines (and, respectively, that of a line and a circle, and of two circles). In these cases, she suggests "to economize the chance of error" by performing an additional construction yielding the same point as the intersection of two lines (or, respectively, of a line and a circle, or of two circles) meeting "at a moderate angle". So, the models depending on the angle between the intersecting lines are problematic in the sense that the accuracy of the construction can be seemingly improved by Hudson's method. Nevertheless, computer experiments on various constructions of a regular pentagon [18, p. 43] show that in practice there is not much difference between the results of the constant-variance model and the two non-constant-variance models mentioned above. Furthermore, computer experiments on the intersection of two random lines [18, Section 5.5] attest similarity between the results of these models. In other words, Hudson's method does not after all provide significant improvement to the accuracy. This allows us to assume that the point actually obtained follows the two-dimensional normal distribution with the correct point as mean and the variance $\sigma^2$ in all directions.

We also assume (likewise, e.g., in [3]) that the compass is "modern", i.e., "noncollapsing" and so permits transferring of lengths by a single operation. Mustonen's geometrography is now made up of the following basic operations.

- B1. Set a point with given coordinates.
- B2. Draw a line through two given points.
- B3. Set a point on a given line with given coordinates.
- B4. Measure the distance between two given points.
- B5. Draw a circle with a given center and a given radius.
- B6. Draw a circle with a given center and going through a given point.
- B7. Set the intersection point of two given lines.
- B8. Set the intersection point of a given circle and a given line that is closer to a given point.
- B9. Set the intersection point of two given circles that is closer to a given point.

These operations are also commands for the GEOM program which operates in the versatile computing environment *Survo*. For more information on this program and the original and international cult status bearing computing system, see [19].

Concerning a solution of a given ruler-and-compass construction, the GEOM program has the following objectives.

- a. To describe it.
- b. To plot its figure.
- c. To compute its Lemoine complexity and inexactness.
- d. To compute its statistical accuracy by simulations.

So, we have two different approaches to geometrography. The traditional way, due to Lemoine, counts numbers of certain operations, while the new way, due to Mustonen, is statistical. Both approaches obviously measure the intricacy of ruler-and-compass constructions but from different points of view. Therefore it is interesting to ask how similar results these geometrographies give. A natural subquestion is to compare the Lemoine complexity and inexactness with each other. Further, one may also hypothesize that the more operations are needed, the more inaccurate the construction will be, and conversely. In Section 6, we will make these comparisons for certain exemplar constructions, but before that, we consider the drawing of a perpendicular to a line in Mustonen's geometrography.

## 4. Example: a perpendicular to a line

We describe by Mustonen's geometrography the ordinary construction for drawing a perpendicular from a given point $P$ to the line going through given points $P_1$ and $P_2$. The 5-tuples $(l_1, l_2, c_1, c_2, c_3)$ in certain steps indicate the Lemoine symbol of that step. Note that the drawing of the "initial" points, lines and circles does not effect on the Lemoine symbol. Also note that certain initial points, lines and circles are kept fixed (without random variation) in the simulation experiments. The GEOM code is to be found in [18, pp. 4–5] but its steps are the following.

S1. Set $P_1 = (4, 4)$.

S2. Set $P_2 = (4, 6)$.

S3. Draw the line $L_1$ through $P_1$ and $P_2$.

S4. Set $P = (5, 4)$.

S5. $(0, 0, 1, 0, 1)$. Draw the circle $C_1$ with center $P$ and radius 2. The fact that the radius need not be exactly 2, explains why the Lemoine symbol is not $(0, 0, 2, 0, 1)$. This remark concerns also S8 and S9 where the radius is the same as here.

S6. Set $Q_1$ as the intersection point of $C_1$ and $L_1$ that is closer to (4,2).

S7. Set $Q_2$ as the intersection point of $C_1$ and $L_1$ that is closer to (4,6).

S8. $(0, 0, 1, 0, 1)$. Draw the circle $C_2$ with center $Q_1$ and radius 2.

