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Flexible Octahedra in the Projective Extension of the Euclidean 3-Space

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Abstract. In this paper we complete the classification of flexible octahedra in the projective extension of the Euclidean 3-space. If all vertices are Euclidean points then we get the well known Bricard octahedra. All flexible octahedra with one vertex on the plane at infinity were already determined by the author in the context of self-motions of TSSM manipulators with two parallel rotary axes. Therefore we are only interested in those cases where at least two vertices are ideal points. Our approach is based on Kokotsakis meshes and reducible compositions of two four-bar linkages.

Key Words: Flexible octahedra, Kokotsakis meshes, Bricard octahedra

MSC 2010: 53A17, 52B10

1. Introduction

A polyhedron is said to be *flexible* if its spatial shape can be changed continuously due to changes of its dihedral angles only, i.e., every face remains congruent to itself during the flex.

1.1. Review

In 1897 R. Bricard [5] proved that there are three types of flexible octahedra[1] in the Euclidean 3-space E^3. These so-called *Bricard octahedra* are as follows:

Type 1: All three pairs of opposite vertices are symmetric with respect to a common line.

Type 2: Two pairs of opposite vertices are symmetric with respect to a common plane which passes through the remaining two vertices.

Type 3: For a detailed discussion of this type we refer to [23]. We only want to mention that these flexible octahedra possess two flat poses.

[1] No face degenerates into a line and no two neighboring faces coincide permanently during the flex.

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Due to *Cauchy's theorem* [8] all three types are non-convex; they even have self-intersections.

As I.K. Sabitov [20] proved the *Bellows Conjecture*, every flexible polyhedron in E^3 keeps its volume constant during the flex. Especially for *Bricard octahedra* it was shown by R. Connelly [9] that all three types have a vanishing volume. Connelly [10] also constructed the first flexible polygonal embedding of the 2-sphere into E^3. A simplified flexing sphere was presented by K. Steffen [26]. Note that both flexing spheres are compounds of *Bricard octahedra*.

R. Alexander [1] has shown that every flexible polyhedron in E^3 preserves its total mean curvature during the flex (see also I. Pak [19, p. 264]). Recently V. Alexandrov [2] showed that the *Dehn invariants* (cf. [12]) of any *Bricard octahedron* remain constant during the flex and that the *Strong Bellows Conjecture* (cf. [11]) holds true for the *Steffen polyhedron*.

H. Stachel [24] proved that all *Bricard octahedra* are also flexible in the hyperbolic 3-space. Moreover Stachel [22] presented flexible cross-polytopes in the Euclidean 4-space.

1.2. Related work and overview

As already mentioned all types of flexible octahedra in E^3 were firstly classified by R. Bricard [5]. His proof presented in [6] is based on properties of a *strophoidal* spatial cubic curve. In 1978 R. Connelly [9] sketched a further algebraic method for the determination of all flexible octahedra in E^3. H. Stachel [21] presented a new proof which uses mainly arguments from projective geometry beside the converse of *Ivory's Theorem*, which limits this approach to flexible octahedra with finite vertices.

A. Kokotsakis [14] discussed the flexible octahedra as special cases of a sort of meshes named after him (see Fig. 1). As recognized by the author in [18] Kokotsakis' very short and elegant proof for *Bricard octahedra* is also valid for type 3 in the projective extension $E^\star$ of E^3 if no two opposite vertices are ideal points. Stachel [23] also proved the existence of flexible octahedra of type 3 with one vertex at infinity and presented their construction.

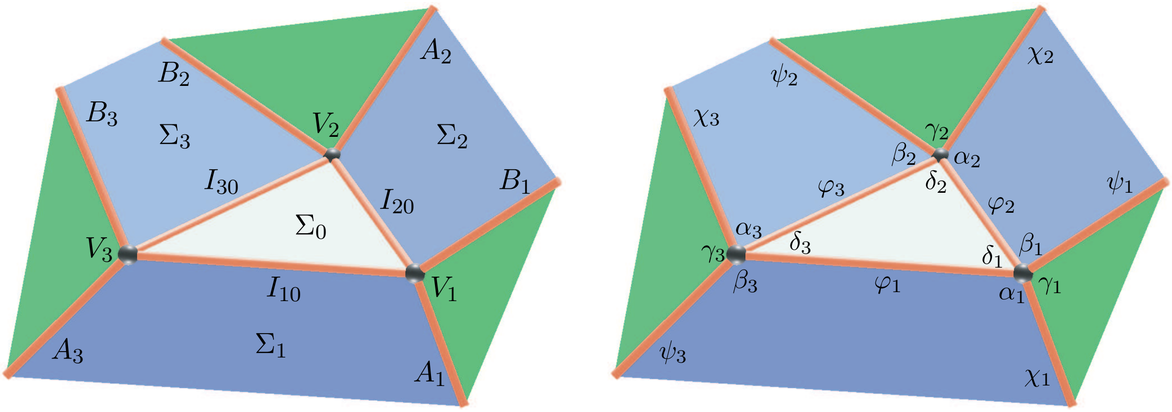

Figure 1: A *Kokotsakis mesh* is a polyhedral structure consisting of a n-sided central polygon $\Sigma_0 \in E^3$ surrounded by a belt of polygons in the following way: Each side I_{i0} of Σ_0 is shared by an adjacent polygon Σ_i, and the relative motion between cyclically consecutive neighbor polygons is a spherical coupler motion. Here a *Kokotsakis* mesh for $n = 3$ is given which determines an octahedron. φ_i, χ_i and ψ_i denote the angles enclosed by neighboring faces.

Moreover the author determined in [18] all flexible octahedra where one vertex is an ideal point.

Up to recent, there are no proofs for BRICARD's famous statement known to the author which enclose the projective extension of E^3 although these flexible structures attracted many prominent mathematicians; e.g., G.T. BENNETT [3], W. BLASCHKE [4], O. BOTTEMA [7], H. LEBESGUE [13] and W. WUNDERLICH [27]. The presented article together with [18] closes this gap.

Our approach is based on a kinematic analysis of *Kokotsakis meshes* as the composition of spherical coupler motions given by STACHEL [25], which is repeated in more detail in Section 2. In Section 3 we determine all flexible octahedra where no pair of opposite vertices are ideal points. The remaining special cases are treated in Section 4.

2. Notation and related results

We inspect a *Kokotsakis mesh* for $n = 3$ (see Fig. 1). If we intersect the planes adjacent to the vertex V_i with a sphere S^2 centered at this point, the relative motion Σ_i/Σ_{i+1} (mod 3) is a spherical coupler motion.

2.1. Transmission by a spherical four-bar mechanism

We start with the analysis of the first spherical four-bar linkage $\mathcal{C}$ with the frame link $I_{10}I_{20}$ and the coupler A_1B_1 according to H. STACHEL [25] (see Figs. 1 and 2).

We set $\alpha_1 := \overline{I_{10}A_1}$ for the spherical length of the driving arm, $\beta_1 := \overline{I_{20}B_1}$ for the output arm, $\gamma_1 := \overline{A_1B_1}$, and $\delta_1 := \overline{I_{10}I_{20}}$. We may suppose $0 < \alpha_1, \beta_1, \gamma_1, \delta_1 < \pi$.

The coupler motion remains unchanged when A_1 is replaced by its antipode $\overline{A}_1$ and at the same time α_1 and γ_1 are substituted by $\pi - \alpha_1$ and $\pi - \gamma_1$, respectively. The same holds for the other vertices. When I_{10} is replaced by its antipode $\overline{I}_{10}$, then also the sense of orientation changes, when the rotation of the driving bar $I_{10}A_1$ is inspected from outside of S^2 either at I_{10} or at $\overline{I}_{10}$.

We use a cartesian coordinate frame with I_{10} on the positive x-axis and $I_{10}I_{20}$ in the xy-plane such that I_{20} has a positive y-coordinate (see Fig. 2). The input angle φ_1 is measured between $I_{10}I_{20}$ and the driving arm $I_{10}A_1$ in mathematically positive sense. The output angle $\varphi_2 = \sphericalangle\, \overline{I}_{10}I_{20}B_1$ is the oriented exterior angle at vertex I_{20}. As given in [25] the constant spherical length γ_1 of the coupler implies the following equation

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0 \tag{1}$$

with $t_i = \tan(\varphi_i/2)$, $c_{11} = 4\,\mathrm{s}\alpha_1\,\mathrm{s}\beta_1 \neq 0$,

$$\begin{aligned} c_{00} &= N_1 - K_1 + L_1 + M_1, \quad & c_{02} &= N_1 + K_1 + L_1 - M_1, \\ c_{20} &= N_1 - K_1 - L_1 - M_1, \quad & c_{22} &= N_1 + K_1 - L_1 + M_1, \end{aligned} \tag{2}$$

$$K_1 = \mathrm{c}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{s}\delta_1, \quad L_1 = \mathrm{s}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{s}\delta_1, \quad M_1 = \mathrm{s}\alpha_1\,\mathrm{s}\beta_1\,\mathrm{c}\delta_1, \quad N_1 = \mathrm{c}\alpha_1\,\mathrm{c}\beta_1\,\mathrm{c}\delta_1 - \mathrm{c}\gamma_1. \tag{3}$$

In this equation s and c are abbreviations for the sine and cosine function, respectively, and the spherical lengths α_1, β_1 and δ_1 are signed.

