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The Gergonne Conic

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Abstract. The notion of Gergonne point was generalized in several ways during the last decades, including the work of Konecny [5]. Given a triangle $V_1V_2V_3$, a point I and three arbitrary directions q_i, we find a distance $x = IQ_1 = IQ_2 = IQ_3$ along these directions, for which the three cevians V_iQ_i are concurrent. If I is the incenter, q_i are the direction of the altitudes, and x is the radius of the incenter, the point of concurrency is the Gergonne point. For arbitrary directions q_i, it is shown that each point I generally yields two solutions, and points of concurrency lie on a conic, which can be called the Gergonne conic.

Key Words: Gergonne point, conics, projectivity, pencil of conics

MSC 2010: 51M04, 51N35

1. Introduction

The well-known center of the triangle is the Gergonne point, the intersection of three cevians defined by the contact points of the inscribed circle [4]. As almost every triangle center, it appears in several configurations (see, e.g., [3]), and the point is also generalized by several authors, for example for higher dimensions in [2] or by embedding in a more general context in [6]. Gergonne points are also generalized by [5], applying circles concentric to the inscribed circle.

The first question naturally arises from this point of view: if the radius of the circle x is altered, what will be the path of the point $G(x)$. Boyd et al. computed the convex coordinates of $G(x)$ in [1], which obviously yields that the path is a hyperbola.

The general problem can be formulated as follows: given a triangle $V_1V_2V_3$, a point I and three arbitrary directions q_i, find a distance $x = IQ_1 = IQ_2 = IQ_3$ along these directions, for which the three cevians V_iQ_i are concurrent. Normally these lines will not meet in one point: instead of one single center G we have three different intersection points: $G_{12} = V_1Q_1 \cap V_2Q_2$, $G_{23} = V_2Q_2 \cap V_3Q_3$ and $G_{31} = V_3Q_3 \cap V_1Q_1$. In this paper we find all the solutions and consequences of this generalized problem.

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2. The general solution

In [3] the authors proved the following theorem:

Theorem 1. *Let V_1, V_2, V_3 and I be four points in the plane in general position. Let q_1, q_2, q_3 be three different oriented lines through I ($V_i \notin q_i$). Then there exist at most two values $x \in \mathbb{R}\backslash\{0\}$ such that finding points Q_i along the lines q_i as $IQ_1 = IQ_2 = IQ_3 = x$, the lines V_iQ_i are concurrent.*

It was shown that altering the value x, the points G_{12},G_{23} and G_{31} separately move on three conics c_i, ($i = 1, 2, 3$), defined in the following way: For a real number x and $i = 1, 2, 3$, let $Q_i(x)$ be a point on q_i through I for which $IQ_i(x) = x$, ($i = 1, 2, 3$). The correspondences $Q_i(x) \leftrightarrow Q_j(x)$ define perspectivities $(q_i) \overline{\barwedge} (q_j)$, ($i \neq j$). Let $l_i(x)$ be a line connecting V_i and $Q_i(x)$. The correspondences $l_i(x) \leftrightarrow l_j(x)$ define projectivities $(V_i) \barwedge (V_j)$, ($i \neq j$). The intersection points of corresponding lines of these projectivities lie on three conics:

$$\begin{aligned} (V_1) \barwedge (V_2) &\Rightarrow c_3 \\ (V_1) \barwedge (V_3) &\Rightarrow c_2 \\ (V_2) \barwedge (V_3) &\Rightarrow c_1\,. \end{aligned} \tag{1}$$

It was shown in [3] that these conics have three common points: I and two points G_1, G_2 that can be real and different, imaginary or coinciding. These common points G_1 and G_2 are solutions of the general problem and thus can be considered as *generalized Gergonne points*, naturally depending on the triangle, the directions and the point I.

Remark 1. According to the projective principles in the proof, the statement remains valid if we substitute the condition $IQ_1 = IQ_2 = IQ_3 = x$ with the more general case, when only the ratios of these lengths have to be preserved.

3. The Gergonne conic

According to [3], for given triangle, directions and point I there are at most two generalized Gergonne points. Now for arbitrary I we collect all these points with respect to a fixed triangle and directions.

To formulate the main result, we need some further notations and lemmas. Let the line $\overline{q}_i$ be parallel to q_i through V_i (c.f. Fig. 1). Finding the points

$$C_3 = \overline{q}_1 \cap \overline{q}_2, \quad C_2 = \overline{q_1} \cap \overline{q_3}, \quad C_1 = \overline{q_2} \cap \overline{q_3},$$

the conic c_i will pass through C_i, because C_i is the intersection point of $\overline{q_j}$ and $\overline{q_k}$ that are the corresponding lines in the projectivities given by (1). Thus, for every I the conic paths c_i are uniquely determined by five points, separately:

$$c_1 = (I, C_1, G_{23}, V_2, V_3), \quad c_2 = (I, C_2, G_{31}, V_3, V_1), \quad c_3 = (I, C_3, G_{12}, V_1, V_2),$$

which have 3 common points: I, G_1, G_2, where G_1, G_2 are the solutions of the problem.