S9. $(0, 0, 1, 0, 1)$. Draw the circle $C_3$ with center $Q_2$ and radius 2.

S10. Set $P'$ as the intersection point of $C_2$ and $C_3$ that is closer to $(3, 4)$.

S11. $(2, 1, 0, 0, 0)$. Draw the line $L$ through $P$ and $P'$.

S12. The desired perpendicular is $L$.

S13. Set $M$ as the intersection point of $L$ and $L_1$. (This is actually not necessary for the construction itself but it is needed for simulation experiments.)

By adding the 5-tuples we get the Lemoine symbol $(2, 1, 3, 0, 3)$, complexity $2+1+3+0+3 = 9$, and inexactness $2 + 3 + 0 = 5$.

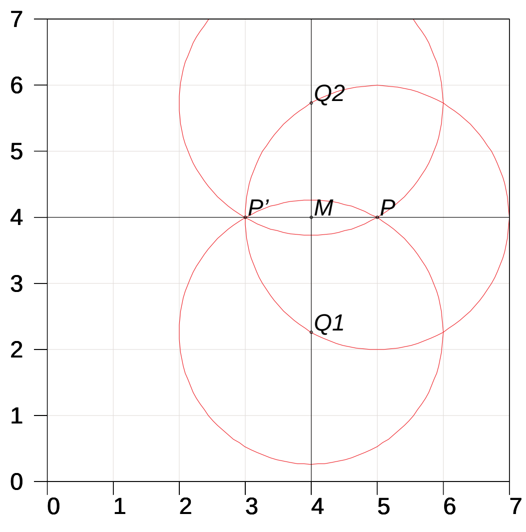


Figure 1: The perpendicular from a given point.

In order to study inaccuracy experimentally, we first subtract from the actual $y$-coordinate of $M$ the correct value 4 obtaining $d = y - 4$. If $d$ appears to be unbiased, i.e., the mean $\mathcal{E}(d)$ is close to zero, then we can measure the inaccuracy by the standard deviation $\mathcal{D}(d)$. We also compute the actual direction angle of $L$, that is

$$\alpha = \arctan \frac{y' - y}{x' - x}$$

where $(x, y)$ and $(x', y')$ are the actual coordinates of $P$ and $P'$, respectively. Since the correct value is $\alpha = 0$, the bias of $\alpha$ is $\mathcal{E}(\alpha)$. If it is close to zero, then we can also use $\mathcal{D}(\alpha)$ as a measure of inaccuracy.

In an experiment, this construction was repeated for 100000 times. In each replicate, when drawing lines and circles, points locating them were replaced by their sampled values using the standard deviation $\sigma = 0.001$. For more details, see [18, Section 2]. The results were $\mathcal{E}(d) = -0.000003$, $\mathcal{E}(\alpha) = 0.000001$, $\mathcal{D}(d) = 0.000957$, $\mathcal{D}(\alpha) = 0.000958$. So $d$ and $\alpha$ seem to be unbiased and $\mathcal{D}(d)$ is close to $\sigma$. (That also $\mathcal{D}(\alpha)$ is close to $\sigma$ is only a meaningless coincidence.)

A natural attempt to improve the accuracy is to increase the radii of the circles. Let the radius of $C_1$ be $r$ and let that of $C_2$ and $C_3$ be $2r$. Results for certain values of $r$ are tabulated in [18, p. 7].

In general, $\mathcal{D}(d)$ remains quite firmly close to $\sigma = 0.001$. For example, if $r = 3$, then $\mathcal{D}(d) = 0.000901$ which seems to be near to the minimum. However, although we in this case get a good $M$, this does not hold for all points of $L$. For example, if the comparison point is $(2, 4)$ instead of $M = (4, 4)$, then the corresponding inaccuracy is 0.00111, about as much above $\sigma$ as $\mathcal{D}(d)$ is below. As $r \to \infty$, it appears that $\mathcal{D}(d) \to \sigma$. For large values of $r$, there seems to be a tendency that $\mathcal{D}(d)$ increases as $r$ increases.