Note that the biquadratic equation Eq. (1) describes a *2-2-correspondence* between points A_1 on the circle $a_1 = (I_{10}; \alpha_1)$ and B_1 on $b_1 = (I_{20}; \beta_1)$ (see Fig. 2). Moreover, this 2-2-correspondence only depends on the ratio of the coefficients $c_{22} : \cdots : c_{00}$ (cf. Lemma 1 of [16]).

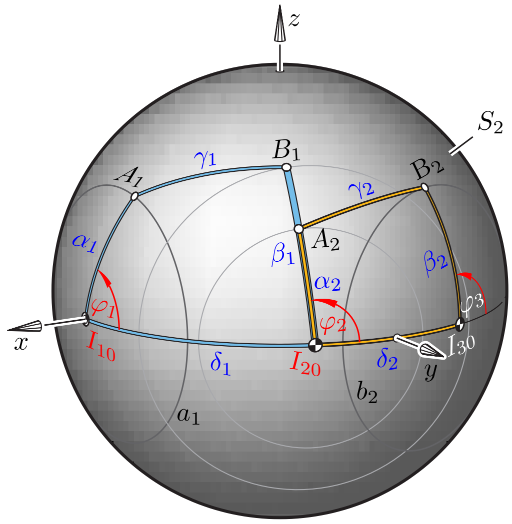

Figure 2: Composition of the two spherical four-bars $I_{10}A_1B_1I_{20}$ and $I_{20}A_2B_2I_{30}$ with spherical side lengths $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1, 2$ (courtesy H. Stachel)

2.2. Composition of two spherical four-bar linkages

Now we use the output angle φ_2 of the first four-bar linkage $\mathcal{C}$ as input angle of a second four-bar linkage $\mathcal{D}$ with vertices $I_{20}A_2B_2I_{30}$ and consecutive spherical side lengths α_2, γ_2, β_2 and δ_2 (Fig. 2). The two frame links are assumed in aligned position. In the case $\sphericalangle I_{10}I_{20}I_{30} = \pi$ the spherical length δ_2 is positive, otherwise negative. Analogously, a negative α_2 expresses the fact that the aligned bars $I_{20}B_1$ and $I_{20}A_2$ are pointing to opposite sides. Changing the sign of β_2 means replacing the output angle φ_3 by $\varphi_3 - \pi$. The sign of γ_2 has no influence on the transmission and therefore we can assume $\gamma_2 > 0$ without loss of generality (w.l.o.g.).

Due to (1) the transmission between the angles φ_1, φ_2 and the output angle φ_3 of the second four-bar with $t_3 := \tan(\varphi_3/2)$ can be expressed by the two biquadratic equations

$$c_{22}t_1^2t_2^2 + c_{20}t_1^2 + c_{02}t_2^2 + c_{11}t_1t_2 + c_{00} = 0, \quad d_{22}t_2^2t_3^2 + d_{20}t_2^2 + d_{02}t_3^2 + d_{11}t_2t_3 + d_{00} = 0. \tag{4}$$

The d_{ik} are defined by equations analogue to Eqs. (2) and (3).

The author already determined in [17] all cases where the relation between the input angle φ_1 of the arm $I_{10}A_1$ and the output angle φ_3 of $I_{30}B_2$ is reducible and where additionally at least one of these components produces a transmission which equals that of a single spherical four-bar linkage $\mathcal{R}$ (= spherical quadrangle $I_{10}I_{r0}B_3A_3$). These so-called reducible compositions with a spherical coupler component can be summarized as follows (cf. Theorem 5 and 6 of [17]):

Theorem 1. *If a reducible composition of two spherical four-bar linkages with a spherical coupler component is given, then it is one of the following cases:*

(a) *One spherical coupler is a spherical isogram which happens in one of the following four cases:*

$$c_{00} = c_{22} = 0, \quad d_{00} = d_{22} = 0, \quad c_{20} = c_{02} = 0, \quad d_{20} = d_{02} = 0,$$

(b) *the spherical couplers are forming a spherical focal mechanism which is analytically given for $F \in \mathbb{R} \setminus \{0\}$ by*

$$\begin{gathered} c_{00}c_{20} = Fd_{00}d_{02}, \quad c_{22}c_{02} = Fd_{22}d_{20}, \\ c_{11}^2 - 4(c_{00}c_{22} + c_{20}c_{02}) = F[d_{11}^2 - 4(d_{00}d_{22} + d_{20}d_{02})], \end{gathered} \tag{5}$$

(c) $c_{22} = c_{02} = d_{00} = d_{02} = 0$ *resp.* $d_{22} = d_{20} = c_{00} = c_{20} = 0$,

(d) $c_{20} = Ad_{02}$, $c_{22} = Ad_{22}$, $c_{02} = Bd_{22}$, $c_{00} = Bd_{02}$, $d_{00} = d_{20} = 0$, $d_{02}d_{22} \neq 0$ *resp.* $d_{02} = Ac_{20}$, $d_{22} = Ac_{22}$, $d_{20} = Bc_{22}$, $d_{00} = Bc_{20}$, $c_{00} = c_{02} = 0$, $c_{20}c_{22} \neq 0$ *with* $A \in \mathbb{R} \setminus \{0\}$ *and* $B \in \mathbb{R}$.

2.3. Geometric aspects of Theorem 1

Spherical isogram:

Now we point out the geometric difference between the two spherical isograms given by $c_{00} = c_{22} = 0$ and $c_{20} = c_{02} = 0$, respectively.

(i) It was already shown in [25] that $c_{00} = c_{22} = 0$ is equivalent to the conditions $\beta_1 = \alpha_1$ and $\delta_1 = \gamma_1$ which determines a spherical isogram.

(ii) $c_{20} = c_{02} = 0$ is equivalent to the conditions $\beta_1 = \pi - \alpha_1$ and $\delta_1 = \pi - \gamma_1$ (cf. [17]). Note that the couplers of both isograms have the same movement because we get item (ii) by replacing either I_{10} or I_{20} of item (i) by its antipode.

Moreover the cosines of opposite angles in spherical isograms (of both types) are equal (cf. [14, §8]).

Spherical focal mechanism:

Here also two cases can be distinguished:

(i) In [16] it was shown that the characterization of the spherical focal mechanism given in Theorem 1 is equivalent to the condition

$$\mathrm{s}\alpha_1\,\mathrm{s}\gamma_1 : \mathrm{s}\beta_1\,\mathrm{s}\delta_1 : (\mathrm{c}\alpha_1\,\mathrm{c}\gamma_1 - \mathrm{c}\beta_1\,\mathrm{c}\delta_1) = \mathrm{s}\beta_2\,\mathrm{s}\gamma_2 : \mathrm{s}\alpha_2\,\mathrm{s}\delta_2 : (\mathrm{c}\alpha_2\,\mathrm{c}\delta_2 - \mathrm{c}\beta_2\,\mathrm{c}\gamma_2).$$

Moreover in this case always $\mathrm{c}\chi_1 = -\mathrm{c}\psi_2$ holds with $\chi_1 = \sphericalangle\, I_{10}A_1B_1$ and $\psi_2 = \sphericalangle\, I_{30}B_2A_2$.

(ii) But in the algebraic characterization of the spherical focal mechanism (5) also a second possibility is hidden, namely:

$$\mathrm{s}\alpha_1\,\mathrm{s}\gamma_1 : \mathrm{s}\beta_1\,\mathrm{s}\delta_1 : (\mathrm{c}\alpha_1\,\mathrm{c}\gamma_1 - \mathrm{c}\beta_1\,\mathrm{c}\delta_1) = \mathrm{s}\beta_2\,\mathrm{s}\gamma_2 : \mathrm{s}\alpha_2\,\mathrm{s}\delta_2 : (\mathrm{c}\beta_2\,\mathrm{c}\gamma_2 - \mathrm{c}\alpha_2\,\mathrm{c}\delta_2).$$

In this case always $\mathrm{c}\chi_1 = \mathrm{c}\psi_2$ holds. Note that we get this case from the first one by replacing either I_{30} or I_{10} by its antipode.

3. The general case of flexible octahedra in $E^\star$

In this section we assume that no two opposite vertices of the octahedron are ideal points. As a consequence there exists at least one face of the octahedron where all three vertices are in E^3. This face corresponds to Σ_0 in Fig. 1. Now the *Kokotsakis mesh* for $n = 3$ is flexible if and only if the transmission of the composition of the two spherical four-bar linkages $\mathcal{C}$ and $\mathcal{D}$ equals that of a single spherical four-bar linkage $\mathcal{R}$ with $I_{r0} = I_{30}$.

