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Closed Space Curves Made from Circles on Polyhedra

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Abstract. Suppose that $\mathcal{P}$ is a polyhedron, all of whose faces are regular polygons such that the incircles of adjoined faces are tangent to each other. Various closed space curves are then determined by linking together portions of the circles. This paper examines such biarc curves, concentrating on those which lie not only on $\mathcal{P}$, but also on a sphere. Thirteen of these are called the *regular polyhedral polyarcs*: two on a tetrahedron, three on a cube, two on an octahedron, four on a dodecahedron, and two on an icosahedron. More general spherical circle-to-circle curves are also considered.

Keywords: biarc, polyarc, regular polyhedra, sphericon, spherical curve, quadrarc

MSC 2010: 51M20, 51M04, 51N20

1. Introduction

On a sheet of paper or in your mind, draw two circles that share exactly one point; label it P. Fold the paper so as to separate the circles. Now, keeping the angle of the fold strictly between 0° and 180°, trace along one of your circles through P onto the other circle. There are two ways to do this, but in only one of them does the motion continue "in the same direction" while passing through P; this is the one we want, so that the resulting two-part curve has no cusp. Now we transfer the paper-folding idea to a cube that has a circle inscribed in each face:

In Fig. 1, imagine a variable point X approaching P from below on the left-hand facial circle. At P, the point X could either continue on the same circle or else continue on a different circle. An essential feature of this situation is that there is *only one* other circle at P. Imagine X continuing toward Q; on arrival there we again find a unique new circle available. Choosing it and continuing in this manner, the point X eventually returns to P,

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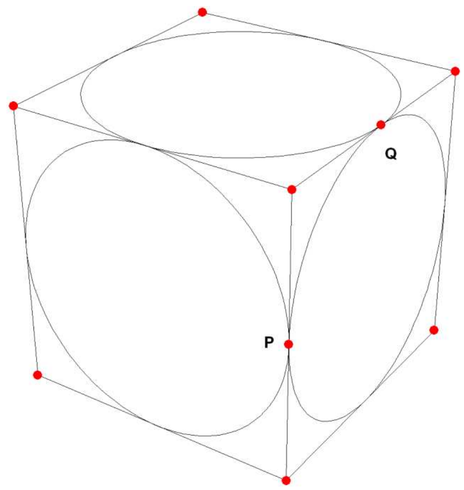

Figure 1: Three facial incircles

having visited all 6 faces. The locus of X is a curve consisting of 6 quarter-circles; the curve lies on a sphere and meets the edges of the cube in 6 coplanar points.

Before looking more closely at these properties, we quickly describe two more curves: instead of using 1/4 of each circle, use 1/2 of each, or use 3/4 of each. These curves, too, lie on a sphere and are bisected by a plane. One could start with a tetrahedron instead of a cube, or any polyhedron whose faces have incircles — that is all that is needed for tracing cusplessly from one face to another. Among such polyhedra are prisms and antiprisms, 13 Archimedean solids, 92 Johnson solids (all shown at MathWorld [18]), and, of course, the 5 Platonic solids. In this article, we shall concentrate on the latter, also known as the 5 regular polyhedra. Many of these polyhedral curves, which appear to be new to the literature, are exhibited in animations in the Gallery [9].

Closed curves made from circles, but which do not lie on polyhedra, have been widely studied ([2, 5, 7, 12, 15]), especially in connection with biarcs. The curves to be introduced in this article are biarcs, in accord with the definition given in Bolton [2]: *A biarc curve consists of mutually tangential circular arc and straight line sections constructed to ensure continuity of the first derivative.* Of particular interest is a result from Fuhs and Stachel [5]: *There are infinitely many connecting biarcs between any two line elements. They all lie on a sphere; all the intermediate contact points trace a circle on the sphere, and the corresponding tangent lines generate a hyperboloid of one sheet.*

2. A baseball curve made from circles

Here we shall examine the cubic quadrarc, or CQ, which resembles the curve on a tennis ball (Fig. 2) and may therefore be regarded one of the many so-called baseball curves. Google finds a remarkable variety of these in physics, biology, and art, as well as sports and mathematics. Some of these curves appear in Allison, Diaz, and Miller [1]; Thompson [17] analyzes baseball cover designs, and López-López [10] reports measurements of softballs, basketballs, and tennis balls, concluding that they all show “the baseball curve” — so that one may say that this is one of the world’s most common space curves — right up there with the helix.

In addition to the construction of the CQ using half-circles on a cube, a second method starts with an intersection of the two cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$ as in Fig. 3 and

Figure 2: One of many "baseball curves" — on a tennis ball

[9, Gallery #1, #2]: simply wrap a string around the intersection as shown.

While viewing Fig. 3, imagine positive x and y axes starting at the center of the figure and pointing to the right and behind, respectively. When looking from the point $(10, 0, 0)$, you can view the CQ as a four-part curve consisting top, right, bottom, and left half-circles, with parametric equations

$$A = A(\theta) = \begin{cases} x = \sin\theta \\ y = -\cos\theta \\ z = 1, \qquad 0 \le \theta < \pi, \end{cases}$$

$$B = B(\theta) = \begin{cases} x = -\sin\theta \\ y = 1 \\ z = \cos\theta, \qquad 0 \le \theta < \pi, \end{cases}$$

$$C = C(\theta) = \begin{cases} x = -\sin\theta \\ y = \cos\theta \\ z = -1, \qquad 0 \le \theta < \pi, \end{cases}$$

$$D = D(\theta) = \begin{cases} x = -\sin\theta \\ y = -1 \\ z = \cos\theta, \qquad 0 \le \theta < \pi. \end{cases}$$