The inaccuracy $\mathcal{D}(\alpha)$ is significant for "moderate" values of $r$ but seems to decrease as $r$ increases, tending to zero as $r \to \infty$. For example, $\mathcal{D}(\alpha) = 0.000365$ if $r = 3$ and $\mathcal{D}(\alpha) = 0.00000129$ if $r = 1000$.

These observations make us conclude that there is no uniformly best way to gain the maximum accuracy for drawing a perpendicular. However, a nominal level for the accuracy can be achieved for any given point of the perpendicular by using sufficiently large circles. Since it is not reasonable to restrict the size of the drawing area, we establish the following interpretation. A perpendicular from a point $P$ to a line $L$ can be drawn without any error in the direction angle, and its distance from the true perpendicular follows from the error distribution of the placement of $P$.

This convention corresponds to using the *square ruler* in constructing the perpendicular. Doing so it is assumed that the first edge of the square ruler can be positioned strictly parallel to $L$ and the second (orthogonal) edge through $P$ like a standard ruler. Interestingly, Lemoine [14] uses this instrument as an extra tool in some of his constructions.

We can also adopt similar conventions concerning drawing parallel lines, angle bisector and finding the midpoint of a line segment. Hence, in order to simplify more complicated constructions, the following composite operations have been added to the list of the operations of the GEOM program.

B10. Draw from a given point a perpendicular to a given line.

B11. Draw from a given point a parallel to a given line.

B12. Draw the bisector of an angle between two given lines.

B13. Determine the midpoint of the line segment defined by two given points.

## 5. Example: a regular pentagon

Our next problem is to inscribe a regular pentagon in a unit circle. In [18], the complexity of various such constructions is studied both in Lemoine's and Mustonen's geometrography. We present three of them in detail but first we recall the *Carlyle circle.* Let us consider the equation $x^2 - sx + p = 0$ where $s$ and $p$ are given signed lengths. Set $A = (0, 1)$ and $B = (s, p)$. Draw the circle with diameter $AB$. If it meets the $x$-axis at $H_1 = (x_1, 0)$ and $H_2 = (x_2, 0)$, then $x_1$ and $x_2$ are the solutions of this equation. This circle has a role also in the case of complex root, see [3, p. 100]. For more details, we refer to [3, Section 2].

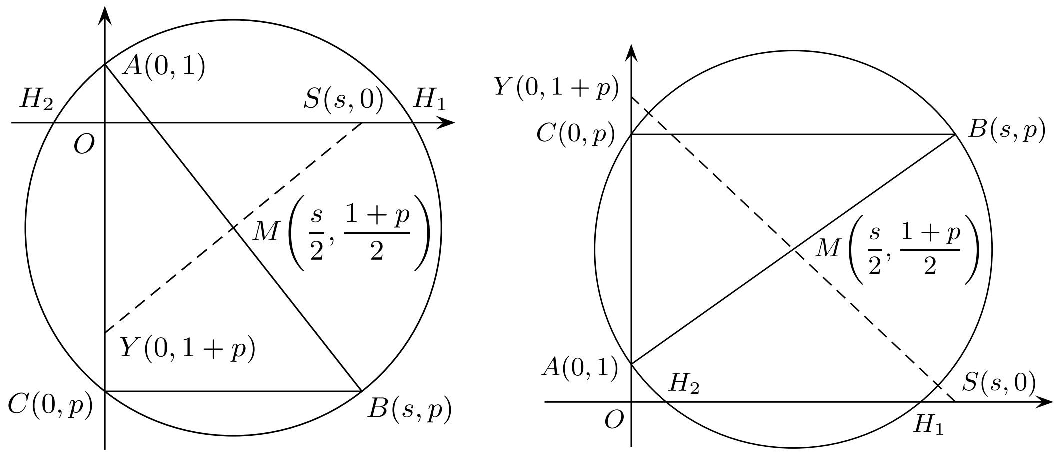


Figure 2: Carlyle circles; $p \leq 0$ (left) and $p > 0$ (right).