Lemma 1. *The vertex V_1 (V_2 or V_3) is the solution for a point I if and only if $I \in t_1$ (t_2 or t_3), where t_1 (t_2 or t_3) is the line which contains the points at equal distance from V_1V_2 (V_2V_3 or V_3V_1) and V_1V_3 (V_2V_1 or V_3V_2) measuring along the directions q_2 (q_3 or q_1) and q_3 (q_1 or q_2), respectively.*

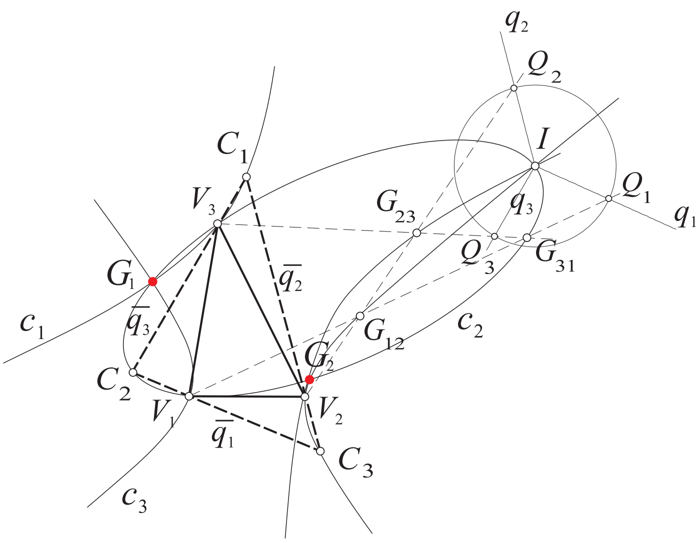


Figure 1: Points C_i, $i = 1, 2, 3$, are defined by lines $\overline{q}_i$ parallel to q_i through V_i

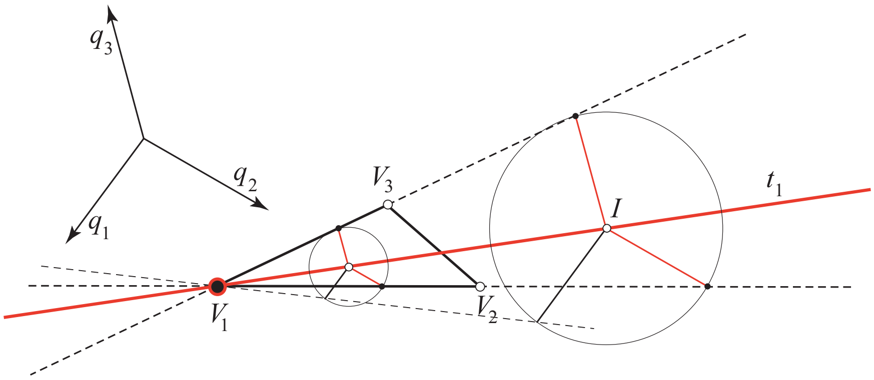


Figure 2: Vertex V_1 of the triangle can also be a generalized Gergonne point for points I being in special position

Proof: The statement follows directly from the definition of the generalized Gergonne point and the properties of a central similarity (see Fig. 2). □

Lemma 2. *If a point $I \neq V_i$ lies on the line V_2V_3 (V_3V_1 or V_1V_2), then the conic c_1 (c_2 or c_3) splits into the line V_2V_3 (V_3V_1 or V_1V_2) and another line which passes through the solutions.*

Proof: If $I \in V_2V_3$, then the first correspondence given by (1) is the perspectivity $(V_1)\,\overline{\wedge}\,(V_2)$. Thus, the conic c_1 splits into the axes of this projectivity and the self corresponding ray V_2V_3. According to the proof of Theorem 1 [3], the axes of the projectivity $(V_1)\,\overline{\wedge}\,(V_2)$ cuts the conics c_2 and c_3 into the solutions G_1, G_2. This proof is also valid for the circular shift of index. □

Lemma 3. *If a point $I \neq C_i$ lies on the line C_2C_3 (C_3C_1 or C_1C_2), then each of the conics c_2, c_3 (c_3, c_1 or c_1, c_2) splits into the line C_2C_3 (C_3C_1 or C_1C_2) and another line which passes through the corresponding vertex V_3, V_2 (V_1, V_3 or V_2, V_1) and one solution G_1.*