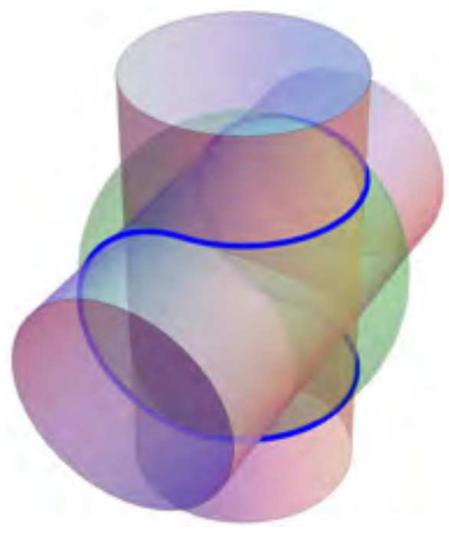

Figure 3: CQ on a sphere, binding two cylinders

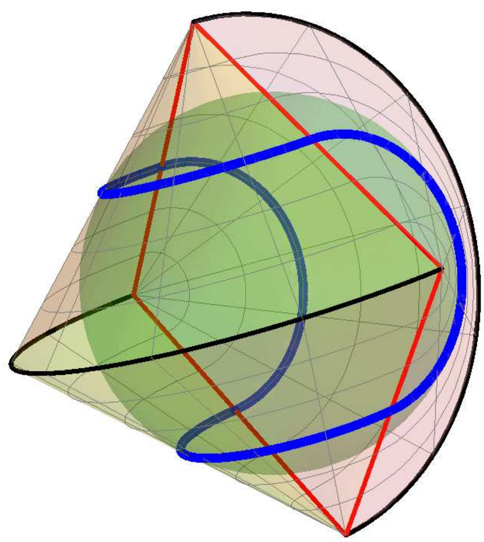

Figure 4: CQ on a sphericon

These equations readily show that the CQ lies on the sphere

$$x^2 + y^2 + z^2 = 2$$

and belongs to the cube having vertices $(\pm 1, \pm 1, \pm 1)$ on the sphere

$$x^2 + y^2 + z^2 = 3.$$

Now for a pleasant surprise: the CQ lies on a sphericon, as shown in Fig. 4 and also at Robert FERRÉOL's wonderful page [3], one of many [4]. The sphericon, a clever piecing together of two half-cones, has a number of interesting properties, as presented in STEWART [16]. Equations for the sphericon are easily found:

$$z = \begin{cases} 2 - \sqrt{x^2 + y^2} & x \geq 0 \quad \text{(top)} \\ -2 + \sqrt{x^2 + y^2} & x \geq 0 \quad \text{(bottom)} \end{cases}$$

$$y = \begin{cases} 2 - \sqrt{x^2 + z^2} & x \leq 0 \quad \text{(front)} \\ -2 + \sqrt{x^2 + z^2} & x \leq 0 \quad \text{(back).} \end{cases}$$

As Eric WEISSTEIN writes [18], the sphericon is not as widely known as it should be. For an animated sphericon-with-CQ, see [9, Gallery #3].

Having seen that the CQ fits nicely on a cube, a sphere, and a sphericon, we ask next what objects fit nicely in the CQ. Right away, one imagines the square $(0, \pm 1, \pm 1)$ in Fig. 3, but let's keep an eye on *all six* of the line segments made from the four *variable* points A, B, C, D in the above parametric equations for the CQ:

$$|AB|^2 = |BC|^2 = |CD|^2 = |DA|^2 = 4 + 2\sin^2\theta\,;$$
$$|AC|^2 = |BD|^2 = 4 + 4\cos^2\theta\,,$$

so that we have an identity:

$$2\,|AB|^2 + |AC|^2 = 16.$$

Putting $|AB|^2 = |AC|^2$ leads to $\tan\theta = \pm\sqrt{2}$, or equivalently,

$$\theta \in \{\arccos\sqrt{1/3},\ \pi - \arccos\sqrt{1/3}\}.$$

So, for these two values of θ, and only these two, the six segments have the same length $(4\sqrt{3}/3)$; thus, the CQ has exactly two inscribed regular tetrahedra.

Again regarding θ as a variable, we can say that the edges AB, BC, CD, DA form a regular spherical quadrilateral (a square when $\theta = 0$). The tetrahedron formed by the four points lends itself to interesting animations [9, Gallery #3, #4].

Continuing with variable θ, let

$$S(\theta) = |AB| + |AC| + |AD| + |BC| + |BD| + |CD|\,.$$

Then $S'(\theta) = 0$ for $\tan\theta = \pm\sqrt{2}$; i.e., the two inscribed regular tetrahedra are the configurations that maximize the sum of edgelengths. For $0 \leq \theta < \pi$, there are two other solutions of the equation $S'(\theta) = 0$; these correspond to configurations for which the sum of edgelengths is locally minimal: at $\theta = 0$, four sides have length 2 and two have length $\sqrt{6}$, and at $\theta = \pi/2$, four have length $\sqrt{6}$ and two have length 2.

Although we may speak of *the* CQ of a cube, it is easy to count a total of 16 CQs, but they are all congruent to any one.

A final note regards the name "cubic quadrarc". The term *quadrarc* was used by GRIDGEMAN [6] and then by ROSIN [14] and KIMBERLING [8], in connection with approximations of ellipses by circular arcs, as at the Ellipse at St. Peter's Square [19] in Rome. Possibly *quadrarc* has not previously been used in the names of space curves.