DeTemple [3]. The vertices of the pentagon are the solutions of the equation $z^5 = 1$; they are $z_k = e^{\frac{2k\pi i}{5}}$ where $k = 0, 1, 2, 3, 4$. Since $z_1 + z_4 = 2\cos\frac{2\pi}{5}$ and $z_2 + z_3 = 2\cos\frac{4\pi}{5}$ are the roots of the equation $x^2 + x - 1 = 0$, the following construction is easily justified [3, p. 101]. We describe it stepwise by using Mustonen's geometrography. For more details and the GEOM code, see [18, Section 3.1].

S1. Set $O = (0, 0)$.
S2. Set $Q = (-1, 0)$.
S3. Draw the circle $C_1$ with center $O$ and radius 1.
S4. Draw the line $L_1$ through $O$ and $Q$.
S5. Set $P_0$ as the intersection point of $C_1$ and $L_1$ that is closer to (1,0).
S6. Draw the line $L_2$ from $O$ perpendicular to $L_1$.
S7. Set $A$ as the intersection point of $C_1$ and $L_2$ that is closer to (0,1).
S8. $(2, 1, 2, 0, 1)$. Set $M$ as the midpoint of $QO$. Since the circle $C_1$ already exists, only another circle (with center $Q$ and radius 1) is needed.
S9. $(0, 0, 2, 0, 1)$. Draw the circle $C_2$ with center $M$ and going through $A$.
S10. Set $H_1$ as the intersection point of $C_2$ and $L_1$ that is closer to $(1, 0)$.
S11. Set $H_2$ as the intersection point of $C_2$ and $L_1$ that is closer to $(-1.5, 0)$.
S12. $(0, 0, 3, 0, 1)$. Draw the circle $C_3$ with center $H_1$ and radius 1.
S13. $(0, 0, 1, 0, 1)$. Draw the circle $C_4$ with center $H_2$ and radius 1. The radius 1, measured in S12, need not be measured again. So placing only one compass leg is enough.

S14. Set $P_1$ as the intersection point of $C_1$ and $C_3$ that is closer to $(0, 1)$.

S15. Set $P_2$ as the intersection point of $C_1$ and $C_4$ that is closer to $(-1, 0.5)$.

S16. Set $P_3$ as the intersection point of $C_1$ and $C_4$ that is closer to $(-1, -0.5)$.

S17. Set $P_4$ as the intersection point of $C_1$ and $C_3$ that is closer to $(0, -1)$.

S18. The desired pentagon is $P_0P_1P_2P_3P_4$.

This construction has the Lemoine symbol (2,1,8,0,4), complexity $2+1+8+0+4 = 15$, and inexactness $2 + 8 + 0 = 10$.

