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# Surfaces of revolution satisfying $\triangle^{III}\boldsymbol{x} = A\boldsymbol{x}$

**Stylianos Stamatakis, Hassan Al-Zoubi**

*Department of Mathematics, Aristotle University of Thessaloniki*
*GR-54124 Thessaloniki, Greece*
*email: stamata@math.auth.gr*

**Abstract.** We consider surfaces of revolution in the three-dimensional Euclidean space which are of coordinate finite type with respect to the third fundamental form *III*, i.e., their position vector $\boldsymbol{x}$ satisfies the relation $\triangle^{III}\boldsymbol{x} = A\boldsymbol{x}$, where $A$ is a square matrix of order 3. We show that a surface of revolution satisfying the preceding relation is a catenoid or part of a sphere.

*Key Words:* Surfaces in the Euclidean space, surfaces of coordinate finite type, Beltrami operator

*MSC 2010:* 53A05, 47A75

## 1. Introduction

Let $\boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$ be a regular parametric representation of a surface $S$ in the Euclidean space $\mathbb{R}^3$ which does not contain parabolic points. For two sufficient differentiable functions $f(u^1, u^2)$ and $g(u^1, u^2)$ the first Beltrami operator with respect to the third fundamental form $III = e_{ij}du^i du^j$ of $S$ is defined by

$$\nabla^{III}(f, g) = e^{ij} f_{|i}\, g_{|j},$$

where $f_{|i} := \dfrac{\partial f}{\partial u^i}$ and $e^{ij}$ denote the components of the inverse tensor of $e_{ij}$. The second Beltrami differential operator with respect to *III* is defined by[1]

$$\triangle^{III} f = \frac{-1}{\sqrt{e}} \left(\sqrt{e}\, e^{ij} f_{|i}\right)_{|j} \tag{1}$$

$(e := \det(e_{ij}))$. In [5] we showed the relation

$$\triangle^{III}\boldsymbol{x} = \nabla^{III}\left(\frac{2H}{K}\boldsymbol{n}\right) - \frac{2H}{K}\boldsymbol{n}, \tag{2}$$

[1]with sign convention such that $\triangle = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ for the metric $ds^2 = dx^2 + dy^2$

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where $\boldsymbol{n}$ is the unit normal vectorfield, $H$ the mean curvature and $K$ the Gaussian curvature of $S$. Moreover we proved that a surface satisfying the condition

$$\triangle^{III}\boldsymbol{x} = \lambda\boldsymbol{x}, \quad \lambda \in \mathbb{R},$$

i.e., a surface $S\colon\ \boldsymbol{x} = \boldsymbol{x}(u^1, u^2)$ *for which all coordinate functions are eigenfunctions* of $\triangle^{III}$ *with the same eigenvalue* $\lambda$, is part of a sphere ($\lambda = 2$) or a minimal surface ($\lambda = 0$). Using terms of B.-Y. Chen's theory of finite type surfaces [1] the above result can be expressed as follows: *A surface $S$ in $\mathbb{R}^3$ is of III-type 1 (or of null III-type 1) if and only if $S$ is part of a sphere (or a minimal surface).*

In general a surface $S$ is said to be *of finite type* with respect to the fundamental form *III* or, briefly, *of finite III-type*, if the position vector $\boldsymbol{x}$ of $S$ can be written as a finite sum of nonconstant eigenvectors of the operator $\triangle^{III}$, that is if

$$\boldsymbol{x} = \boldsymbol{c} + \boldsymbol{x}_1 + \boldsymbol{x}_2 + \ldots + \boldsymbol{x}_m, \quad \triangle^{III}\boldsymbol{x}_i = \lambda_i\boldsymbol{x}_i, \quad i = 1, \ldots, m, \tag{3}$$

where $\boldsymbol{c}$ is a constant vector and $\lambda_1, \ldots, \lambda_m$ are eigenvalues of $\triangle^{III}$. When there are exactly $k$ nonconstant eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ appearing in (3) which all belong to different eigenvalues $\lambda_1, \ldots, \lambda_k$, then $S$ is said to be of *III-type $k$*; when $\lambda_i = 0$ for some $i = 1, \ldots, k$, then $S$ is said to be of *null III-type $k$*.