3. Two more curves on a cube

In Section 1, three curves are defined using incircles on a cube. One of these, the CQ, uses 1/2 of each circle. The curve which uses 1/4 of each circle we shall call the *short cubic hexarc* (SCH, as in Figs. 5 and 6), and the one which uses 3/4 of each circle, the *long cubic hexarc* (LCH, as in Fig. 7). Next, referring to the configuration in Fig. 1, we take the vertices of the cube be $(\pm 1, \pm 1, \pm 1)$. There are actually 16 mutually congruent SCHs and 16 mutually congruent LCHs on the cube.

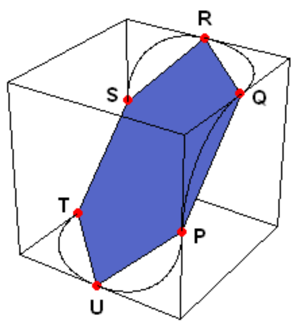

Figure 5: SCH on a sphere

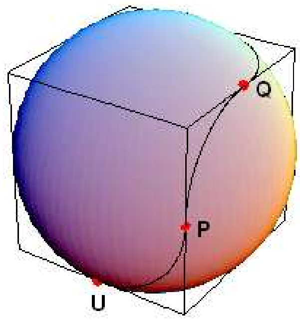

Figure 6: SCH in a cube, with inscribed regular hexagon

In order to study a single example, take P and Q in Fig. 1 to be two of six points in which the SCH meets edges of the cube, and write

$$P = (1, 1, 0) \qquad Q = (0, 1, 1) \qquad R = (-1, 0, 1)$$
$$S = (-1, -1, 0) \quad T = (0, -1, -1) \quad U = (1, 0, -1).$$

These points lie on the plane $x - y + z = 0$ and hence on the great circle formed by intersecting this plane with the sphere $x^2 + y^2 + z^2 = 2$. If the whole configuration is moved rigidly so that the circle becomes horizontal, we may regard it as an equator, $\mathbb{E}$, and the SCH can be described as follows: start at P, stay on 1/4 of an incircle above $\mathbb{E}$, thus arriving at Q; then stay on 1/4 of an incircle below $\mathbb{E}$, thus arriving at R, and continue in this manner, oscillating above and below $\mathbb{E}$ until returning to P.

Figure 6 and [9, Gallery #5] show the six points and call attention to the trivial fact that they are the vertices of a regular hexagon:

The LCH can be described relative to the same equator, using 3/4 of each incircle, and here, too, we have the same inscribed hexagon. Another shape inscribed in the LCH is the cube octahedron [11], as shown in Fig. 7 and [9, Gallery #6].

4. Tetrahedral quadrarcs

Let $ABCD$ be a regular tetrahedron, and let A', B', C', D' be the incircles of the face-triangles of $ABCD$, namely, BDC, CDA, DAB, ABC, respectively. Let

$$M_{BC},\ M_{CA},\ M_{AB},\ M_{CD},\ M_{DB},\ M_{DA}$$

be the respective midpoints of sides BC, CA, AB, CD, DB, DA. *The short tetrahedral quadrarc,* STQ, is the union of these four circular arcs: on A', the shorter arc from M_{DB} to M_{BC}; on

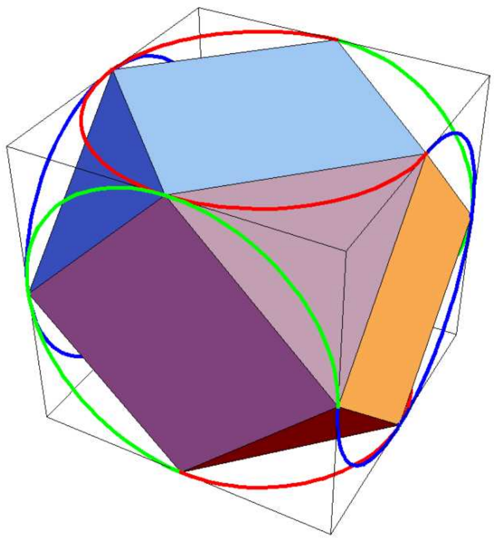

Figure 7: LCH in a cube, with inscribed cube octahedron

B', the shorter arc from M_{BC} to M_{CA}; on C', the shorter arc from M_{CA} to M_{AD}; on D', the shorter arc from M_{AD} to M_{DB}. The *long tetrahedral quadrarc,* LTQ, is a complement of the STQ in the sense that it is defined by taking the other of the two arcs on each of the circles. These definitions of the STQ and LTQ obviously agree with the method given in Section 1, applied to a tetrahedron instead of a cube. While a cube admits 16 of each of the curves CQ, SCH, LCH, a tetrahedron admits a total of 3 STQs and 3 LTQs.