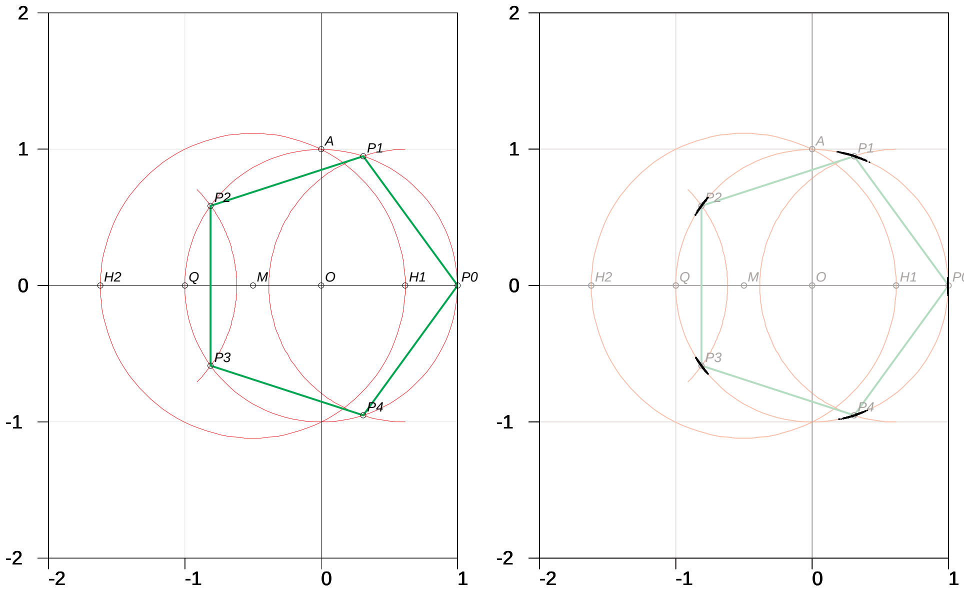


Figure 3: DeTemple's construction of a regular pentagon.

To study the complexity in MUSTONEN's geometrography, let $e$ be the actual length of $P_0P_1$. If the mean $\mathcal{E}(e)$ is close to the correct value $\frac{1}{2}\sqrt{10-2\sqrt{5}} = 1.1755705...$, then $e$ can be considered unbiased, and we can measure the inaccuracy of this side by the standard deviation

$$\epsilon = \mathcal{D}(e).$$

Furthermore, let $d^2$ be the squared total error of the locations of $P_0, P_1, P_2, P_3$ and $P_4$. We measure the inaccuracy of the entire construction by

$$\delta = (\tfrac{1}{5}\mathcal{D}(d^2))^{\frac{1}{2}}.$$

In an experiment [18], this construction was repeated for 1000000 times with $\sigma = 0.001$. The result $\mathcal{E}(e) = 1.1755738$ showed that we can consider $e$ unbiased. Similar results were obtained also in other experiments, and so we can use the measure $\epsilon$ throughout. The results were $\epsilon = 0.00227$ and $\delta = 0.00276$. It seems that $\delta$ is closely proportional to $\sigma$, cf. [18, Table 2].

The right-hand side of Figure 3 shows the distribution of vertices in an experiment when 1000 replicates with $\sigma = 0.01$ were generated and plotted. The variance of $P_1$ and $P_2$ is greater than that of $P_3$ and $P_4$.

Ptolemy (see, e.g., [1], [3]). First, determine $H_1$. The construction is otherwise similar to DeTemple's but it is based on the fact that a side of a regular pentagon is the larger part when cutting a diagonal in the golden ratio. Second, take $AH_1$ as the length of the sides of the pentagon. The Lemoine symbol is $(2, 1, 8, 0, 5)$. So the complexity $2+1+8+0+5 = 16$ is one unit greater than DeTemple's, whereas the inexactness $2 + 8 + 0 = 10$ is equal. For more details and the GEOM code, see [18, Section 3.2].

Again, this construction was repeated in [18] for 1000000 times with $\sigma = 0.001$. The results $\epsilon = 0.00271$ and $\delta = 0.00373$ are worse than DeTemple's. Especially $P_2$ and $P_3$ are more inaccurate since they are constructed relying on $P_1$ and $P_4$ which already contain inaccuracies.

Labelle [13]. For brevity, we describe Labelle's construction of a regular pentagon directly by stating its steps. For more details and the GEOM code, we refer to [18, Section 3.3].