It was shown in [18] that the items (c) and (d) of Theorem 1 as well as the spherical focal mechanism of type (i) do not yield a solution for this problem. Moreover it should be noted that the composition of two spherical isograms of any type also forms a spherical focal mechanism as Eq. (5) holds, and then the spherical four-bar linkage $\mathcal{R}$ also has to be a spherical isogram. This implies the following necessary conditions already given in [18]:

Lemma 1. *If an octahedron in the projective extension of E^3 is flexible where no two opposite vertices are ideal points, then its spherical image is a composition of spherical four-bar linkages $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{R}$ of the following type:*

1. *$\mathcal{C}$ and $\mathcal{D}$, $\mathcal{C}$ and $\mathcal{R}$ as well as $\mathcal{D}$ and $\mathcal{R}$ are forming spherical focal mechanism of type* (ii)*,*
2. *$\mathcal{C}$ and $\mathcal{D}$ are forming a spherical focal mechanism of type* (ii) *and $\mathcal{R}$ is a spherical isogram,*
3. *$\mathcal{C}$, $\mathcal{D}$ and hence also $\mathcal{R}$ are spherical isograms.*

3.1. Flexible octahedra of type 3 with vertices at infinity

In contrast to the proof for type 1 and type 2 A. Kokotsakis showed without any limiting argumentation with respect to $E^\star$ that the third case of Lemma 1 corresponds with the *Bricard octahedron* of type 3 if no two opposite vertices are ideal points. Therefore the following angle conditions given in [14] also have to hold in our case:

$$\delta_i = \gamma_i, \quad \alpha_i = \beta_i, \quad \delta_3 + \gamma_3 = \pi, \quad \alpha_i + \beta_i = \pi \quad \text{for } i = 1, 2, \tag{6}$$

where the angles are denoted according to Fig. 1(b). For $\beta_1 + \alpha_2 = \pi$ and $\beta_2 + \alpha_3 = \pi$ two of the remaining 3 vertices are ideal points. These conditions already imply $\beta_3 + \alpha_1 = \pi$ and therefore also the third remaining vertex has to be an ideal point. Moreover all three vertices are collinear which follows directly from the existence of the two flat poses. This already yields a contradiction (cf. footnote 1). Together with Theorem 2 of [18] this proves the following statement:

Theorem 2. *A flexible octahedron of type 3 with one finite face can have not more than one vertex at infinity.*

Remark 1. For the construction of these flexible octahedra see H. Stachel [23].

3.2. Flexible octahedra with a face or an edge at infinity

We can even generalize the observation that if three vertices are ideal points then they have to be collinear in order to get a flexible structure:

Theorem 3. *In the projective extension of E^3 there do not exist flexible octahedra where one face is at infinity if the other 3 vertices are finite.*

Proof: Given are the finite vertices V_1, V_2, V_3 and the three ideal points U_1, U_2, U_3 (see Fig. 3(a)). W.l.o.g. we can assume that the face $[V_1, U_2, U_3]$ is fixed. Since $[U_1, U_2, U_3]$ is a face of the octahedron, also the direction of U_1 is fixed.

Now the points V_2 and V_3 have to move on circles about their footpoints F_2 and F_3 with respect to $[V_1, U_3]$ and $[V_1, U_2]$, respectively. Note that F_2, V_2, V_3, F_3 can also be seen as an RSSR mechanism (cf. [15]) with intersecting rotary axes in V_1. We split up the vector V_2V_3 in a component $\mathbf{u}$ in direction U_1 and a component orthogonal to it. Now the octahedron is flexible if the length of the component $\mathbf{u}$ is constant during the RSSR motion. It can easily be seen that a spherical motion of $[V_2, V_3]$ with center V_1 and this distance property can only be a composition of a rotation about a parallel to $[V_2, V_3]$ through V_1 and a rotation about $[V_1, U_1]$.

Then we consider one of the two possible configurations where V_1, V_2, V_3, U_1 are coplanar. Due to our considerations the velocity vectors of V_2 and V_3 with respect to the fixed system are orthogonal to this plane as they can only be a linear combination of the velocity vectors

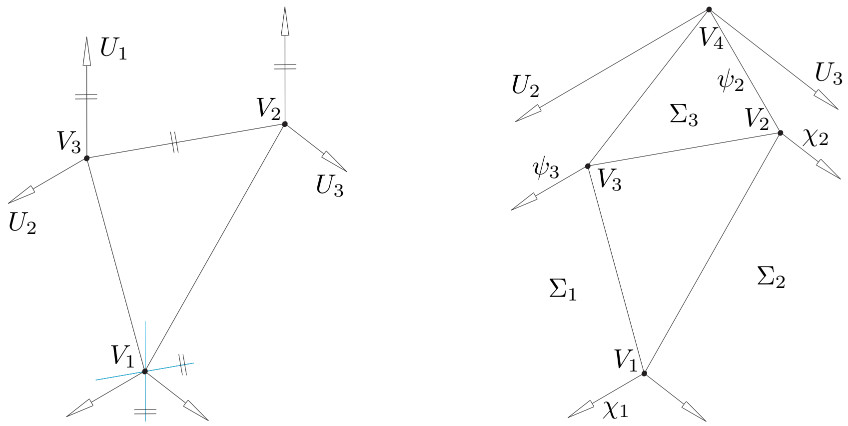

Figure 3: Both figures can be seen as a parallel projection of a spatial structure but on the other hand also as a planar configuration, because such structures has to possess two flat poses.

implied by the rotation about $[V_1, U_1]$ or about a parallel to $[V_2, V_3]$ through V_1. In order to guarantee that these vectors are tangent to the circles of the RSSR mechanism, the two rotary axes also have to lie within the plane V_1, V_2, V_3, U_1. Therefore U_1, U_2, U_3 are collinear and this again contradicts the definition of an octahedron. □

Moreover we can also prove the following theorem:

Theorem 4. *In the projective extension of E^3 there do not exist flexible octahedra with a finite face and one edge at infinity.*

Proof: We assume that $V_1, \ldots, V_4$ are finite and that U_2, U_3 are ideal points. We consider again V_1, U_2, U_3 as the fixed system. Now we split up the octahedron into two parts: in a mechanism which consists of V_1, U_2, V_3, V_4 and in one which is determined by V_1, V_2, U_3, V_4 (see Fig. 3(b)). Note that both mechanisms have the kinematic structure of a serial 2R chain.

We consider the configuration where the 2R chain V_1, U_2, V_3, V_4 is singular, i.e., these four points are coplanar where the carrier plane is denoted by ε. Now this mechanism can only induce a velocity to V_4 which is orthogonal to ε. The other 2R chain also implies a velocity to V_4 and its direction is orthogonal to U_3. In order to guarantee that the directions of the two velocities in V_4 are fitting together (which is a necessary condition for the flexibility) the point U_3 has to be located in ε. Therefore the points V_1, U_2, U_3, V_3, V_4 are within ε which equals the plane of the fixed system.

In the following we show that also the point V_2 has to lie in ε if the octahedron is of type 1 or type 2, respectively:

Type 1: In this case the spherical image of the motion transmission from Σ_1 to Σ_2 via V_3 and V_2 is a spherical focal mechanism of type (ii). Therefore the condition $\mathrm{c}\chi_2 = \mathrm{c}\psi_3$ (see Fig. 3(b)) holds which implies that also the other 2R chain has to be in a singular configuration.

Type 2: We have to distinguish three subcases depending on the vertices V_i $(i = 1, 2, 3)$ in which the spherical image of the motion transmission corresponds to a spherical isogram:

- $i = 1$: Now the spherical image of the motion transmission from Σ_1 to Σ_2 via V_3 and V_2 is a spherical focal mechanism of type (ii) which equals the above discussed case.
- $i = 3$: Now the spherical image of the motion transmission from Σ_1 to Σ_3 via V_1 and V_2 is a spherical focal mechanism of type (ii) which implies $c\chi_1 = c\psi_2$. As χ_1 equals 0 or π this already yields that all 6 vertices are coplanar.
- $i = 2$: This case can be done analogously as the above one if we start with a singular configuration of the 2R chain V_1, V_2, U_3, V_4.

Moreover, as there always exist two singular configurations of a 2R chain, a flexible octahedron where one edge is an ideal line has to have two flat poses.

In order to admit two flat poses, either V_4 has to be an ideal point (cf. Theorem 3) or V_2, U_2, V_3, U_3 have to be located on a line which again yields a contradiction as U_2 coincides with U_3. This already finishes the proof. □

Remark 2. The two geometric/kinematic proofs of Theorems 3 and 4 demonstrate the power of geometry in the context of flexibility because purely algebraic proofs for these statements seem to be a complicated task.

4. Special cases of flexible octahedra in $E^\star$

In the first part of this section we determine all flexible octahedra with at least three vertices on the plane at infinity. These so called trivial flexible octahedra are the content of the next theorem:

Theorem 5. *In the projective extension of E^3 any octahedron is flexible where at least two edges are ideal lines but no face coincides with the plane at infinity.*

Proof: Under consideration of footnote 1 there are only two types of octahedra fulfilling the requirements of this theorem. These two types are as follows:

a. two pairs of opposite vertices are ideal points,

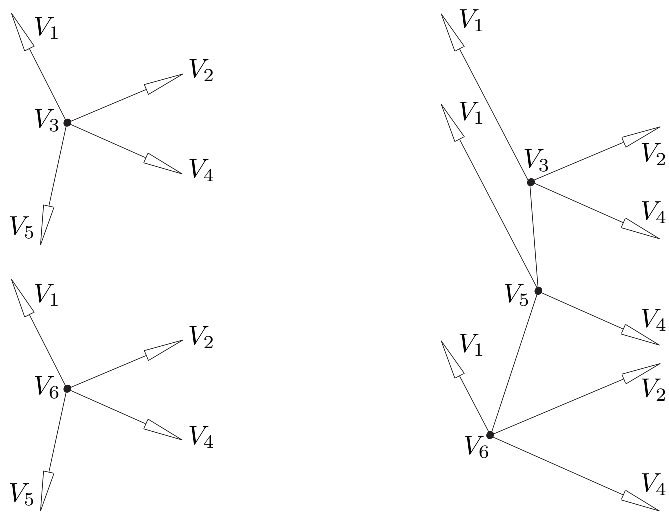

Figure 4: The degenerated flexible octahedra of type (a) have a 4-parametric self-motion in contrast to those of type (b) which possess a constrained one

b. three vertices are ideal points where two of them are opposite ones.