Proof: Let $I \in C_2C_3$, and let x_1 be the distance between the points I and V_1. In this case the previously described correspondence $l_1(x) \leftrightarrow l_2(x)$ leads to the following:

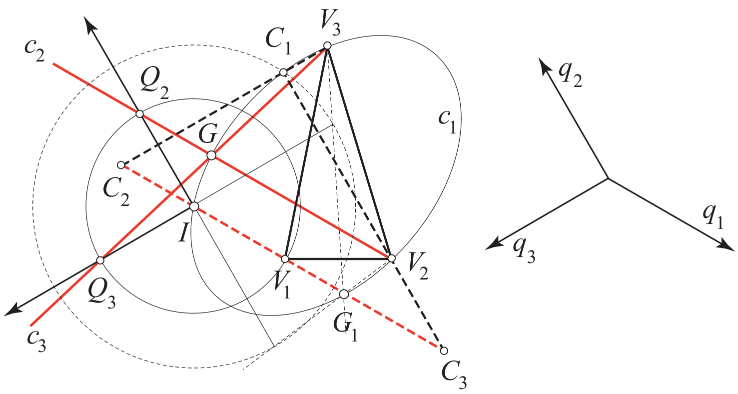


Figure 3: If point I is on the line C_2C_3 then conics c_2 and c_3 split up into pairs of lines

– the line C_2C_3 corresponds with all lines $l_2(x) \in (V_2)$, $x \in R$,

– the line $l_2(x_1)$ corresponds with all lines $l_1(x) \in (V_1)$, $x \in R$.

Thus, the conic c_3 splits up into the lines C_2C_3 and V_2Q_2, where $IQ_2 = x_1$ and the line IQ_2 is parallel to q_2. In the same way: the conic c_2 splits up into the lines C_2C_3 and V_3Q_3, where $IQ_3 = x_1$ and the line IQ_3 is parallel to q_3. It is clear that the intersection point of the lines V_2Q_2 and V_3Q_3 lies on the conic c_1 and is the solution G. The second solution $G_1 \neq I$ is the intersection point of the conic c_1 and the line C_2C_3 (see Fig. 3). This proof is also valid for the circular shift of index. □

Lemma 4. *If G is the solution for I, then it is the solution for every $I^j \in l$, where l is the line through I and G. Consequently if G is the solution for I_1 and I_2 as well, then the points G, I_1 and I_2 are collinear.*

Proof: The statement follows directly from the definition of the generalized Gergonne point and the properties of a central similarity (see Fig. 4). □

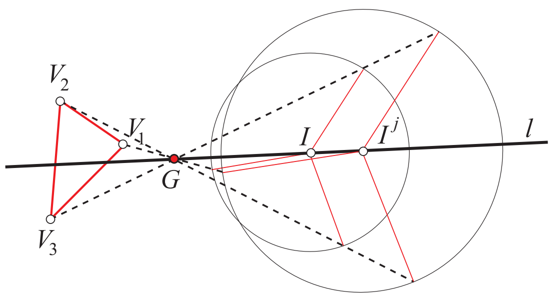


Figure 4: If G is the solution for I, then it is the solution for every point I^j on the line IG

By these lemmas we can prove our main result, that for given triangle and directions all the generalized Gergonne points lie on one conic, which can be called *Gergonne conic*.

Theorem 2. *For given $V_1, V_2, V_3, q_1, q_2, q_3$, all solutions lie on one conic Γ which passes through V_1, V_2, V_3.*

Proof: Let I, G_1 and l be as in Lemma 4 (see Fig. 5). For every $I^j \in l$ we consider the conics $c_2^j = (V_1, V_3, C_2, G_1, I^j)$ and $c_3^j = (V_1, V_2, C_3, G_1, I^j)$. These conics have 4 common