S1. Set $O = (0, 0)$.
S2. Set $P_0 = (1, 0)$.
S3. Draw the circle $C_1$ with center $O$ and going through $P_0$.
S4. Draw the line $L_1$ through $O$ and $P_0$.
S5. Set $Q$ as the intersection point of $C_1$ and $L_1$ that is closer to $(-1, 0)$.
S6. $(0, 0, 2, 0, 1)$. Draw the circle $C_2$ with center $P_0$ and going through $Q$.
S7. $(0, 0, 2, 0, 1)$. Draw the circle $C_3$ with center $Q$ and going through $O$.
S8. Set $A$ as the intersection point of $C_2$ and $C_3$ that is closer to $(-1, 1)$.
S9. $(0, 0, 2, 0, 1)$. Draw the circle $C_4$ with center $A$ and going through $P_0$.
S10. Set $B$ as the intersection point of $C_2$ and $L_1$ that is closer to $(3, 0)$.
S11. Set $C$ as the intersection point of $C_2$ and $C_4$ that is closer to $(1, 2)$.
S12. $(0, 0, 2, 0, 1)$. Draw the circle $C_5$ with center $B$ and going through $C$.
S13. Set $D$ as the intersection point of $C_2$ and $C_4$ that is closer to $(-1, -1)$.
S14. $(0, 0, 2, 0, 1)$. Draw the circle $C_6$ with center $B$ and going through $D$.
S15. Set $P_1$ as the intersection point of $C_1$ and $C_5$ that is closer to $(0, 1)$.
S16. Set $P_2$ as the intersection point of $C_1$ and $C_6$ that is closer to $(-1, 0.5)$.
S17. Set $P_3$ as the intersection point of $C_1$ and $C_6$ that is closer to $(-1, -0.5)$.
S18. Set $P_4$ as the intersection point of $C_1$ and $C_5$ that is closer to $(0, -1)$.
S19. The desired pentagon is $P_0P_1P_2P_3P_4$.

The Lemoine symbol is now $(0, 0, 10, 0, 5)$. Hence the complexity $0 + 0 + 10 + 0 + 5 = 15$ and inexactness $0 + 10 + 0 = 10$ are the same as for DeTemple's construction.

An experiment in [18] repeating Labelle's construction for 1000000 times with $\sigma = 0.001$ gave $\epsilon = 0.00245$ and $\delta = 0.00282$. From this viewpoint, Labelle's construction is almost as good as DeTemple's.

## 6. Comparing complexity characteristics

In addition to the above constructions, we consider those due to Hirano [8], Richmond [22] and *"Mohr-Mascheroni"* [10]. Their GEOM codes (and figures) are found in [18, Sections 3.4–3.6]. The lastly mentioned is of greater interest because it uses only the compass. Actually, this construction is probably not due to Mohr and Mascheroni, but it is named

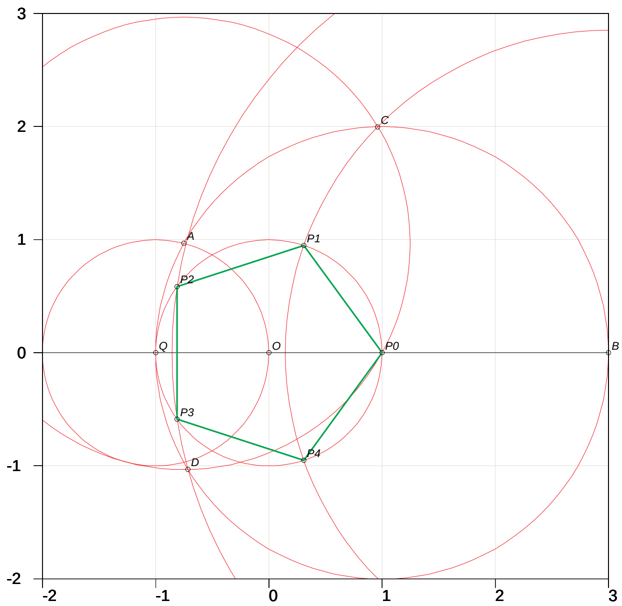


Figure 4: Labelle's construction of a regular pentagon.

here after them because the Mohr-Mascheroni theorem (see, e.g., [16, Section 3]) states that all ruler-and-compass constructions can be done by the compass only. Furthermore, Mustonen [18, Section 9] improved Richmond's construction. We include also this one, called *Richmond-Mustonen* construction, in our comparison of constructions. All in all, we thus have seven constructions of a regular pentagon. For each, we compute the Lemoine complexity $C$, Lemoine inexactness $I$, and $\epsilon$ and $\delta$ defined above. We also compute the "trivial complexity" $T = C - I$, that is, the sum of how many lines and how many circles must be drawn. These characteristics are the following.