It can immediately be seen from Fig. 4(a) and (b), that these two degenerated cases are flexible. A detailed proof is left to the reader. □

Due to the Theorems 3, 4 and 5 the only open problem is the determination of all flexible octahedra where only one pair of opposite vertices consists of ideal points. For the discussion of these octahedra we need some additional considerations which are prepared in the next two subsections.

4.1. Central triangles with one ideal point

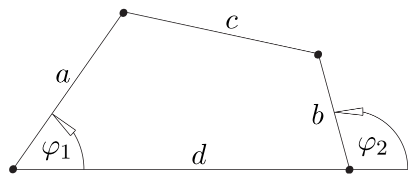

Figure 5: Planar four-bar mechanism with driving arm a, follower b, coupler c and base d

Given is an octahedron where two opposite vertices are ideal points and the remaining four vertices are in E^3. The four faces through an ideal point constitute a 4-sided prism where the motion transmission between opposite faces equals the one of the corresponding planar four-bar mechanism (orthogonal cross section of the prism). It can easily be seen that the input angle φ_1 and the output angle φ_2 of a planar four-bar linkage (see Fig. 5) are related by

$$p_{22}t_1^2t_2^2 + p_{20}t_1^2 + p_{02}t_2^2 + p_{11}t_1t_2 + p_{00} = 0 \qquad (7)$$

with $t_i := \tan(\varphi_i/2)$, $p_{11} = -8ab$ and

$$\begin{aligned} p_{22} &= (a-b+c+d)(a-b-c+d), \quad & p_{20} &= (a+b+c+d)(a+b-c+d),\\ p_{02} &= (a+b+c-d)(a+b-c-d), \quad & p_{00} &= (a-b+c-d)(a-b-c-d). \end{aligned} \qquad (8)$$

W.l.o.g. we can assume $a, b, c, d > 0$. Moreover in [18] the following lemma was proven:

Lemma 2. *If a reducible composition of one planar and one spherical four-bar linkage with a spherical coupler component is given, then one of the algebraic conditions characterizing the four cases of Theorem 1 is fulfilled.*

A closer study of the items (a)–(d) in Theorem 1 with respect to Lemma 2 was also done in [18], where we assumed that V_1 denotes the ideal point. In the following we sum up the achieved results:

ad (a) The conditions $c_{00} = c_{22} = 0$ imply $a = b$ and $c = d$, i.e., the planar four-bar mechanism is a parallelogram or an antiparallelogram. Note that opposite angles in the parallelogram and in the antiparallelogram are equal.
In contrast, $c_{20} = c_{02} = 0$ has no solution under the assumption $a, b, c, d > 0$.

ad (b) In this case we only get a solution if the relation

$$2ac : 2bd : (a^2 - b^2 + c^2 - d^2) = \mathrm{s}\beta_2\,\mathrm{s}\gamma_2 : \mathrm{s}\alpha_2\,\mathrm{s}\delta_2 : (\mathrm{c}\beta_2\,\mathrm{c}\gamma_2 - \mathrm{c}\alpha_2\,\mathrm{c}\delta_2)$$

holds. Moreover this condition implies $\mathrm{c}\chi_1 = \mathrm{c}\psi_2$.

ad (c) The case $d_{22} = d_{20} = c_{00} = c_{20} = 0$ does not yield a solution because $c_{00} = c_{20} = 0$ cannot be fulfilled for $a, b, c, d > 0$.
The other case $d_{00} = d_{02} = c_{22} = c_{02} = 0$ implies $\mathrm{c}\varphi_1 = \mathrm{c}\psi_1$ and $\mathrm{c}\chi_2 = \mathrm{c}\varphi_3$ or as second possibility $\mathrm{c}\varphi_1 = \mathrm{c}\psi_1$ and V_2, V_3, V_5, V_6 are coplanar.

ad (d) The case $c_{20} = Ad_{02}$, $c_{22} = Ad_{22}$, $c_{02} = Bd_{22}$, $c_{00} = Bd_{02}$, $d_{00} = d_{20} = 0$, $d_{02}d_{22} \neq 0$ does not yield a solution.
The other case $d_{02} = Ac_{20}$, $d_{22} = Ac_{22}$, $d_{20} = Bc_{22}$, $d_{00} = Bc_{20}$, $c_{00} = c_{02} = 0$, $c_{20}c_{22} \neq 0$ implies the relations $\mathrm{c}\varphi_2 = \mathrm{c}\chi_1$ and $\mathrm{c}\varphi_1 = \mathrm{c}\chi_3$.

4.2. Preparatory lemmata

In order to give the proof of the main theorem in Section 4.3 in a compact form we prove the following two preparatory lemmata:

Lemma 3. *A planar base polygon of a 4-sided prism*[2] *remains planar during the flex if and only if one of the following cases hold:*

1. *The edges of the prism are orthogonal to the planar base,*
2. *the planar quadrilateral is a deltoid and the edges are orthogonal to the deltoid's line of symmetry,*
3. *the planar quadrilateral is an antiparallelogram and its plane of symmetry is parallel to the edges of the prism,*
4. *the planar quadrilateral is a parallelogram.*

Proof: We consider the orthogonal cross section of a prism which is an ordinary four-bar mechanism as given in Fig. 5. We denote with s and l the shortest and longest bar, respectively, and with p and q the length of the remaining bars. As item 1 is trivial we assume that the edges of the prism are not orthogonal to the planar base. For the used notation of the following case study please see Fig. 6:

1. $s + l < p + q$: Due to *Grashof's theorem* we get a double-crank mechanism if we fix the shortest bar s. Considering all four poses where the sides coincide with the frame link already implies the contradiction.
2. $s + l > p + q$: If we fix any of the four bars we always get a double-rocker mechanism. W.l.o.g. we can assume that d is the longest bar. As a consequence the following inequalities hold:
$$d + a > b + c \quad \text{and} \quad d + b > a + c. \tag{9}$$
Therefore there exists a configuration where the edges e_1, e_2, e_5 are coplanar ($\tau_1 = 0 \Rightarrow e_5$ is between e_1 and e_2). This implies that the points V_1, V_2, V_5 have to be collinear which is the case if $\lambda_1 = \mu_1$ and $\lambda_5 = \mu_2$ hold. The analogous consideration for the edges e_1, e_2, e_4 yields $\lambda_2 = \mu_2$ and $\lambda_1 = \mu_4$.
Now $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$ and the coplanarity condition of V_1, V_2, V_4, V_5 yield that $\tau_1 = 0$ implies $\tau_2 = 0$. Therefore there exists a flat pose which contradicts our assumption $s + l > p + q$.
3. $s + l = p + q$: Here we assume that the prism only has one flat position. In this case we have to distinguish two subcases:

[2] We exclude those prisms where always two pairs of neighboring sides coincide during the flex, as they are not of interest for the problem under consideration (cf. footnote 1).

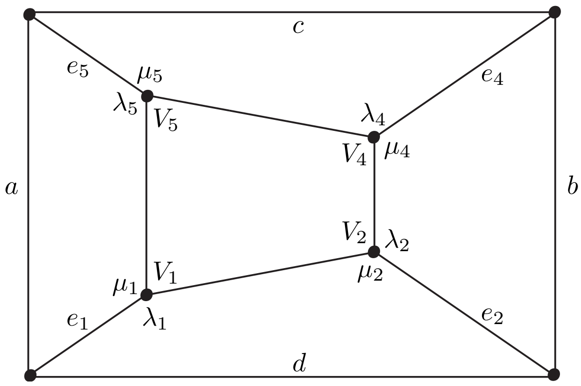

Figure 6: Perspective view of an orthogonal cross section of the prism (= four-bar linkage a, b, c, d) and of its four coplanar vertices V_1, V_2, V_4, V_5. Note that the dihedral angles along the prism edges e_i are denoted by τ_i. Moreover the face angles of the prism at V_i are denoted by μ_i and λ_i, respectively.

a. l and s are neighboring bars: W.l.o.g. we set $l = d$, $s = b$, $p = c$ and $q = a$. Due to the inequalities

$$l + q > s + p \quad \text{and} \quad p + q > l - s, \tag{10}$$

there exist the following two special poses of the prism illustrated in Fig. 7. These two poses imply $\lambda_2 = \lambda_4$, $\mu_4 = \mu_5$ and $\lambda_2 = \mu_2$, $\lambda_1 = \mu_4$, respectively. Together with the coplanarity condition of V_1, V_2, V_4, V_5 these conditions yield one of the following three cases:

i. V_1, V_2, V_4 are always collinear which contradicts footnote 2,

ii. V_2, V_4, V_5 are always collinear which contradicts footnote 2,

iii. $[V_1, V_2]$ and $[V_4, V_5]$ are parallel. This already yields the contradiction as a four-bar mechanism where two opposite bars are always parallel during the motion can only be a parallelogram.

b. l and s are opposite bars: W.l.o.g. we set $l = d$, $s = c$, $p = a$ and $q = b$. Due to the inequalities

$$l + p > s + q \quad \text{and} \quad l + q > s + p \tag{11}$$

there exist the following two special poses of the prism illustrated in Fig. 8. These two poses imply $\lambda_2 = \lambda_4$, $\mu_4 = \mu_5$ and $\lambda_4 = \lambda_5$, $\mu_1 = \mu_5$, respectively. Together with the coplanarity condition of V_1, V_2, V_4, V_5 these conditions yield one of the following three cases:

i. V_2, V_4, V_5 are always collinear which contradicts footnote 2,

ii. V_1, V_4, V_5 are always collinear which contradicts footnote 2,

iii. $[V_1, V_5]$ and $[V_2, V_4]$ are parallel. This yields the same contradiction as the corresponding case given above.