The only known surfaces of finite *III*-type are parts of spheres, the minimal surfaces and the parallel of the minimal surfaces (which are actually of null *III*-type 2, see [5]).

In this paper we want to determine the connected surfaces of revolution $S$ in $\mathbb{R}^3$ which are *of coordinate finite III-type*, i.e., their position vectorfield $\boldsymbol{x}(u^1, u^2)$ satisfies the condition

$$\triangle^{III}\boldsymbol{x} = A\boldsymbol{x}, \quad A \in M(3,3), \tag{4}$$

where $M(m,n)$ denotes the set of all matrices of the type $(m,n)$.

Coordinate finite type surfaces with respect to the first fundamental form $I$ were studied in [2] and [3]. In the last paper O. Garay showed that the only complete surfaces of revolution in $\mathbb{R}^3$, whose component functions are eigenfunctions of their Laplacian are the catenoids, the spheres and the circular cylinders, while F. Dillen, J. Pas and L. Verstraelen proved in [2] that the only surfaces in $\mathbb{R}^3$ satisfying

$$\triangle^{I}\boldsymbol{x} = A\boldsymbol{x} + B, \quad A \in M(3,3), \quad B \in M(3,1),$$

are the minimal surfaces, the spheres and the circular cylinders.

Our main result is the following

**Proposition 1.** A surface of revolution $S$ satisfies (4) if and only if $S$ is a catenoid or part of a sphere.

We first show that the mentioned surfaces indeed satisfy the condition (4).

A. On a catenoid the mean curvature vanishes, so, by virtue of (2), $\triangle^{III}\boldsymbol{x} = 0$. Therefore a catenoid satisfies (4), where $A$ is the null matrix in $M(3,3)$.

B. Let $S$ be part of a sphere of radius $r$ centered at the origin. Then

$$H = \frac{1}{r}, \quad K = \frac{1}{r^2}, \quad \boldsymbol{n} = -\frac{1}{r}\,\boldsymbol{x}.$$

So, by (2), it is $\triangle^{III}\boldsymbol{x} = 2\boldsymbol{x}$. Therefore $S$ satisfies (4) with $A = 2I_3$, where $I_3$ is the identity matrix in $M(3,3)$.

## 2. Proof of the main theorem

Let $C$ be the profile curve of a surface of revolution $S$ of the differentiation class $C^3$. We suppose that

(a) $C$ lies on the $(x_1, x_3)$-plane,

(b) the axis of revolution of $S$ is the $x_3$-axis and

(c) $C$ is parametrized by its arclength $s$.

Then $C$ admits the parametric representation

$$\boldsymbol{r}(s) = (f(s),\, 0,\, g(s))\,, \quad s \in J$$

($J \subset \mathbb{R}$ open interval), where $f(s), g(s) \in C^3(J)$. The position vector of $S$ is given by

$$\boldsymbol{x}(s,\theta) = (f(s)\cos\theta,\ f(s)\sin\theta,\ g(s))\,, \quad s \in J, \quad \theta \in [0, 2\pi).$$

Putting $f(s)' := \frac{df(s)}{ds}$ we have because of (c)

$$f'^2 + g'^2 = 1 \quad \forall s \in J. \tag{5}$$

Furthermore it is $f' \cdot g' \neq 0$, because otherwise $f = \text{const.}$ or $g = \text{const.}$ and $S$ would be a circular cylinder or part of a plane, respectively. Hence $S$ would consist only of parabolic points, which has been excluded. In view of (5) we can put

$$f' = \cos\varphi, \quad g' = \sin\varphi, \tag{6}$$

where $\varphi$ is a function of $s$. Then the unit normal vector of $S$ is given by

$$\boldsymbol{n} = (-\sin\varphi\cos\theta,\ -\sin\varphi\sin\theta,\ \cos\varphi).$$