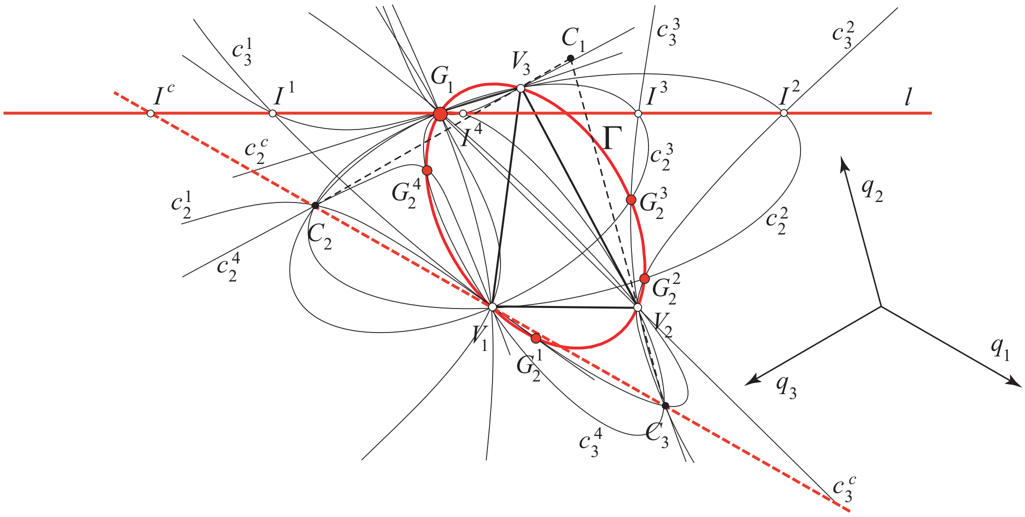


Figure 5: The conic Γ contains all the solutions for the given triangle and directions. Further details in the proof of Theorem 2. (This figure is computed and plotted by the software *Mathematica*)

points: V_1, G_1, I_j and (according to the previous theorem) the other solution G_2^j. The correspondence $c_2^j \leftrightarrow c_3^j$ is $(1,1)$ correspondence between the pencils of conics (V_1, V_3, C_2, G_1) and (V_1, V_2, C_3, G_1). According to the Chasles's formula [8], the intersection points of the corresponding conics lie on one plane curve of the order $1 \cdot 2 + 1 \cdot 2 = 4$. In our case this quartic splits up into the lines l and C_2C_3 and one conic Γ. Namely, for $I^c = C_2C_3 \cap l$ conics c_2^c and c_3^c split up into C_2C_3 and the lines V_3G_1, V_2G_1 (according to Lemma 3); every $I^j \in l$ is the part of $c_2^j \cap c_3^j$; all solutions G_2^j lie on one residual conic Γ which passes through V_1 because it is the intersection point for all pairs (c_2^j, c_3^j). Since Γ is also the result of the correspondences $c_1^j \leftrightarrow c_2^j$ and $c_1^j \leftrightarrow c_3^j$, we can conclude that it also passes through V_2 and V_3.

If $\overline{G} \neq G_1$ is any other solution for any other point $\overline{I} \neq I$, then, according to Lemma 4, $\overline{G}$ is the solution for the intersection point $\overline{\overline{I}} = \overline{l} \cap l$. According to the previous consideration, $\overline{G}$ also lies on the conic Γ. □

Lemma 5. *On every line $l = IG_1$ there exists a unique point I^Δ for which the solutions coincide, i.e., $G_2^\Delta = G_1$.*

Proof: We consider the projectivity $(l) \barwedge (G_1)$[1] where corresponding elements are $I^j \leftrightarrow G_1G_2^j$. The point I^Δ which corresponds with the tangent line of the conic Γ at its point G_1 has coinciding solutions in G_1. □

Theorem 3. *For given $V_1, V_2, V_3, q_1, q_2, q_3$, all lines $l = IG$ (where G is the solution for I) form one envelope conic Δ with tangent lines C_iC_j.*

Proof: Let G_1 and G_2 be two different solutions for I and $l_1 = IG_1$, $l_2 = IG_2$. For every point $G^j \in \Gamma$ let $I_1^j \in l_1$ and $I_2^j \in l_2$ be the points for which G^j is the solution. (I_1^j and I_2^j are

[1]In the proof of Theorem 2, with $I^j \leftrightarrow G_2^j$, we defined a projective correspondence between the range (l_1) and the 2nd order range (Γ). Since the cross ratio of the points on one conic can be defined as the cross ratio of lines which join them with any other point on this conic, we can conclude that $I^j \leftrightarrow G_1G_2^j$ defines the projectivity between (l_1) and (G_1).