4. $s + l = p + q$: Now we assume that the prism has two flat positions. Then a, b, c, d can only form a deltoid, a parallelogram or an antiparallelogram. For these three cases we show by the following short computation that the base remains planar during the flex if and only if item 2, 3 or 4 of Lemma 3 holds.

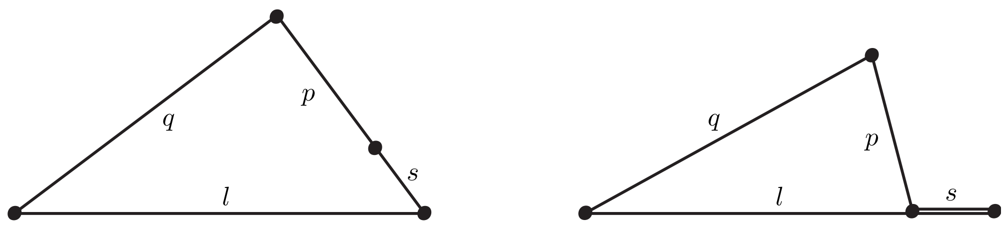

Figure 7: Special poses of the four-bar linkage ($l = 5$, $s = 1$, $p = 2$, $q = 4$) where l and s are neighboring bars

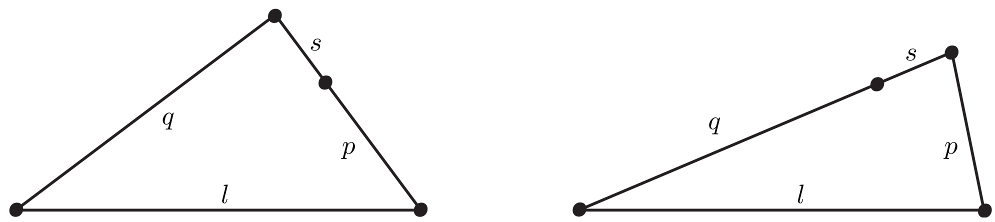

Figure 8: Special poses of the four-bar linkage ($l = 5$, $s = 1$, $p = 2$, $q = 4$) where l and s are opposite bars

W.l.o.g. we can assume that the prism has z-parallel edges and that V_1 coincides with the origin. Then the remaining points have coordinates:

$$V_2 = \begin{pmatrix} d \\ 0 \\ h_2 \end{pmatrix}, \quad V_4 = \begin{pmatrix} d + bc\varphi_2 \\ bs\varphi_2 \\ h_4 \end{pmatrix}, \quad V_5 = \begin{pmatrix} ac\varphi_1 \\ as\varphi_1 \\ h_5 \end{pmatrix} \tag{12}$$

with $a, b, c, d > 0$. Therefore beside Eq. (7) the coplanarity condition $\det(V_2, V_3, V_4) = 0$ has to hold, which can be written under consideration of $t_i := \tan(\varphi_i/2)$ for $i = 1, 2$ as follows:

$$a[dh_4 + h_2(b-d)]t_1t_2^2 + a[dh_4 - h_2(d+b)]t_1 + b(ah_2 - dh_5)t_2 - b(ah_2 + dh_5)t_1^2t_2 = 0. \tag{13}$$

Moreover due to footnote 2 we can assume that t_1 or t_2 is not constant zero during the flex. In the next step we compute the resultant R of Eq. (7) and Eq. (13) with respect to t_1.

- Deltoid: W.l.o.g. we can set $a = d$ and $b = c$. Moreover we can assume $c \neq d$ because otherwise we get a rhombus which is discussed later on as a special case of the parallelogram case. Now R can only vanish without contradiction for $(h_5 - h_2)[q_1(c - d)t_2^2 + q_2(d + c)] = 0$ with

  $$q_1 = (h_2 + h_5)c + (2h_4 - h_2 - h_5)d, \quad q_2 = (h_2 + h_5)c - (2h_4 - h_2 - h_5)d.$$

  Therefore we have to distinguish two cases:

  ⋆ $q_1 = q_2 = 0$: This factors can only vanish without contradiction for $h_2 = -h_5$ and $h_4 = 0$ which already yields item 2 of Lemma 3.

  ⋆ $h_2 = h_5$: Now Eq. (7) and Eq. (13) have the common factor $dt_1 \neq 0$. Then the resultant of the remaining factors with respect to t_1 can only vanish without contradiction (w.c.) for $\overline{q}_1(c - d)t_2^2 + \overline{q}_2(d + c) = 0$ with

  $$\overline{q}_1 = h_5c + (h_4 - h_5)d, \quad \overline{q}_2 = h_5c - (h_4 - h_5)d.$$

$\overline{q}_1 = \overline{q}_2 = 0$ implies $h_4 = h_5 = 0$, a contradiction.

- Parallelogram/antiparallelogram: Now we set $a = b$ and $c = d$. Then R can only vanish w.c. for $(h_5 - h_4 + h_2)[\hat{q}_1(b-d)t_2^2 + \hat{q}_2(b+d)] = 0$ with

$$\hat{q}_1 = (h_5 - h_4 - h_2)b + (h_2 - h_4 - h_5)d, \quad \hat{q}_2 = (h_5 - h_4 - h_2)b - (h_2 - h_4 - h_5)d.$$

Therefore we have to distinguish three cases:

- $\star$ $b = d$ and $\hat{q}_2 = 0$: This implies $h_2 = h_5$. Now Eq. (7) and Eq. (13) have the common factor $d^2 t_1 \neq 0$. Then the resultant of the remaining factors with respect to t_1 can only vanish w.c. for $h_4 = 2h_2$. Then the common factor of Eq. (7) and Eq. (13) yields $t_1 - t_2$ which implies a special solution of item 4 of Lemma 3.
- $\star$ $\hat{q}_1 = \hat{q}_2 = 0$: This two conditions already imply $h_4 = 0$ and $h_2 = h_5$. As the common factor of Eq. (7) and Eq. (13) equals $t_1(b+d) - t_2(b-d)$ this case yields item 3 of Lemma 3.
- $\star$ $h_5 - h_4 + h_2 = 0$: If this condition is fulfilled the common factor of Eq. (7) and Eq. (13) equals $t_1 - t_2$ and therefore we get item 4 of Lemma 3. □

Lemma 4. *A planar four-bar mechanism with $l + s \geq p + q$ which is no parallelogram or antiparallelogram always has a configuration with parallel arms if l is one of these arms. Moreover such a four-bar mechanism has also a configuration where the coupler is parallel to the base. These two configurations coincide ($\Rightarrow$ folded pose) if and only if the four-bar linkage is a deltoid.*

Proof: We use the notation of the four-bar mechanism from Fig. 5. Now there are the following two possibilities such that the arms a, b are parallel:

1. They are located on the same side with respect to the base-line d. Therefore $\varphi_1 = \varphi_2$ holds and the corresponding equation of Eq. (7) reads as

$$(a-b-c+d)(a-b+c+d)t_1^2 + (a-b+c-d)(a-b-c-d) = 0 \tag{14}$$

As a consequence we get a real solution of the problem if

$$-(a-b-c+d)\underbrace{(a-b+c+d)}_{>0}(a-b+c-d)\underbrace{(a-b-c-d)}_{<0} \geq 0 \tag{15}$$

holds.[3] Therefore we get a solution in one of the following four cases:

$$(i) \quad a+d > b+c \quad \text{and} \quad a+c > b+d, \qquad (iii) \quad a+d = b+c, \tag{16}$$
$$(ii) \quad a+d < b+c \quad \text{and} \quad a+c < b+d, \qquad (iv) \quad a+c = b+d. \tag{17}$$

Now one of the cases (i) or (ii) is fulfilled if one of the arms a, b is the longest bar of the mechanism. Clearly, we can also assume in the special cases (iii) and (iv) w.l.o.g. that one of the arms a, b is the longest bar of the mechanism. This proves the first part of the lemma.

[3]Note that $(a-b+c+d)(a-b-c-d) = 0$ would yield that the mechanism is rigid.

2. They are not located on the same side with respect to the base-line d, hence $\varphi_1 = \varphi_2 + \pi$. Now the corresponding equation of Eq. (7) reads as:

$$(a+b-c+d)(a+b+c+d)t_1^2 + (a+b-c-d)(a+b+c-d) = 0 \tag{18}$$

Therefore we get a real solution of the problem if

$$-\underbrace{(a+b-c+d)}_{>0}\underbrace{(a+b+c+d)}_{>0}(a+b-c-d)\underbrace{(a+b+c-d)}_{>0} \geq 0 \tag{19}$$

holds. As a consequence we get a solution if $a+b \leq c+d$ holds. As due to case 1, one of the arms is the longest bar, this is only possible for the special case $a+b = c+d$. But on the other hand there exists a pose where the coupler and the base are parallel for $c+d \leq a+b$. Now this equation is always fulfilled which proves the second part of the lemma.

If $c+d = a+b$ and condition (iii) or (iv) are fulfilled we get a folded pose. It can easily be seen that the solution of the linear system of equations is a deltoid. □

4.3. Main theorem

In this section we give the complete classification of flexible octahedra with two opposite vertices at infinity.