The components $h_{ij}$ and $e_{ij}$ of the the second and the third fundamental tensors in (local) coordinates are the following

$$\begin{aligned} h_{11} &= \varphi', & h_{12} &= 0, & h_{22} &= f\sin\varphi, \\ e_{11} &= \varphi'^2, & e_{12} &= 0, & e_{22} &= \sin^2\varphi, \end{aligned} \tag{7}$$

hence [4]

$$\frac{2H}{K} = h_{ij}e^{ij} = \frac{1}{\varphi'} + \frac{f}{\sin\varphi}\,. \tag{8}$$

From (1) and (7) we find for a sufficiently differentiable function $u = u(s,\theta)$ defined on $J \times [2\pi, 0)$

$$\triangle^{III} = -\frac{u''}{\varphi'^2} + \left(\frac{\varphi''}{\varphi'^2} - \frac{\cos\varphi}{\sin\varphi}\right)\frac{u'}{\varphi'} - \frac{u_{|\theta\theta}}{\sin^2\varphi}\,. \tag{9}$$

Considering the following functions of $s$

$$P_1 = R\sin\varphi - \frac{\cos\varphi}{\varphi'}R', \quad P_2 = -R\cos\varphi - \frac{\sin\varphi}{\varphi'}R', \tag{10}$$

where we have put for simplicity $R := \frac{2H}{K}$, and applying (9) on the coordinate functions $x_i$, $i = 1, 2, 3$, of the position vector $\boldsymbol{x}$ we find

$$\triangle^{III}x_1 = P_1\cos\theta, \quad \triangle^{III}x_2 = P_1\sin\theta, \quad \triangle^{III}x_3 = P_2\,. \tag{11}$$

So we have:

(a) *The coordinate functions $x_1, x_2$ are both eigenfunctions of $\triangle^{III}$ belonging to the same eigenvalue if and only if for some real constant $\lambda$ holds*

$$\lambda f = R\sin\varphi - \frac{\cos\varphi}{\varphi'}\,R'.$$

(b) *The coordinate function $x_3$ is an eigenfunction of $\triangle^{III}$ if and only if for some real constant $\mu$ holds*

$$\mu g = -R\cos\varphi - \frac{\sin\varphi}{\varphi'}\,R'.$$

We denote by $a_{ij}$, $i, j = 1, 2, 3$, the entries of the matrix $A$. By using (11) condition (4) is found to be equivalent to the following system

$$\left\{\begin{aligned} P_1\cos\theta &= a_{11}f\cos\theta + a_{12}f\sin\theta + a_{13}\,g \\ P_1\sin\theta &= a_{21}f\cos\theta + a_{22}f\sin\theta + a_{23}\,g \\ P_2 &= a_{31}f\cos\theta + a_{32}f\sin\theta + a_{33}\,g \end{aligned}\right. \tag{12}$$

Since $\sin\theta$, $\cos\theta$ and 1 are linearly independent functions of $\theta$, we obtain from $(12_3)$ $a_{31} = a_{32} = 0$. On differentiating $(12_1)$ and $(12_2)$ twice with respect to $\theta$ we have

$$\left\{\begin{aligned} P_1\cos\theta &= a_{11}f\cos\theta + a_{12}f\sin\theta \\ P_1\sin\theta &= a_{21}f\cos\theta + a_{22}f\sin\theta \end{aligned}\right.$$

Thus $a_{13}g = a_{23}g = 0$, so that $a_{13}$ and $a_{23}$ vanish. The system (12) is equivalent to the following

$$\left\{\begin{aligned} (P_1 - a_{11}f)\cos\theta - a_{12}f\sin\theta &= 0 \\ (P_1 - a_{22}f)\sin\theta - a_{21}f\cos\theta &= 0 \\ P_2 - a_{33}g &= 0 \end{aligned}\right.$$

But $\sin\theta$ and $\cos\theta$ are linearly independent functions of $\theta$, so we finally obtain $a_{12} = a_{21} = 0$, $a_{11} = a_{22}$ and $P_1 = a_{11}f$. Putting $a_{11} = a_{22} = \lambda$ and $a_{33} = \mu$ we see that the system (12) reduces now to the following equations

$$P_1 = \lambda f, \quad P_2 = \mu g. \tag{13}$$

On account of (10) and (13) we are left with the system

$$\left\{\begin{aligned} R &= \lambda f\sin\varphi - \mu g\cos\varphi \\ R' &= -\varphi'(\lambda f\cos\varphi + \mu g\sin\varphi) \end{aligned}\right. \tag{14}$$