An arbitrary regular tetrahedron is similar to the tetrahedron having vertices

$$A = (3, 3, 3), \quad B = (3, -3, -3), \quad C = (-3, 3, -3), \quad D = (-3, -3, 3),$$

and we shall use this tetrahedron to develop parametric equations for the STQ and LTQ. The incircle D' of face-triangle ABC passes through the points

$$M_{AB} = (3, 0, 0), \; M_{BC} = (0, 0, -3), \; M_{CA} = (0, 3, 0),$$

so that the center O_D of D' is given by

$$O_D = (\sqrt{3}, \, \sqrt{3}, \, -\sqrt{3}).$$

Let S be the sphere that has center O_D and passes through $(3, 0, 0)$, and let U denote the sphere $x^2 + y^2 + z^2 = 9$. Since $S \cap U$ is necessarily a circle, and since the three points M_{AB}, M_{BC}, M_{CA} lie on this circle as well as on the circle D', the two circles are identical. Similarly, the circles A', B', C' lie on U. To summarize: *the vertices of the tetrahedron $ABCD$ lie on the sphere*

$$x^2 + y^2 + z^2 = 27\,,$$

while the STQ and LTQ lie on the sphere

$$x^2 + y^2 + z^2 = 9\,.$$

It is easy to check that the four incircles are given by the following parametric equations:

$$\text{circle } D': \begin{cases} x = 1 + 2\cos t, \\ y = 1 - \cos t + \sqrt{3}\sin t, \\ z = -1 + \cos t + \sqrt{3}\sin t; \end{cases}$$

$$\text{circle } B': \begin{cases} x = -1 + \cos t - \sqrt{3}\sin t, \\ y = 1 + 2\cos t, \\ z = 1 - \cos t - \sqrt{3}\sin t; \end{cases}$$

$$\text{circle } C': \begin{cases} x = -1 - 2\cos t, \\ y = -1 + \cos t - \sqrt{3}\sin t, \\ z = -1 + \cos t + \sqrt{3}\sin t; \end{cases}$$

$$\text{circle } A': \begin{cases} x = 1 - \cos t + \sqrt{3}\sin t, \\ y = -1 - 2\cos t, \\ z = 1 - \cos t - \sqrt{3}\sin t; \end{cases}$$

where, for all four circles, $0 \le t \le 2\pi$. Remarkably, each circle passes through six points having only integers for coordinates:

Table 1: All-integer coordinates on the four incircles

t	on circle D'	on circle B'	on circle C'	on circle A'
0	$(3,0,0)$	$(0,3,0)$	$(-3,0,0)$	$(0,-3,0)$
$\pi/3$	$(2,2,1)$	$(-2,2,-1)$	$(-2,-2,1)$	$(2,-2,-1)$
$2\pi/3$	$(0,3,0)$	$(-3,0,0)$	$(0,-3,0)$	$(3,0,0)$
π	$(0,0,-3)$	$(-2,-1,1)$	$(0,0,-3)$	$(2,1,1)$
$4\pi/3$	$(-1,2,-2)$	$(0,0,3)$	$(1,-2,-2)$	$(0,0,3)$
$5\pi/3$	$(2,-1,-2)$	$(1,2,2)$	$(-2,1,-2)$	$(-1,-1,2)$

Among objects inscribed in an STQ is a square, as a degenerate member of a family of inscribed tetrahedra, two of which are regular. For a proper introduction to this family, consider the circles D', B', C', A'. Borrowing from the above parametric equations, let

$$\widehat{D} = \widehat{D}(t) = (1 + 2\cos t,\ 1 - \cos t + \sqrt{3}\sin t,\ -1 + \cos t + \sqrt{3}\sin t),$$

and likewise for $\widehat{B}$, $\widehat{C}$, and $\widehat{A}$. Let $f(t) = \left|\widehat{B}\widehat{D}\right|^2$ and $g(t) = \left|\widehat{C}\widehat{D}\right|^2$, so that

$$\begin{aligned} f(t) &= (2 + \cos t + \sqrt{3}\sin t)^2 + (-3\cos t + \sqrt{3}\sin t)^2 + (-2 + 2\cos t + 2\sqrt{3}\sin t)^2. \\ g(t) &= (2 + 4\cos t)^2 + (2 - 2\cos t + 2\sqrt{3}\sin t)^2, \end{aligned}$$

as in Figs. 8 and 9.

The congruence of the graphs of f and g illustrates the identity

$$\left|\widehat{B}\widehat{D}\right|^2 + \left|\widehat{C}\widehat{D}\right|^2 = 72.$$

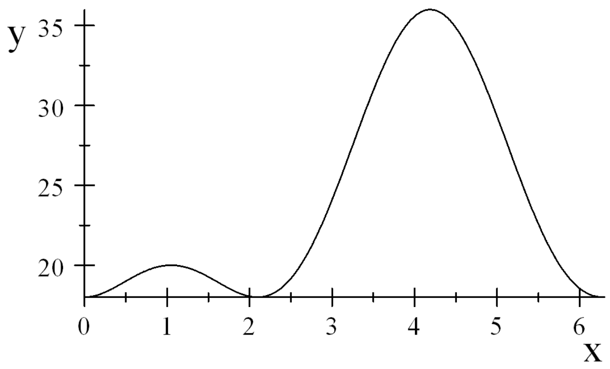

Figure 8: Graph of f

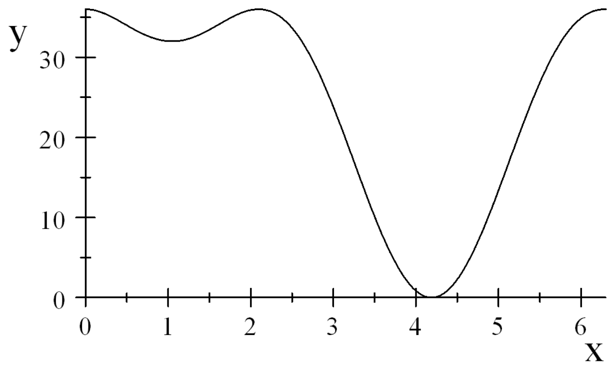

Figure 9: Graph of g

Putting $f(t) = g(t)$ leads to two solutions:

$$t \in \left\{ \arccos \frac{1-\sqrt{3}-3^{3/4}\sqrt{2}}{4},\ 2\pi - \arccos \frac{1-\sqrt{3}+3^{3/4}\sqrt{2}}{4} \right\},$$

with midpoint $4\pi/3$. Thus, there are exactly two regular tetrahedra inscribed in the STQ. Both have sidelength 6. Similar results hold for the LTQ.