Theorem 6. *In the projective extension of E^3 any octahedron, where exactly two opposite vertices (V_3, V_6) are ideal points, is flexible in one of the following cases:*

(I) *The remaining two pairs of opposite vertices (V_1, V_4) and (V_2, V_5) are symmetric with respect to a common line as well as the edges of the prisms through V_3 and V_6, respectively.*

(II) (i) *One pair of opposite vertices (V_2, V_5) is symmetric with respect to a plane which contains the remaining pair of opposite vertices (V_1, V_4). Moreover also the edges of the prisms through V_3 and V_6 are symmetric with respect to this plane.*

(ii) *The remaining 4 vertices V_1, V_2, V_4, V_5 are coplanar and form an antiparallelogram and its plane of symmetry is parallel to the edges of the prisms through V_3 and V_6, respectively.*

(III) *This type is characterized by the existence of two flat poses and consists of two prisms where the orthogonal cross sections are congruent antiparallelograms. For the construction of these octahedra see Fig. 12.*

(IV) *The remaining 4 vertices V_1, V_2, V_4, V_5 are coplanar and form*

(i) *a deltoid and the edges of the prisms through V_3 and V_6 are orthogonal to the deltoids line of symmetry,*

(ii) *a parallelogram.*

Proof: For the notation used in this proof we refer to Fig. 9. Moreover the corresponding prisms through the points V_3 and V_6 are denoted by Π_3 and Π_6, respectively. The faces through the remaining vertices V_i in E^3 always form 4-sided pyramids Λ_i for $i = 1, 2, 4, 5$.

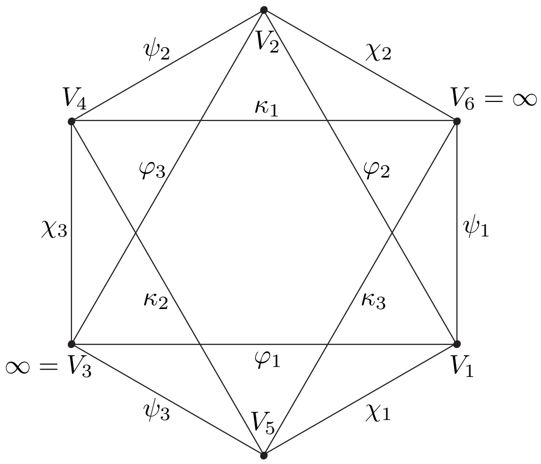

Figure 9: Schematic sketch of the octahedron $V_1, \ldots, V_6$ with dihedral angles $\varphi_i, \psi_i, \chi_i, \kappa_i$, $i = 1, 2, 3$

We can stop the discussion of cases if the points V_1, V_2, V_4, V_5 are permanently coplanar during the flex because then by Lemma 3 we can only get a solution of type (II,ii) and (IV) or special cases of them. The following proof is split into three parts:

1st Part:

In this part we apply the conditions of case (d) of Theorem 1 over the octahedron in such a way that the corresponding two cosine equalities hold if any of the 8 faces is considered as central triangle. Up to the relabeling of vertices this yields the following case:

$$\mathrm{c}\varphi_3 = \mathrm{c}\chi_2, \quad \mathrm{c}\varphi_1 = \mathrm{c}\chi_3, \quad \mathrm{c}\kappa_3 = \mathrm{c}\psi_3, \quad \mathrm{c}\kappa_1 = \mathrm{c}\psi_1. \tag{20}$$

If additionally $\mathrm{c}\chi_2 = \mathrm{c}\kappa_3$ holds we get a special case of item (A) of the 3rd part treated later.

Therefore we can assume w.l.o.g. that the orthogonal cross section of Π_3 and Π_6 are deltoids (and not parallelograms or antiparallelograms). Moreover it can easily be seen that a flat pose of Π_3 or Π_6 implies a flat pose of the whole octahedron. Therefore the spherical image of $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5$ are spherical deltoids or isograms.

1. If Λ_1 or Λ_4 are of isogram type then $\mathrm{c}\chi_3 = \mathrm{c}\kappa_1$ holds, which already implies that the orthogonal cross sections of Π_3 and Π_6 are similar deltoids. Now we distinguish two cases:
   a. In the first case we assume that in both flat poses $V_3 \neq V_6$ holds. Due to the similarity the intersection points V_1, V_2, V_4, V_5 of corresponding prism edges are located on a line. As two such flat poses exist the line can only be orthogonal to the edges of the prism. Therefore V_1, V_2, V_4, V_5 are coplanar during the flex and we are done due to Lemma 3.
   b. If in one of the flat poses $V_3 = V_6$ holds then the deltoids are congruent. As a consequence there exists an Euclidean motion such that Π_3 and Π_6 coincides. Moreover we can assume w.l.o.g. that this is a rotation about $[V_2, V_5]$. Due to the rotational symmetry and the symmetry of the deltoid the line spanned by the intersection points V_1 and V_4 of the other edges has to intersect the rotational axis $[V_2, V_5]$ (see Fig. 10(a)). Therefore V_1, V_2, V_4, V_5 are coplanar during the flex and we are done due to Lemma 3.

2. If Λ_1 and Λ_4 are of deltoid type then $\mathrm{c}\psi_2 = \mathrm{c}\kappa_2$ and $\mathrm{c}\chi_1 = \mathrm{c}\varphi_2$ hold. We distinguish two cases:

a. If $c\psi_2 = c\varphi_2$ holds, then Λ_2 and Λ_5 are of isogram type. In the flat poses $V_2 = V_5$ holds and we see that the corresponding faces of the pyramids Λ_2 and Λ_5 are congruent. This already implies with $c\psi_2 = c\varphi_2$ that the orthogonal cross sections of Π_3 and Π_6 are parallelograms/antiparallelograms which yields the contradiction.

b. In the other case Λ_2 and Λ_5 are of deltoid type. This already implies that in the flat poses $V_3 = V_6$ holds. Therefore the orthogonal cross sections of Π_3 and Π_6 are congruent deltoids ($\Rightarrow c\chi_3 = c\kappa_1$). This yields a contradiction as Λ_1 and Λ_4 are of isogram type.

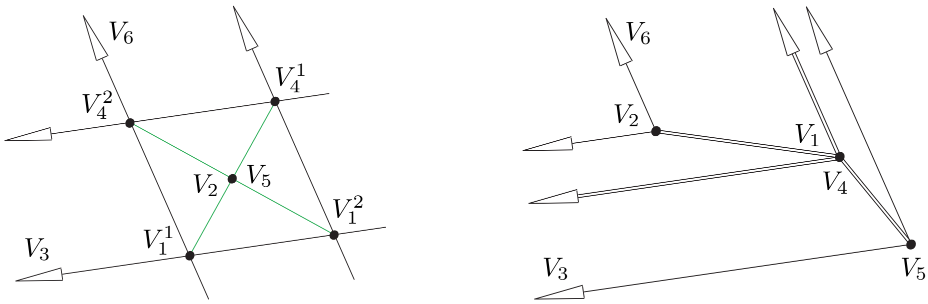

Figure 10: (a) Rotation of Π_3 about the projecting line $[V_2, V_5]$. The connecting lines $[V_1^1, V_4^1]$ or $[V_1^2, V_4^2]$ of possible intersection points intersect $[V_2, V_5]$. (b) Flat pose of the octahedron where Λ_1 and Λ_4 are congruent.

2nd Part:

As for the one case of item (c) of Theorem 1 the four points $\in E^3$ are already coplanar during the flex, we are done due to Lemma 3.

Therefore we apply the conditions of the other case of item (c) of Theorem 1 over the octahedron in such a way that the corresponding two cosine equalities hold if any of the 8 faces is considered as central triangle. Up to the relabeling of vertices this yields the following case:

$$c\varphi_2 = c\psi_2, \quad c\varphi_1 = c\chi_3, \quad c\kappa_2 = c\chi_1, \quad c\kappa_1 = c\psi_1. \tag{21}$$

Moreover as the cosines of the dihedral angles through V_1 and V_4 are pairwise the same we can apply *Kokotsakis' theorem* (*Satz über zwei Vierkante*) given in [14, §12] which implies that the pyramids Λ_1 and Λ_4 are congruent. Now we have to distinguish two cases because they can be congruent with respect to an orientation preserving or a non-orientation preserving isometry:

1. Orientation preserving isometry: As $[V_1, V_6] \parallel [V_4, V_6]$ and $[V_1, V_3] \parallel [V_4, V_3]$ has to hold the rigid body motion can only be a composition of a half-turn about a line l orthogonal to the plane $[X, V_3, V_6]$ plus a translation along the axis, where X stands for any point of E^3. Moreover l has to be located within the plane $[V_1, V_2, V_5]$ because otherwise there does not exist a translation such that the remaining pairs of corresponding edges intersect in V_2 and V_5, respectively. This already yields that V_1, V_2, V_4, V_5 are coplanar during the flex and we are done due to Lemma 3.
2. Non-orientation preserving isometry: Here we are left with three possibilities:
   a. The Euclidean motion is a composition of a reflection in $\varepsilon := [X, V_3, V_6]$ and a translation parallel to this plane. If $[V_1, V_2, V_5]$ is orthogonal to ε then V_1, V_2, V_4, V_5 are coplanar during the flex and we are done due to Lemma 3.