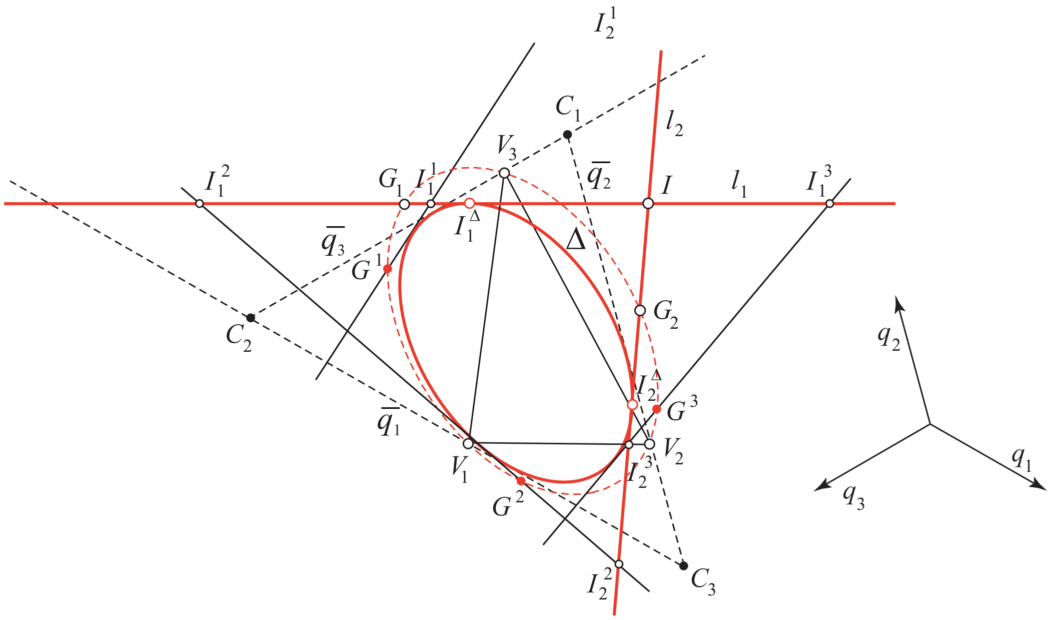


Figure 6: The lines connecting any point I with its solutions G_1 and G_2 are tangent to the conic Δ at I_1^Δ and I_2^Δ, respectively

uniquely determined: $I_1^j = l_1 \cap (V_2, V_3, C_1, G_1, G^j)$, $I_2^j = l_1 \cap (V_2, V_3, C_1, G_2, G^j)$.) According to Lemma 4 points I_1^j, I_2^j and G^j are collinear.

There are two projective correspondences $(l_1) \barwedge (\Gamma)$ and $(l_2) \barwedge (\Gamma)$ which induce the projectivity $(l_1) \barwedge (l_2)$ with corresponding pairs $I_1^j \leftrightarrow I_2^j$. This projectivity determines one envelope conic Δ and it is clear, according to Lemma 4, that every line which joins any point with its solution is the tangent line of it (see Fig. 6).

According to the Lemmas 3 and 4: if $I \in C_iC_j$, one solution always lies on the line C_iC_j and it is the solution for every $I \in C_iC_j$, $I \neq C_i$, $I \neq C_j$.

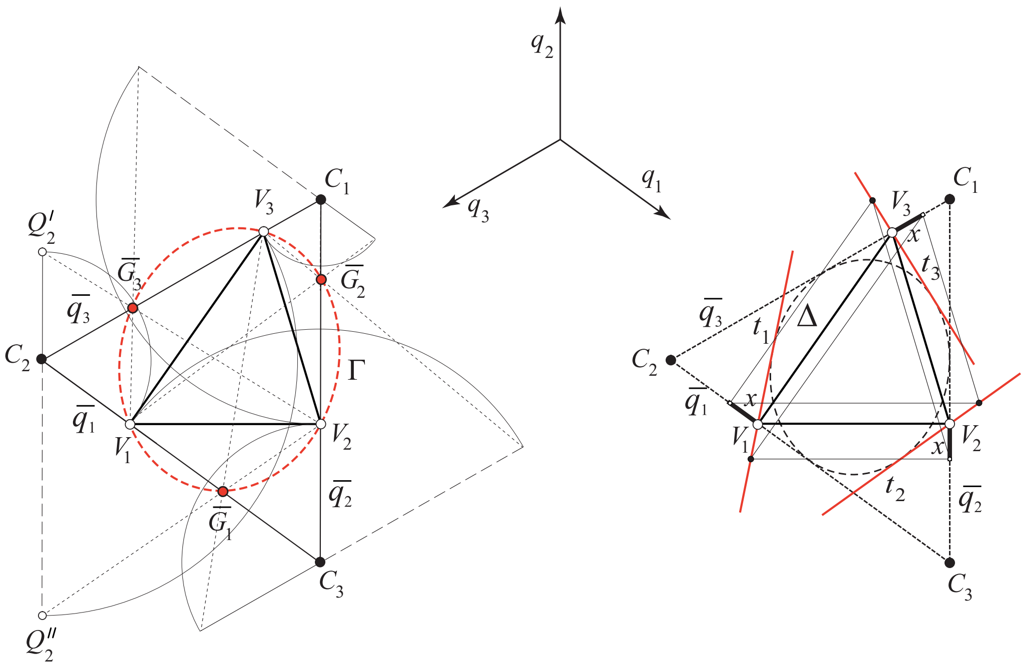
The touching point of Δ and l_1 (l_2) corresponds with $I \in l_2$ ($I \in l_1$), thus this touching point is I_1^Δ (I_2^Δ) from Lemma 5, i.e., the point with coinciding solutions $G_1 \in l_1$ ($G_2 \in l_2$). □