On differentiating $(14_1)$ with respect to $s$ we find, by virtue of (6),

$$R' = \frac{\lambda - \mu}{2}\sin\varphi\cos\varphi. \tag{15}$$

We distinguish the following cases:

**Case I.** Let $\lambda = \mu$.
Then (15) reduces to $R' = 0$.

*Subcase Ia.* Let $\lambda = \mu = 0$. From $(14_1)$ we obtain $R = 0$, i.e., $H = 0$. Consequently $S$, being a minimal surface of revolution, is a catenoid.

*Subcase Ib.* Let $\lambda = \mu \neq 0$.

Then from (6), $(14_2)$ and $R' = 0$ we have $f \cdot f' + g \cdot g' = 0$, i.e., $(f^2 + g^2)' = 0$. Therefore $f^2 + g^2 =$ const. and $S$ is obviously part of a sphere.

**Case II.** Let $\lambda \neq \mu$.
From $(14_2)$, (15) we find firstly

$$\frac{1}{\varphi'} = \frac{2\,(\lambda f \cos\varphi + \mu g \sin\varphi)}{(\mu - \lambda)\sin\varphi\cos\varphi}. \tag{16}$$

From this and (8) we obtain

$$R = \frac{\lambda + \mu}{(\mu - \lambda)\sin\varphi}\, f + \frac{2\mu}{(\mu - \lambda)\cos\varphi}\, g.$$

Hence, by virtue of $(14_1)$,

$$af + bg = 0, \tag{17}$$

where

$$a = \lambda\sin\varphi + \frac{\lambda + \mu}{(\lambda - \mu)\sin\varphi}, \quad b = \frac{2\mu}{(\lambda - \mu)\cos\varphi} - \mu\cos\varphi. \tag{18}$$

We note that $\mu \neq 0$, since for $\mu = 0$ we have

$$a = \frac{\lambda\sin^2\varphi + 1}{\sin\varphi}, \quad b = 0,$$

and relation (17) becomes

$$\frac{\lambda\sin^2\varphi + 1}{\sin\varphi}\, f = 0,$$

whence it follows $\lambda\sin^2\varphi + 1 = 0$, a contradiction.

On differentiating (17) with respect to $s$ and taking into account (16) we obtain

$$a_1\,\frac{f}{\sin\varphi} + b_1\,\frac{g}{\cos\varphi} = 0, \tag{19}$$

where

$$a_1 = \lambda(\lambda - \mu)^2\sin^4\varphi + (\lambda - \mu)(\lambda\mu - \lambda^2 + 3\lambda + \mu)\sin^2\varphi - (\lambda + \mu)(3\lambda - \mu), \tag{20}$$

$$b_1 = \mu\left[(\lambda - \mu)^2\sin^4\varphi + (\lambda - \mu)\,(\mu - \lambda + 4)\sin^2\varphi - 2\,(\lambda + \mu)\right]. \tag{21}$$

By eliminating now the functions $f$ and $g$ from (17) and (19) and taking into account (18), (20) and (21) we find

$$\lambda(\lambda - \mu)^2\sin^4\varphi + (\lambda - \mu)(\lambda\mu - \lambda^2 + 5\lambda + \mu - 2)\sin^2\varphi + (\lambda + \mu)(\mu - 3\lambda + 4) = 0.$$

Consequently

$$\lambda\,(\lambda - \mu)^2 = 0, \quad (\lambda - \mu)\left(\lambda\mu - \lambda^2 + 5\lambda + \mu - 2\right) = 0, \;\; (\lambda + \mu)\,(\mu - 3\lambda + 4) = 0.$$

From the first equation we have $\lambda = 0$. Then, the other two become as follows

$$\mu - 2 = 0, \quad \mu + 4 = 0,$$

which is a contradiction.

So the proof of the theorem is completed.

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