In any other case the translation vector has to be the zero vector ($\Rightarrow V_2$ and V_5 are located on ε) such that the other corresponding edges intersect in V_2 and V_5, respectively. As the orthogonal cross section of Π_3 is at least a deltoid, the flat poses of this prism imply flat poses of the whole structure as all vertices are located on ε. Therefore the spherical images of Λ_1 and Λ_4 have to be spherical deltoids:

i. Under $c\psi_2 = c\kappa_2$ the flat poses immediately imply that V_1, V_2, V_4, V_5 have to be coplanar during the flex and we are done due to Lemma 3.

ii. For the other possibility $c\chi_3 = c\kappa_1$ the points V_3 and V_6 coincide in the flat poses and therefore the deltoidal cross sections of Π_3 and Π_6 are congruent. This case was already discussed in item (1b) of the 1st part.

b. The Euclidean motion is a composition of a reflection in $\varepsilon := [X, V_3, V_6]$ and a half-turn about a line l orthogonal to ε. Applying such a transformation all pairs of corresponding edges of the pyramids are parallel. Therefore V_2 and V_5 are also ideal points which contradicts our assumptions.[4]

c. Under the assumption that $\sphericalangle V_3XV_6$ is constant $\pi/2$ during the flex the Euclidean motion could also be composed of a reflection in one of the planes $\varepsilon_1 := [l, V_3]$ or $\varepsilon_2 := [l, V_6]$ plus a translation parallel to it. This case cannot yield a solution as any octahedron with $\sphericalangle V_3XV_6 = \text{const.} \neq 0$ has to be rigid. The proof is left to the reader.

Kokotsakis' theorem cannot be applied if the spherical images of Λ_1 and Λ_4 are isograms. In this case ($c\psi_2 = c\kappa_2$, $c\chi_3 = c\kappa_1$) such an octahedron already has two flat poses. Now the orthogonal cross sections of Π_3 and Π_6 are deltoids, parallelograms or antiparallelograms and the spherical images of Λ_2 and Λ_5 are spherical isograms or spherical deltoids. As not both spherical images of Λ_2 and Λ_5 can be isograms (otherwise we get item (B) of the 3rd part) we can assume w.l.o.g. that Λ_2 has a deltoidal spherical image.

1. If at least one further structure of Π_3 and Π_6 is of deltoid type then V_1 has to coincide with V_4 in the flat pose (see Fig. 10(b)). This already shows that also in this case Λ_1 and Λ_4 are congruent and therefore we can apply the same argumentation as given above.
2. If the orthogonal cross sections of Π_3 and Π_6 are parallelograms or antiparallelograms then we can only get a special case of item (A) of the following 3rd part.

3rd Part:

We are left with the possibilities given in item (a) and (b) of Theorem 1. W.l.o.g. we take V_1, V_2, V_3 as representative triangle. Then the motion transmission from Σ_3 to Σ_2 via V_3 and V_1 is reducible if

- the orthogonal cross section of Π_3 is a parallelogram or an antiparallelogram,
- the spherical image of Λ_1 is an isogram,
- case (b) holds.

Analogous possibilities hold for the motion transmission from Σ_1 to Σ_2 via V_3 and V_2. Now combinatorial aspects show that one of the following cases has to hold:

A. the orthogonal cross section of Π_3 is a parallelogram or an antiparallelogram,

B. the spherical image of Λ_1 and Λ_2 are isograms,

C. both motion transmissions are reducible due to case (b),

[4] We get a special case of a flexible octahedron of Theorem 5.

D. the motion transmission from Σ_1 to Σ_2 (or Σ_3 to Σ_2) is reducible due to case (b) and the spherical image of Λ_1 (or Λ_2) is an isogram.[5]

Therefore the remaining flexible octahedra with opposite vertices on the plane at infinity can only belong to one of these four cases. As a consequence the reducible composition implied by these flexible octahedra has to be of the same type independent of the choice of the central triangle. This yields the following conditions:

ad A. $c\varphi_1 = c\chi_3,\ c\varphi_3 = c\psi_3,\ c\psi_1 = c\kappa_1,\ c\chi_2 = c\kappa_3.$

ad B. $c\varphi_1 = c\psi_1,\ c\varphi_3 = c\chi_2,\ c\kappa_1 = c\chi_3,\ c\kappa_3 = c\psi_3,\ c\varphi_2 = c\psi_2 = c\kappa_2 = c\chi_1.$

ad C. $c\varphi_1 = c\kappa_1,\ c\varphi_3 = c\kappa_3,\ c\psi_3 = c\chi_2,\ c\chi_3 = c\psi_1.$

ad D. $c\varphi_1 = c\psi_1,\ c\varphi_3 = c\kappa_3,\ c\psi_3 = c\chi_2,\ c\chi_3 = c\kappa_1,\ c\kappa_2 = c\psi_2,\ c\chi_1 = c\varphi_2.$

In the following these four cases are discussed in detail:

ad (C) If the orthogonal cross sections of Π_3 and Π_6 are parallelograms or antiparallelograms then we get a special case of item (A). Therefore we can assume that this is not the case.

This assumption together with the property that the cosines of the dihedral angles of Π_3 and Π_6 are equal, already imply that these prisms are related by an Euclidean similarity transform. Now we consider the orthogonal cross section (four-bar mechanism a, b, c, d) of one of these prisms:

1. $l + s < p + q$: If we choose s as base then *Grashof's theorem* is fulfilled and we get a double-crank mechanism. Such a mechanism has two poses where the coupler is parallel to the base.
   a. If in one of these two poses the parallel planes of both prisms do not coincide or if $V_3 = V_6$ holds then the condition that the corresponding edges of the prisms intersect each other in this pose, already yields that the coupler and the base must have the same length. But this already contradicts $l + s < p + q$.
   b. If in one of the two flat poses the parallel planes of both prisms coincide (but $V_3 \neq V_6$), then this already implies that V_1, V_2, V_4, V_5, V_3 and V_4, V_5, V_1, V_2, V_6 are congruent. Moreover it can be seen from this pose that the pyramids Λ_1 and Λ_4 are congruent with respect to an orientation preserving isometry. Due to $c\chi_3 = c\psi_1$ and $c\kappa_1 = c\varphi_1$ this property has to hold during the whole flex.[6] As the corresponding rigid body motion also has to interchange the ideal points V_3 and V_6 we are left with two possibilities:
      i. The rigid body motion is a composition of a half-turn about one of the two bisectors of $\sphericalangle V_3XV_6$ plus a translation along this axis. If the axis is located within the plane $[V_1, V_2, V_5]$ the points V_1, V_2, V_4, V_5 are coplanar during the flex and we are done due to Lemma 3. In any other case the translation vector has to be the zero vector such that the other corresponding edges intersect in V_2 and V_5, respectively. This yields solution (I).
      ii. The angle $\sphericalangle V_3XV_6$ is constant $\pi/2$ during the flex. Then a 90°-rotation about a line orthogonal to $[X, V_3, V_6]$ plus a translation along the axis yields a further possibility. This case cannot yield a solution for the same reason as case (2c) of the 2nd part.

[5]Note that we get the case in the parentheses from the other one just by a relabeling.
[6]The same holds for the pyramids Λ_2 and Λ_5.

2. $l+s \geq p+q$: Now there exist the two special poses of Lemma 4. Analogous considerations as in the case $l+s < p+q$ yield one of the following cases:
   a. $a = b$, $c = d$: Now the four-bar mechanism a, b, c, d is a parallelogram or an antiparallelogram. We get a special case of item (A).
   b. We get the above discussed item (1b) and therefore solution (I).
   c. The orthogonal cross sections of Π_3 and Π_6 are similar deltoids. This can only yield a case discussed in item (1) of the 1st part.

ad (D) If the spherical images of Λ_2 and Λ_5 are isograms we get item (B). Therefore we can assume w.l.o.g. that this is not the case and we can apply *Kokotsakis' theorem* which yields that Λ_2 and Λ_5 are congruent. Again we have to distinguish two cases:

1. Non-orientation preserving isometry: We can transform the two pyramids into each other by a reflection in one of the bisecting planes ε_i $(i = 1, 2)$ of $\sphericalangle\, V_3XV_6$ plus a translation parallel to ε_i.[7]
   If $[V_2, V_1, V_4]$ is orthogonal to ε_i then V_1, V_2, V_4, V_5 are coplanar during the flex (cf. Lemma 3).
   In any other case the translation vector has to be the zero vector ($\Rightarrow V_1$ and V_4 are located on ε_i) such that the other corresponding edges intersect in V_1 and V_4. We get solution (II,i).
2. Orientation preserving isometry: As the corresponding rigid body motion also has to interchange the ideal points V_3 and V_6 we are left with two possibilities:
   a. The rigid body motion is a composition of a half-turn about one of the two bisectors of $\sphericalangle\, V_3XV_6$ plus a translation along this rotary axis. In order to guarantee that the remaining vertices V_2 and V_5 exist, the corresponding edges have to intersect the axis of rotation. This already shows that all vertices of E^3 are coplanar during the flex (cf. Lemma 3).
   b. The angle $\sphericalangle\, V_3XV_6$ is constant $\pi/2$ during the flex. Then the $90°$-rotation about a line l orthogonal to $[X, V_3, V_6]$ plus a translation along l yields a further possibility. This case cannot yield a solution for the same reason as case (2c) of the 2nd part.

ad (B) In this case the spherical images of the faces through each vertex $\in E^3$ constitute an isogram. Now the conditions $c\varphi_2 = c\psi_2 = c\kappa_2 = c\chi_1$ yield for φ_2 equal 0 or π that the octahedron has two flat poses. Therefore the orthogonal cross section of the prisms Π_3 and Π_6 can only be a deltoid, a parallelogram or an antiparallelogram.

It can easily be seen that the deltoid case does not fit with both folded positions of the spherical focal mechanism composed of two spherical isograms. Therefore Π_3 and Π_6 have to be of parallelogram type or antiparallelogram type.