Figure 10 (animated at [9, Gallery #8]) shows a surface whose boundary is an STQ. In fact, the surface can be generated as the union of a family of STQs centered at a common point. We call it an *STQ conic surface* because it is a union of four one-third sectors of cones. This type of surface is considered more generally in Section 7.

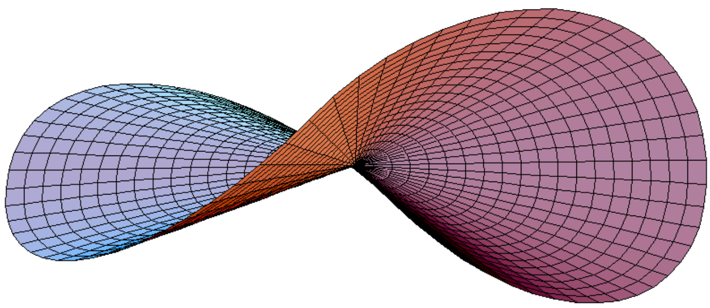

Figure 10: STQ conic surface

5. Curves on other regular polyhedra

The faces of a regular octahedron $ABCDEF$ are equilateral triangles. Applying the method of Section 1 to the facial incircles results in a *short octahedral hexarc* (SOH) and a *long octahedral hexarc* (LOH). To count the SOHs on $ABCDEF$, first note that the number of faces not visited by a given SOH is 2, and that they are opposites. There are 4 ways to choose such a pair. For each pair, we pick any face and observe that there are 3 ways to choose 1/3 of the incircle and that each of these ways determines the rest of an SOH. Accordingly, there are 12 SOHs; as each LOH is complementary to an SOH, there are 12 LOHs.

For a closer look, suppose that

$$\begin{aligned} A &= (0,0,1), \quad B=(0,-1,0), \quad C=(1,0,0), \\ D &= (0,1,0), \quad E=(-1,0,0), \quad F=(0,0,-1), \end{aligned}$$

so that the octahedron is inscribed in the sphere $x^2+y^2+z^2=1$. Any SOH passes through 6 of the 12 edges. One of these, for example, passes through these midpoints of edges:

$$(0,-1/2,-1/2), \quad (1/2,0,-1/2), \quad (1/2,1/2,0)$$

and their reflections in $(0,0,0)$. The six midpoints lie on the plane $x-y+z=0$ and are the vertices of a regular hexagon. All of this holds also for the LOH, and both curves (or, a total of 24 curves, if you wish) lie on the sphere $x^2+y^2+z^2=1/2$. Figure 11 and [9, Gallery #9, #10, #19] show half of an LOH lying on both an octahedron and a sphere.

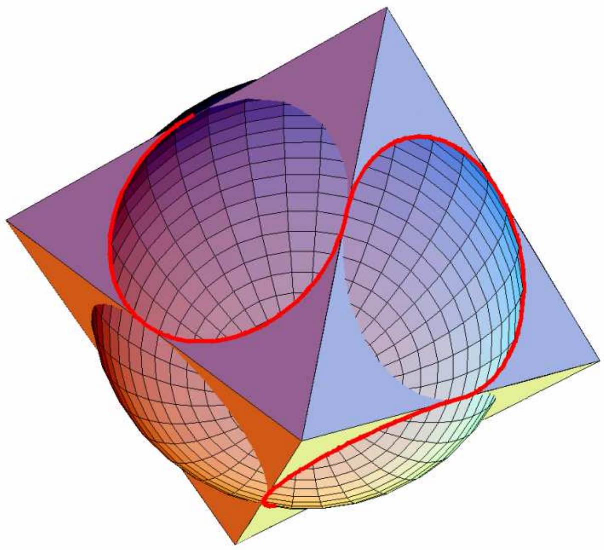

Figure 11: LOH on a sphere

The naming of the seven preceding *regular polyhedral polyarcs* (CQ, SCH, LCH, STQ, LTQ, SOH, TOH) represents roughly half of the possible results of the curve-production rules in Section 1 when applied to regular polyhedra. It is easy to see that for each rule, there is a *cycle number*; that is, the number of faces visited by the curve. A complete list of these follows:

Table 2: Cycle numbers for regular polyhedra

polyhedron	*cycle numbers*
tetrahedron	$4,4$
cube	$6,4,6$
octahedron	$6,6$
dodecahedron	$10,6,6,10$
icosahedron	$10,10$

For example, using 1/5 of each incircle on a dodecahedron results in a curve that visits 10 faces, as does the use of 4/5 of each incircle; hence the names SDD (*short dodecahedral decarc*) and LDD, shown in Fig. 12 and [9, Gallery #11, #12, #13]. Using 2/5 of each circle results in a curve that visits only 6 faces, and likewise when using 3/5; hence the designations SDH and LDH [9, Gallery #22, #21]. The final two curves are the SID and LID [9, Gallery #14, #15].