Corollary 1. *The conic Δ separates the possible positions of I in the way mentioned in the proof of Theorem 1 in [3].*

Proof: Since lines which join points I with their solutions are the tangent lines of Δ it is clear that solutions are real and different for $I \in Ext\Delta$, real and coinciding for $I \in \Delta$ and imaginary for $I \in Int\Delta$. □

4. Construction of conics Γ and Δ

Since V_1, V_2, V_3 lie on Γ it is enough to construct only two other points on it. Arbitrary number of points $G \in \Gamma$ can be constructed according to Lemma 3. Here we will describe the construction of the points $\overline{G}_i \in \Gamma$, $\overline{G}_i \neq V_i$, $\overline{G}_i \in \overline{q}_i$. They can be constructed as the solutions for the points C_1, C_2 or C_3. For example: If $I = C_2 = \overline{q}_1 \cap \overline{q}_3$ the conic c_2 splits into the lines $\overline{q}_1$, $\overline{q}_3$. The conic c_1 splits into the lines $\overline{q}_3$ and V_2Q_2'', where Q_2'' is on the same distance from C_2 as the point V_3 measuring along the direction q_2. The conic c_3 splits into the lines $\overline{q}_1$ and V_2Q_2', where Q_2' is on the same distance from C_2 as the point V_1 measuring along the direction



a b

Figure 7: Construction of conics Γ and Δ

q_2. The intersection points of c_1, c_2, c_3 are $\overline{G}_1$ and $\overline{G}_3$. The point $\overline{G}_2$ can be constructed, in the same way, as one solution for the point C_1. According to Lemma 4 the point $\overline{G}_1$ ($\overline{G}_2$ or $\overline{G}_3$) is a common solution for the points C_2, C_3 (C_3, C_1 or C_1, C_2), see Fig. 7a.

Since the lines $\overline{q}_i$ are the tangent lines of Δ it is enough to construct only two other tangent lines of it. According to Theorem 3 the tangent lines of the conic Δ join points with their solutions. Thus, the construction of the lines t_i which is described in Lemma 1 determines the conic Δ. The lines t_i can be constructed in an elementary way, see Fig. 7b.

5. Special cases

So far we considered that the directions q_i and the point I are in general position. This way, according to Theorem 1, the number of solutions is at most two. But at the previously mentioned result of Konecny, if the directions are perpendicular to the corresponding sides of the triangle and I is the incenter, there are infinitely many solutions. To overcome this problem the following lemma and theorem clarify the special cases.

Lemma 6. *For given V_1, V_2, V_3, q_1, q_2, q_3 there exists a point I with more than two solutions if and only if the points C_i coincide with one point C (the lines $\overline{q}_i$ are concurrent). In this case there are infinitely many solutions for the point I. These solutions lie on the conic Γ_I which passes through the points V_1, V_2, V_3, I and C. For every point $I^j \neq I$, point C is one solution and the other solution G^j is the intersection point of conic Γ_I and the line II^j.*

Proof: If there are three solutions G_1, G_2, G_3 for I, then the conics c_i coincide because each pair of them passes through the same five points. Thus there exists one conic Γ_I which passes through seven points G_1, G_2, G_3, V_1, V_2, V_3 and I, and every point on it is a solution for I. Since the conic Γ_I contains points C_i ($C_i \in c_i$) we can conclude that the points C_i must

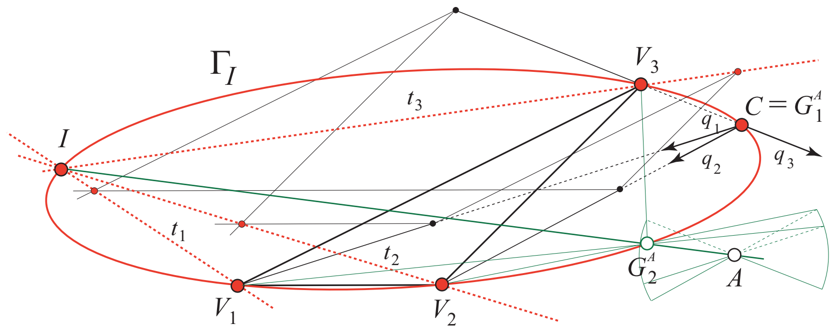


Figure 8: Construction of the point I with infinitely many solutions for given C, V_1, V_2, V_3 and Gergonne points for arbitrary point A