| *Construction* | $C$ | $I$ | $T$ | $\epsilon$ | $\delta$ |
|---|---|---|---|---|---|
| DeTemple | 15 | 10 | 5 | 0.00227 | 0.00276 |
| Ptolemy | 16 | 10 | 6 | 0.00271 | 0.00373 |
| Labelle | 15 | 10 | 5 | 0.00245 | 0.00282 |
| Hirano | 19 | 13 | 6 | 0.00405 | 0.00390 |
| Richmond | 45 | 26 | 19 | 0.00219 | 0.00478 |
| Richmond-Mustonen | 33 | 20 | 13 | 0.00219 | 0.00329 |
| "Mohr-Mascheroni" | 34 | 21 | 13 | 0.00520 | 0.00902 |

The correlations between these characteristics are given below. (In computing them, the values of $\epsilon$ and $\delta$ were expressed with four significant digits.)

| *Variable* | $C$ | $I$ | $T$ | $\epsilon$ | $\delta$ |
|---|---|---|---|---|---|
| $C$ | 1 | 0.997 | 0.996 | 0.075 | 0.521 |
| $I$ | 0.997 | 1 | 0.986 | 0.130 | 0.550 |
| $T$ | 0.996 | 0.986 | 1 | 0.008 | 0.482 |
| $\epsilon$ | 0.075 | 0.130 | 0.008 | 1 | 0.818 |
| $\delta$ | 0.521 | 0.550 | 0.482 | 0.818 | 1 |

The correlations between $C$ and $I$, and, respectively, $C$ and $T$, are perceptibly large. Also the very large correlation between $I$ and $T$ is interesting. The correlation between $C$ and $I$ has also been computed [18, p. 2] based on the data of 89 constructions taken from Lemoine [14]. The result 0.996 confirms the strong parallelism between these characteristics. (It is somewhat surprising that Lemoine does not at all discuss this matter.)

Also the correlations between $\delta$ and, respectively, $C$, $I$, $T$ are significant but the parallelism is not total as we see if we compare Richmond's and "Mohr-Mascheroni" constructions. Then $C$, $I$ and $T$ prefer "Mohr-Mascheroni" while $\delta$ prefers Richmond (and so does also $\epsilon$). On the other hand, Richmond-Mustonen is better than "Mohr-Mascheroni" according to all the characteristics except $T$ which judges them equal.

The correlations between $\epsilon$ and, respectively, $C$, $I$, $T$ are very small. The explanation is that $\epsilon$ in fact measures the inaccuracy of a different construction: to draw a line segment whose length is that of the pentagon. If all the constructions were modified to draw only this line segment, then these correlations would be larger. The large correlation between $\epsilon$ and $\delta$ is easy to understand.

## 7. Another view to inaccuracy

Hu and Wallner [11] consider error propagation through affine transformations in the Euclidean plane. They define that a *fat point* $A$ is the set $A$ of points. However, we can as well interpret that the fat point $A$ is a random variable with uniform distribution on the set $A$. Assuming that the distribution is two-dimensionally normal, we result in Mustonen's geometrography. Then, for example, in an initial placing of the compass, the mean of this variable is the correct point and the standard deviation is a given positive number. If the random variable describes a point obtained by a geometric construction, these characteristics are found experimentally.

To be more precise, distributions of such random variables fill the whole plane unless they are somehow cut. If this causes problems, then it is natural to restrict to disks where the variable point lies with probability larger than a given number. If the distributions are thereafter uniformized in these disks, we end up in a complete analogy with [11]. However, it may be more useful to remain in the original normal distributions or cut distributions, because more primal information then preserves.

Anyway, if a *fat line* consists of, for example, a set of parallel ordinary lines whose distribution we know in the dimension of a normal vector, or of a set of ordinary lines coinciding a point and a set with a known distribution, we can statistically control the error propagation through reflections in this fat line. The same is true for the intersections of such fat lines and, hence, we should also be able to determine, e.g., the area in which an image of a set in affine transformations generated by such fat lines should lie with a certain probability. As a matter of fact, combining the aspects of [11] and [18] seems to yield a whole set of research questions – which should yet be reasonably solvable already on the basis of undergraduate studies in mathematics.

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