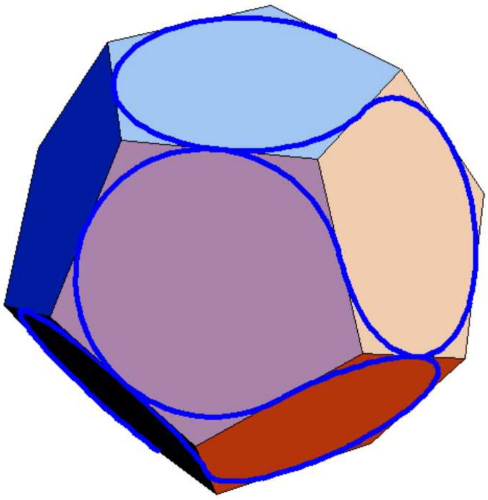

Figure 12: LDD on a dodecahedron

It is well known that the golden ratio, $\tau = (1+\sqrt{5})/2$, plays a role in features of the dodecahedron and its dual the icosahedron. For example, the following 20 points are the vertices of a dodecahedron:

$$(\pm 1, \pm 1, \pm 1) \quad (0, \pm 1/\tau, \pm\tau) \quad (\pm\tau, 0, \pm 1/\tau) \quad (\pm 1/\tau, \pm\tau, 0), \tag{1}$$

The following 12 points are the vertices of an icosahedron:

$$(0, \pm\tau, \pm 1) \quad (\pm 1, 0, \pm\tau) \quad (\pm\tau, \pm 1, 0). \tag{2}$$

Duality will be very helpful for describing the facial incircles of the two shapes: the vectors from the origin to the vertices (2) pass through the centers of the 12 faces (and hence of the 12 facial incircles) of the dodecahedron, and the vectors to the vertices (1) pass through the centers of the 20 faces of the icosahedron!

Every point is of the form

$$(\rho\cos\alpha,\ \rho\cos\beta,\ \rho\cos\gamma),$$

so that if a variable point $(\rho\cos\alpha_t\,,\ \rho\cos\beta_t\,,\ \rho\cos\gamma_t)$ moves around a facial incircle, then, for suitably chosen angles α', β', γ', the point

$$(\rho\cos(\alpha_t+\alpha'),\ \rho\cos(\beta_t+\beta'),\ \rho\cos(\gamma_t+\gamma'))$$

moves around another such circle. Therefore, our problem consists of two subproblems:

(i) to find parametric equations for any one facial incircle, and

(ii) to tell how to choose α', β', γ'.

For (i), it is not hard to verify that one of the facial incircles of the dodecahedron (1) is given by

$$\begin{cases} x = \dfrac{-5 - 3\sqrt{5} - 2\sqrt{5}\cos\theta}{10} \\ y = \dfrac{(5+\sqrt{5})(-1+\cos\theta)}{10} \\ z = \sqrt{\dfrac{5+\sqrt{5}}{10}}\,\sin\theta\,. \end{cases}$$

Similarly, one facial incircle of the icosahedron (2), is given by

$$\begin{cases} x = \dfrac{(1+\sqrt{5})(-1+\cos\theta)}{6} \\ y = \dfrac{4 + 2\sqrt{5} + (\sqrt{5}-1)\cos\theta}{6} \\ z = \dfrac{-\sqrt{3}\sin\theta}{3}\,. \end{cases}$$

Now turning to subproblem (ii), starting with the dodecahedron (1), let U be one of the vertices (2) of the icosahedron, and let Γ_U be the corresponding incircle on a face of the dodecahedron. Write the direction angles of U as α_U, β_U, γ_U. Let V be another of the vertices (1) and let Γ_V be its incircle, and write the direction angles of V as α_V, β_V, γ_V. Then the required angles α', β', γ' are $\alpha_V - \alpha_U$, $\beta_V - \beta_U$, $\gamma_V - \gamma_U$, respectively. The same method applies in reverse if we start with the icosahedron.

6. Curves that visit all faces

We have seen that the curves STQ, LTQ, SCH, LCH, visit all the faces of their polyhedron. This property fails for the other 9 regular polyarcs, as exemplified by the SOH [9, Gallery #20] and indicated by Table 2. However, each regular polyhedron has at least one hybrid curve — one composed of more than one sector-size of circles — which *does* visit all the faces. We begin with the octahedron, for which the two allowable sector-sizes, 1/3 and 2/3, correspond to S and L. The simple closed smooth curve indicated by SSLLSSLL visits every face exactly once [9, Gallery #15, #17].

A more comprehensive inspection follows for the cube. Write "1" for a quadrarc, "2" for a half-circle, and "3" for 3/4 of a circle. There are $3^6 = 729$ sextuples such as $(2, 2, 1, 3, 3, 1)$, and each one codes a curve in the obvious manner. However, the curve coded by $(1, 3, 3, 1, 2, 2)$ is clearly equivalent to that coded by its reversal, $(2, 2, 1, 3, 3, 1)$. Avoiding such double-counting leaves 378 curves. Not all of these return to their starting point, but 17 of them do. Another kind of duplication is exemplified by $(1, 3, 1, 1, 3, 1)$ and $(1, 1, 3, 1, 1, 3)$. After expelling all but one of each such cyclic permutations, we have 9 curves. Only 3 of these visit every face:

$$(1, 1, 1, 1, 1, 1), \quad (3, 3, 3, 3, 3, 3), \quad (1, 2, 3, 3, 2, 1),$$

which represent the SCQ, LCQ, and a hybrid. Note that the SCQ and LCQ are a complementary pair, in the sense that each curve is what remains after the other curve is removed from the union of the facial incircles. The hybrid, $(1, 2, 3, 3, 2, 1)$, has complement $(3, 2, 1, 1, 2, 3)$, which, as a code, represents essentially the same curve as $(1, 2, 3, 3, 2, 1)$, so that the hybrid [9, Gallery #25] is self-complementary — as is any curve having a palindromic code.