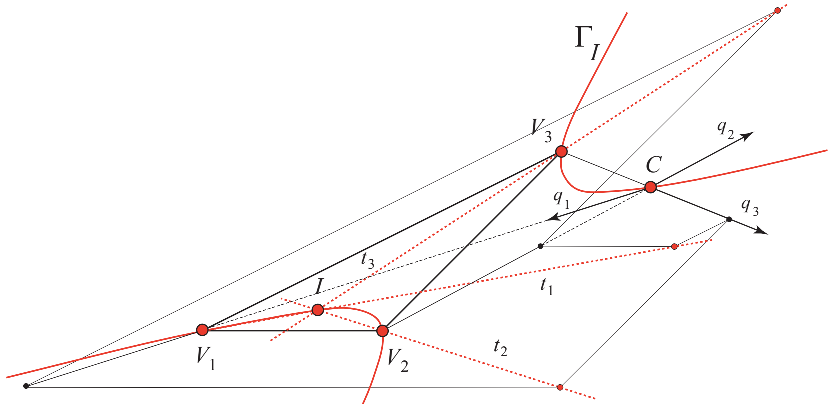


Figure 9: Construction of the point I and the conic Γ_I in the case when the fundamental elements C, V_i, q_i differ from those in Fig. 8 only in the orientation of q_2

coincide because in the opposite case the line C_1C_2 which passes through V_3 would cut Γ_I in three points, which is impossible.

If the lines $\overline{q}_i$ are concurrent with the intersection point C, then C is the solution for every I^j (C corresponds with the points at infinity on the lines q_i through I^j in the projectivities given by (1)). Let I^1, I^2 be two different points with different solutions $G^1 \neq C$, $G^2 \neq C$. According to Lemma 4, G_1, G_2 are the solutions for the intersection point I of the lines I^1G^1, I^2G^2. Thus, there are three solutions for I (G^1, G^2, C), and according to the previous consideration there are infinitely many solutions for I which lie on the conic Γ_I.

Now, it is clear that for every point $I^j \neq I$ one solution is C and the other G^j must lie on the line I^jI and the conic Γ_I.

Since in this case the lines which join any point with its solutions pass through I and C, the envelope conic Δ splits up into the pencils of lines (I) and (C). □

Let the lines V_1V_2, V_1V_3, V_2V_3 be denoted by a_3, a_2, a_1, respectively, and let $d_b(P, a)$ denote the distance from a point P to a line a measuring along a direction b.

Theorem 4. *Let V_1, V_2, V_3 and C be four points in the plane in general position and let three oriented lines q_i coincide with the lines CV_i, $i = 1, 2, 3$. Then there exists a point I such that*

$$d_{q_i}(I, a_j) = d_{q_j}(I, a_i), \quad i \neq j, \ i, j = 1, 2, 3 \tag{2}$$

and every point $G \in \Gamma_I$ (where Γ_I is the conic through the points V_1, V_2, V_3, C, I) is a generalized Gergonne point for I. For every other point P there exist two generalized Gergonne points, $G_1^P = C$ and $G_2^P = \Gamma_I \cap PI$.