Note that opposite edges of a pyramid with an isogram as spherical image are symmetric with respect to a common line in a flat pose. The same holds for the flat pose of a prisms with a parallelogram or antiparallelogram as orthogonal cross section. Beside the scaling factor these two properties already determine the octahedron in the flat pose up to 3 parameters, namely the angles ζ, η, ν (see Fig. 11). Now this structure is flexible if we flex one of the prisms out of the flat pose in such a way that the orthogonal cross section is a parallelogram because then we get a special case of type (IV,ii).

[7]The only possible rotation is a half-turn about a line orthogonal to $[X, V_3, V_6]$. But this rotation is only the transition between the two possible reflections.

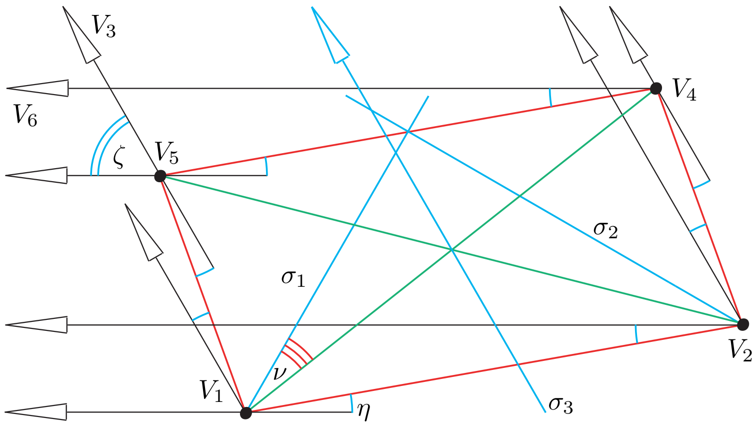

Figure 11: Illustration of the 3 free design parameters ζ, η, ν beside the scaling factor

In the other case (antiparallelogram) the octahedron is not even infinitesimally flexible. According to Kokotsakis (cf. [14, §3 and §13]) this condition is fulfilled if the bisectors σ_i $i = 1, 2, 3$ have a point in common.[8] It can easily be seen (cf. Fig. 11) that this is the case if ν is zero. This already implies the construction of type (III) octahedra which equals the construction of Bricard's type 3 octahedra with two opposite vertices at infinity (see Fig. 12).

Remark 3. Note that in each flat pose of a type-(III)-octahedron flex a bifurcation is possible into a type-(IV,ii)-octahedron flex.

ad (A) In the first case we assume that the orthogonal cross section of Π_3 is a parallelogram. Then we consider one of the two possible configurations where Π_6 is in a flat pose. In this pose it can immediately be seen that V_1, V_2, V_4, V_5 is a parallelogram.[9] Then the flex of Π_3 already implies type (IV,ii).

Therefore we can assume for the last case that the orthogonal cross section of both prisms are antiparallelograms. We have to distinguish two cases:

1. In both flat poses of Π_3, Π_6 is also flat and has the same carrier plane ε as the folded prism Π_3. Therefore this is an octahedron with two flat poses. As a consequence the spherical images of the pyramids $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5$ can only be spherical isograms or spherical deltoids. Assume the triangle V_1, V_2, V_3 as central triangle. If Λ_1 is of isogram type then we have a focal mechanism composed of Λ_1 and Π_3 as Eq. (5) holds.[10] Moreover, this is a reducible composition with a spherical coupler component. The corresponding spherical coupler can only be of isogram type because the deltoid case does not fit with both folded positions of the focal mechanism.

   As a consequence of this consideration all pyramids $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5$ are either isograms (which yields case (B)) or they are all of deltoid type. For the latter case we have to distinguish two principal cases:

   a. $c\kappa_2 = c\chi_1 = c\varphi_2 = c\psi_2$: In this case the points V_1, V_2, V_4, V_5 have to be collinear in both flat poses which already yield that these points are coplanar during the flex.

[8] The σ_i's are the limit of the intersection of two opposite faces of the respective pyramids and prism, respectively.

[9] This parallelogram can even degenerate into a folded one.

[10] The orthogonal cross section of Π_3 (an antiparallelogram) cannot have the additional property of a deltoid as then we get a flipped over rhombus which contradicts footnote 1.

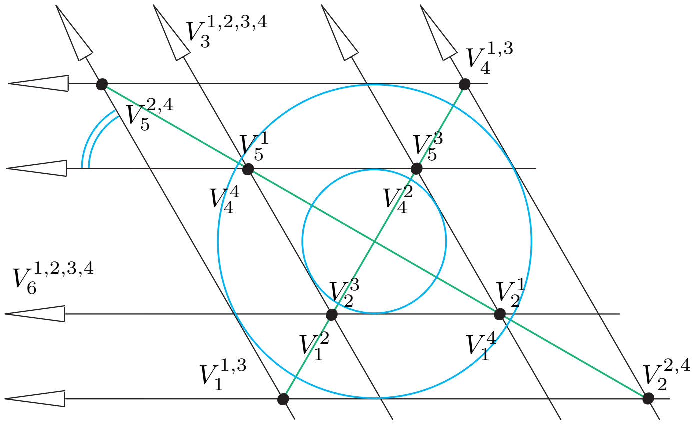

Figure 12: Construction of flexible octahedra of type (III): In the above given construction four flexible octahedra $V_1^i, \ldots, V_6^i$ $(i = 1, 2, 3, 4)$ are hidden, where those with indices $i = 1, 2$ are of type (III): The sides of the three quadrangles spanned by two pairs of opposite vertices touch three concentric circles (which cannot degenerate into the midpoint).
The octahedra with indices $i = 3, 4$ cannot be of type (III) because in the second flat pose the points $V_1^i, V_2^i, V_4^i, V_5^i$ also have to form a rhombus. This is only possible if the orthogonal cross sections of Π_3 and Π_6 are flipped-over rhombi which contradicts footnote 1. Therefore the octahedra $i = 3, 4$ can only have a trivial flex (the relative motion of Π_3 and Π_6 is a rotation with axis $[V_1^i, V_2^i, V_4^i, V_5^i]$; cf. footnote 1) beside the flexibility of type (IV,ii).

b. $c\chi_3 = c\kappa_1$, $c\psi_2 = c\varphi_2$, $c\chi_1 = c\kappa_2$: This case can only yield special cases of the 2nd part as Eq. (21) holds under consideration of $c\varphi_1 = c\chi_3$, $c\varphi_3 = c\psi_3$, $c\chi_2 = c\kappa_3$, and $c\psi_1 = c\kappa_1$.[11]

2. Assuming there exists a flat pose of Π_3 while Π_6 is not in a flat pose sharing the same carrier plane ε of the folded Π_3. Then we can reflect Π_6 on ε and we get Π_6' with the ideal point V_6'. If $\Pi_6 = \Pi_6'$ holds then this already implies that the points V_1, V_2, V_4, V_5 are coplanar during the flex.
Therefore we can assume w.l.o.g. $\Pi_6 \neq \Pi_6'$. If $V_1, \ldots, V_6$ is a flexible octahedron then also the octahedron $V_1, V_2, V_4, V_5, V_6, V_6'$ has to be flexible due to the symmetry. For the same reason the pyramids $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5$ of the octahedron $V_1, V_2, V_4, V_5, V_6, V_6'$ are of deltoid type, which already implies that the points V_1, V_2, V_4, V_5 are coplanar during the flex.

This finishes the proof of the necessity of the conditions given in Theorem 6.

The sufficiency for the flexibility of both types of item (IV) as well as of type (II,ii) follows directly from Lemma 3. As the types (I), (II) and (III) can be constructed from the corresponding types of Bricard flexible octahedra by a limiting process, the sufficiency for these types follows immediately from the flexibility of Bricard's octahedra. This finishes the proof of Theorem 6. □

[11] The remaining possibility $c\chi_2 = c\varphi_3$, $c\psi_2 = c\kappa_2$, $c\chi_1 = c\varphi_2$ can be done analogously because it can be obtained from this case by an appropriate relabeling.

5. Conclusion and future research

In this paper we completed the classification of flexible octahedra in the projective extension of the Euclidean 3-space. If all vertices are finite we get the well known Bricard flexible octahedra. There exist flexible octahedra of type 1 (cf. [18, Theorem 2]) and type 3 (cf. [18, Theorem 4]) with one vertex at infinity. Moreover there do not exist further flexible octahedra with one vertex in the plane at infinity (cf. [18, Theorem 3]).

All flexible octahedra with at least three vertices at infinity are trivially flexible and listed in Theorem 5 (see also Theorem 3).

Finally we presented in Theorem 6 all types of flexible octahedra with two vertices at infinity (see also Theorem 4). The types (I), (II) and (III) of this theorem can be generated from the corresponding *Bricard octahedra* by a limiting process. The remaining octahedra of type (IV) do not have a flexible analogue in E^3; they are flexible without self-intersection.

For a practical application one can think of an open serial chain composed of prisms $\Pi_0, \dots \Pi_n$ where each pair of neighboring prisms Π_i, Π_{i+1} $(i = 0, \dots, n-1)$ forms a flexible octahedron of Theorem 6. Note that such a structure admits a constrained motion. Moreover, if we additionally assume that $\Pi_0 = \Pi_n$ holds, we get a closed serial chain which is in general rigid. It would be interesting under which geometric conditions such structures are still flexible. Clearly, some aspects of this question are connected with the problem of nR overconstrained linkages (e.g. the spatial $4R$ overconstrained linkage is the Bennett mechanism). Finally, it should be noted that the *Renault style polyhedron* presented by I. Pak [19] can be seen as a trivial example for the case $n = 4$.

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