Consider next the icosahedron. A computer search of the 2^{20} possible codes found only two polyarcs that visit each of the 20 faces:

$$(1, 1, 1, 2, 1, 1, 2, 1, 1, 1, 1, 1, 1, 2, 1, 1, 2, 1, 1, 1)$$

and its complement. The former [9, Gallery #23] is free of self-touchpoints, whereas its complement has six.

We turn now to the dodecahedron. The faces are pentagons, so that each code is a 12-tuple using the numbers $1, 2, 3, 4$, so that there are 4^{12} codes. Of these, 512 return to their starting point headed in the "right" direction. Expelling cyclic permutations leaves 52 codes, but these include duplicates, since some of the codes are mere reversals of others. Expelling such duplicates leaves 33 codes, listed here, along with the number of self-touchpoints for each.

(1, 1, 1, 1, 1, 1, 3, 4, 2, 2, 4, 3),	2	(1, 1, 1, 1, 2, 1, 1, 3, 2, 1, 3, 3),	1
(1, 1, 1, 1, 3, 3, 1, 1, 1, 1, 3, 3),	0	(1, 1, 1, 1, 3, 3, 1, 2, 4, 4, 4, 3),	4
(1, 1, 1, 1, 3, 3, 2, 4, 4, 2, 3, 3),	4	(1, 1, 1, 2, 3, 1, 1, 3, 2, 2, 4, 3),	2
(1, 1, 1, 2, 4, 3, 1, 1, 1, 2, 4, 3),	3	(1, 1, 1, 2, 4, 3, 2, 4, 4, 3, 4, 3),	7
(1, 1, 1, 2, 4, 4, 4, 4, 2, 2, 4, 3),	6	(1, 1, 2, 1, 2, 3, 1, 1, 2, 1, 2, 3),	1
(1, 1, 2, 1, 2, 4, 3, 3, 4, 2, 1, 2),	4	(1, 1, 2, 1, 2, 4, 4, 4, 3, 1, 2, 3),	5
(1, 1, 2, 2, 4, 4, 2, 3, 3, 1, 2, 3),	4	(1, 1, 2, 4, 3, 1, 2, 2, 1, 3, 4, 2),	4
(1, 1, 2, 4, 4, 2, 3, 4, 2, 2, 4, 3),	6	(1, 1, 3, 2, 1, 3, 3, 1, 2, 4, 4, 3),	4
(1, 1, 3, 2, 2, 3, 1, 1, 3, 2, 2, 3),	2	(1, 1, 3, 2, 2, 4, 3, 2, 4, 4, 3, 3),	6
(1, 1, 3, 2, 2, 4, 4, 4, 4, 2, 2, 3),	6	(1, 1, 3, 4, 2, 1, 3, 3, 1, 2, 3, 4),	4
(1, 2, 2, 1, 3, 4, 3, 4, 4, 3, 4, 3),	8	(1, 2, 2, 2, 4, 3, 1, 2, 2, 2, 4, 3),	4
(1, 2, 4, 3, 1, 2, 4, 3, 1, 2, 4, 3),	6	(1, 2, 4, 3, 3, 3, 1, 2, 4, 3, 3, 3),	6
(1, 2, 4, 3, 3, 4, 2, 1, 2, 4, 4, 3),	8	(1, 2, 4, 4, 2, 1, 3, 4, 2, 2, 4, 3),	6
(1, 2, 4, 4, 4, 3, 1, 2, 4, 4, 4, 3),	9	(1, 2, 4, 4, 4, 3, 2, 4, 4, 2, 3, 3),	8
(1, 2, 4, 4, 4, 4, 4, 4, 2, 1, 3, 3),	8	(2, 2, 4, 3, 2, 4, 4, 3, 4, 4, 4, 4),	11
(2, 2, 4, 4, 4, 4, 2, 2, 4, 4, 4, 4),	10	(2, 3, 3, 2, 4, 4, 2, 3, 3, 2, 4, 4),	8
(2, 3, 4, 3, 4, 4, 2, 3, 4, 3, 4, 4),	11		

Among these, several stand out, such as $(1, 1, 1, 1, 3, 3, 1, 1, 1, 1, 3, 3)$ because it uses only two sector-sizes and is the only curve in the list that is not tangent to itself. Its complement, $(4, 4, 4, 4, 2, 2, 4, 4, 4, 4, 2, 2)$ has at least one self-touchpoint on at least one edge of each face. These curves and others are bi-partitite, in an obvious sense, whereas $(1, 2, 4, 3, 1, 2, 4, 3, 1, 2, 4, 3)$ is tripartite. The remarkable curve $(1, 1, 1, 1, 3, 3, 1, 1, 1, 1, 3, 3)$ is animated at [9, Gallery #18, #24].

7. Spherical circle-to-circle curves

The 13 regular polyarcs that belong to the 5 regular polyhedra have been considered in Sections 1-5. All of them stem from the observation that we can move from one inscribed circle to another smoothly across an edge — and that there is only one way to do so. The resulting quadrarcs, hexarcs, and decarcs all lie on spheres. We shall now introduce a more general family of curves, starting with a sphere rather than a polyhedron. Let $\mathcal{S}$ denote the sphere

$$x^2 + y^2 + z^2 = \rho^2.$$

Suppose that $0 < \theta < \pi$, and for every integer n, let E_n be the point on the equator given by

$$E_n = (\rho \cos n\theta, \rho \sin n\theta, 0).$$

Let f be a number satisfying $0 < f < 1$.