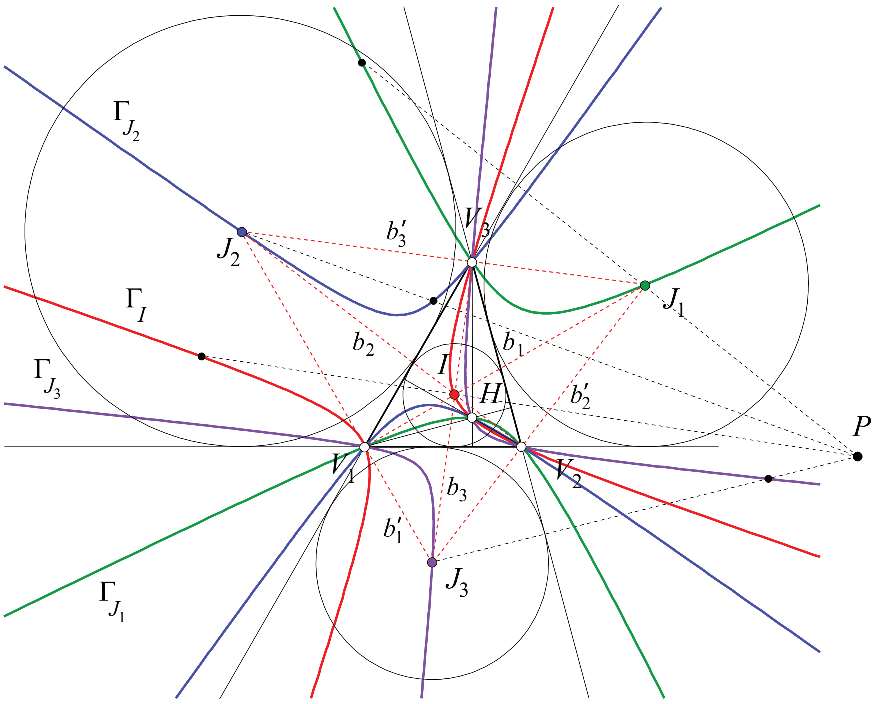


Figure 10: For an arbitrary point P the generalized Gergonne point is the intersection of the corresponding hyperbola Γ and the line PI or PJ_i

Proof: According to Lemma 6 in such a case the envelope conic Δ splits into the pencils of lines (C) and (I), where I is the point with infinitely many generalized Gergonne points on the conic Γ_I. Thus, according to Theorem 3 the lines V_1I, V_2I, V_3I are the lines t_1, t_2, t_3 from Lemma 1, respectively, and their intersection point I has the property (2). The other statements follow directly from Lemma 6 (see Figs. 8 and 9). □

Since that for every q_i, $i = 1, 2, 3$, there are two different orientations, there are 2^3 possibilities for the choosing of the oriented lines q_i. But, these choices in pairs determine the same way for the finding generalized Gergonne points (if the orientations are opposite for every $i = 1, 2, 3$). Thus, for the given non-oriented lines q_i there are four different possibilities for finding the generalized Gergonne points.

Corollary 2. *If the point C from Theorem 4 is the orthocenter H of $\triangle V_1V_2V_3$, then the point I is the incenter or the excenter J_i, $i = 1, 2, 3$, of $\triangle V_1V_2V_3$. The conics Γ_I, Γ_{J_i} are Feuerbach hyperbolas.*

Proof: If the lines q_i, $i = 1, 2, 3$, are the altitudes of $\triangle V_1V_2V_3$, the line t_i is the internal angle bisector b_i or the external angle bisector b_i' of $\triangle V_1V_2V_3$, depending on the orientation of q_i. The proof of this statement is elementary and is based on triangle congruence theorems. Thus, the point I from Theorem 4 is the incenter or excenter of $\triangle V_1V_2V_3$. The conics Γ_I and Γ_{J_i} pass through the ortocenter H are rectangular hyperbola which are called Feuerbach hyperbolas (see Fig. 10). □

An even more specialized situation is when I and C, from Theorem 4, coincide. This is the equilateral triangle $V_1V_2V_3$ with the center C and the positive orientation on q_i given by $\overrightarrow{CV_i}$. In this case, for any point P there are infinitely many generalized Gergonne points which lie on the line PC.

6. Further research

Conics Γ, Δ and c_i, $i = 1, 2, 3$, play central role in the construction. The affine types of these conics however, may only be determined by analytical approach or by closer study of the type of projective pencils determined by cevians. It is also a topic of further research how the types of solutions depend on the ratios mentioned in Remark 1. The exact representation of the length of the radius by the given data can also be discussed analytically in a further study.

References

[1] J.N. Boyd, P.N. Raychowdhury: *The Gergonne point generalized through convex coordinates*. Int. J. Math. Sci. **22**, 423–430 (1999).

[2] M. Hajja, P. Walker: *The inspherical Gergonne center of a tetrahedron*. J. Geometry Graphics **8**, 23–32 (2004).

[3] M. Hoffmann, S. Gorjanc: *On the generalized Gergonne point and beyond*. Forum Geom. **8**, 151–155 (2008).

[4] C. Kimberling: *Gergonne Point*. `http://faculty.evansville.edu/ck6/tcenters/class/gergonne.html`.

[5] V. Konecny: *Problem department*. Math. Mag. **69**, 130–131 (1990).

[6] B. Odehnal: *Generalized Gergonne and Nagel Points*. Beiträge zur Algebra und Geometrie **51**, 477–491 (2010).

[7] O. Veblen, J.W. Young: *Projective Geometry*. Ginn, Boston/MA 1938.

[8] H. Wieleitner: *Theorie der ebenen algebraischen Kurven höherer Ordnung*. G.J. Göschensche Verlagshandlung, Leipzig 1905.

[9] B. Wojtowicz: *Desargues' Configuration in a Special Layout*. J. Geometry Graphics **7**, 191–199 (2003).

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