Let C_n be the circle on $\mathcal{S}$ that passes through E_n and E_{n+1}, situated so that the "fraction" of C_n lying *above* the equator is f if n is odd, and situated so that the fraction of C_n *below* the equator is f if n is even. The curve SCCC(ρ, θ, f) (where SCCC abbreviates *spherical circle-to-circle curve*) is now defined as the union of the f-sectors. For example, to follow this curve starting at $E_1 = (\rho, 0, 0)$, move along the sector of circle C_1 that lies above the equator, then pass through E_2 onto the sector of C_2 below the equator, then pass through E_3 onto the sector of C_3 above the equator, and so on. Or, start at E_1 and move along f-sectors of circles C_0, C_{-1}, C_{-2}, and so on.

For many choices of θ and f, no two of the points E_n coincide. On the other hand, if $\theta = 2\pi/n$ for some positive integer n, then the SCCC returns to E_1 after one trip around the sphere. If n is even, then the reunion is 1-smooth; if n is odd, the first reunion is a cusp but after one more trip around, the SCCC is closed and smooth. Clearly the SCCC is closed if and only if $\theta = 2\pi q$ for some nonzero rational number q.

To see that an SCCC is smooth at each point E_n, it suffices merely to observe that the two circles meeting at E_n are both tangent there to a certain line L, namely, the line of intersection of the planes containing C_{n-1} and C_n. The 13 regular polyhedral polyarcs, and also the lune, are easily accounted for as SCCCs:

Table 3: Spherical circle-to-circle curves

Curve	(ρ, θ, f)	*Curve*	(ρ, θ, f)
lune	$(\rho,\ \pi,\ f)$	LOH	$(\rho,\ \pi/3,\ 2/3)$
SCH	$(\rho,\ \pi/2,\ 1/4)$	SDD	$(\rho,\ \pi/5,\ 1/5)$
CQ	$(\rho,\ \pi/2,\ 1/2)$	LDD	$(\rho,\ \pi/5,\ 4/5)$
LCH	$(\rho,\ \pi/2,\ 3/4)$	SDH	$(\rho,\ \pi/3,\ 2/5)$
STQ	$(\rho,\ \pi/4,\ 1/3)$	LDH	$(\rho,\ \pi/3,\ 3/5)$
LTQ	$(\rho,\ \pi/4,\ 2/3)$	SID	$(\rho,\ \pi/5,\ 1/3)$
SOH	$(\rho,\ \pi/3,\ 1/3)$	LID	$(\rho,\ \pi/5,\ 2/3)$

Each SCCC generates a conic surface, as exemplified by Fig. 10. To construct the surface from a given SCCC, merely draw a line segment from the origin to each point on the SCCC. Or, take the union of the set of r dilations of the SCCC about its center, for $0 \le r \le 1$. For unbounded surfaces SCCC conic surfaces, use rays instead of segments, or dilate using all positive real r.

Returning to the 13 polyarcs in Table 3, we have already noted that each has a midcircle; this is obvious in the context of Section 7. Now suppose, for a given regular polyhedron, that two distinct polyarcs pass through the midpoint of an edge, which is a point on two of the facial incircles. The midcircles of these polyarcs belong to planes, and we call attention to the acute angle between these planes. For example, Fig. 13 shows edges of the planes for the SDD and LDD, and from this configuration we find that the angle is $\pi - \arctan 2$.

Table 4 shows acute angles for pairs of polyarcs on a dodecahedron.

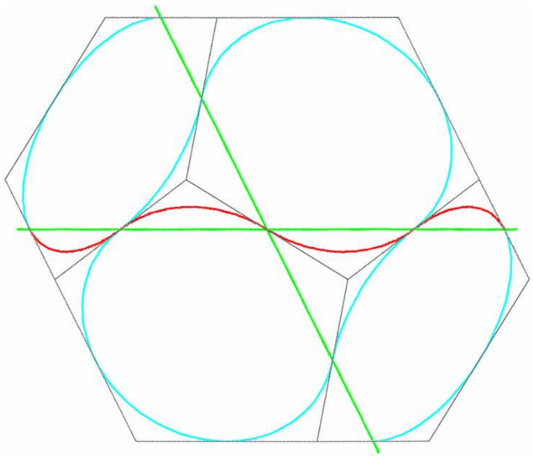

Figure 13: SDD and LDD. Intersecting midcircle planes are indicated as lines. The horizontal line belongs to the SDD.

Table 4: Acute angle between midcircle planes

Polyarc pairs	*Angle*
SDD and LDD	$\pi - \arctan 2 \approx 2.034444$
SDD and SDH	$\arctan(3 - \sqrt{5}) \approx 0.65236$
SDD and LDH	$\arctan(3 + \sqrt{5}) \approx 1.3821$
SDH and LDH	$\arcsin(2/3) \approx 0.72973$
SDH and LDD	$\arccos[(\sqrt{5} - 2\sqrt{5})/15] \approx 1.7204$
LDH and LDD	$\arccos[(\sqrt{5} + 2\sqrt{5})/15] \approx 1.1071$

8. Summary

In Sections 2-6, we have presented a class of biarcs, each of which lies on a sphere and on a regular polyhedron. These thirteen polyhedral polyarcs are classified as indicated by their names, which are listed in full at the beginning of the Gallery [9]. The number of faces of the polyhedron visited by each polyarc is given by Table 2. Other biarcs that visit all faces of polyhedra and also lie on a sphere are described in Section 6, and in Section 7, a more general class of spherical biarcs are described, with attention to an equator and the angle at which such a curve crosses the equator.

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