Journal for Geometry and Graphics
Volume 15 (2011), No. 1, 93–99.

Building a Torus with Villarceau Sections

María García Monera, Juan Monterde*

Dep. de Geometria i Topologia, Universitat de València
Avd. Vicent Andrés Estellés, 1, E-46100-Burjassot (València), Spain
email: magarmo2@alumni.uv.es, monterde@uv.es

Abstract. A new method to build a ring torus is given using only pieces obtained by Villarceau sections. These pieces have the interesting characteristic that all of them are congruent, so we can build a ring torus using an even number of copies of a unique piece. A template of the moon-shaped piece is given.

Keywords: Villarceau section, double tangential plane, sliceforms, ring torus.

MSC 2010: 51N05, 65D17

1. Introduction

Sliceforms were described briefly in Cundy and Rollett's book [1] and they are geometric models constructed from interlocking sets of planar pieces. Slicing is a technique for building models of surfaces that recently have received some attention ([3, 4]).

The usual way to build surfaces is using sections obtained with planes parallel to the coordinate planes. Therefore, the first step to design a sliceform is to create two sets of slotted pieces ("slot-from-the-top" and "slot-from-the-bottom" sets) that intersect at right angles. In the case of the torus such sections include curves often known as *ovals of Booth*, and also *lemniscates of Booth*, after James Booth (1810–1878). But such kind of sliced model do not use one of the most important properties of the ring torus, it is a surface of revolution.

In this note, we propose a construction of the ring torus thanks to the existence of a special section in that surface of revolution (see [2, 5]). The proposed construction has three main properties.

- First, it uses a remarkable section, the Villarceau circles.
- Second, all the pieces of both sets of slotted pieces are congruent. The unique difference between both sets are the slots.
- And third, the fact that the ring torus is a surface of revolution is clear.

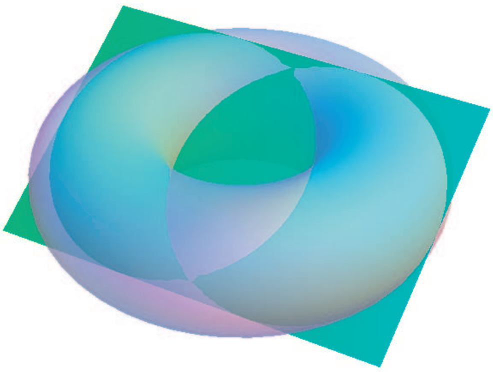
* Corresponding author. This work has been partially supported by a Spanish Ministry of Science and Innovation grant MTM2009-08933.

ISSN 1433-8157/$ 2.50

Notice that the Villarceau circles intersect with all parallels of the surface of revolution. This fact implies that when the circles rotate around the axis they generate the complete surface. However, a similar construction for a sphere with a section produced by an oblique plane does not generate all the surface because the section does not intersect all the parallels.

2. The Villarceau circles

Let Ψ be the ring torus given by $\left(\sqrt{x^2+y^2}-a\right)^2+z^2=b^2$, $a>b$, $a,b\in\mathbb{R}^+$. The tangent plane to Ψ at the point $\left(0,\ \frac{a^2-b^2}{a},\ \frac{b}{a}\sqrt{a^2-b^2}\right)$ has as implicit equation $z=\frac{b}{\sqrt{a^2-b^2}}\,y$. Moreover, it is a double tangent plane because it is also tangent at $\left(0,\ -\frac{a^2-b^2}{a},\ -\frac{b}{a}\sqrt{a^2-b^2}\right)$ (see Fig. 1, left).



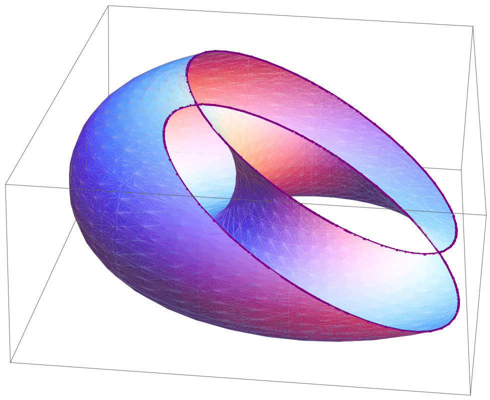


Figure 1: Left: the double tangent plane; Right: the Villarceau circles as the section of the torus by the double tangent plane

As it was observed by Yvon Villarceau in 1848, the intersection of this double tangent plane with Ψ gives rise to two circles of the same radius a and with centers $(\pm b,0,0)$ (see Fig. 1, right). The part inside the torus of each Villarceau circle is like a moon-shaped region (see Fig. 2).

An important property of the Villarceau circles is that they are *loxodromic curves* of the torus. The angle between a Villarceau circle and the parallel curves of the torus is constant.

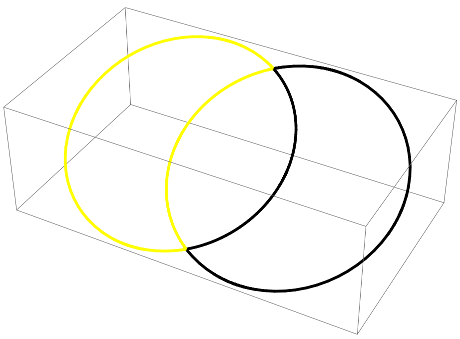


Figure 2: The two moon-shaped pieces which are the main ingredients of our construction

Let us define the angle φ such that

$$\tan\varphi = \frac{b}{\sqrt{a^2 - b^2}}\,.$$

Thus, the implicit equation of the tangent plane defining the Villarceau circles is $z = y\tan\varphi$. A simple computation shows that φ is the loxodromic angle.

The proposed construction consists in applying a rotation with z-axis to both moon-shaped pieces.

3. Computation of the intersection segments

When the rotation process is applied to just one of the two moon-shaped pieces there are no intersections (see Fig. 3, left), but when the two moon-shaped pieces are rotated with the z-axis as axis of rotation they intersect themselves (see Fig. 3, right). Moreover, thanks to the loxodromic property of the Villarceau circles, the angle of intersection of any pair of circles is always the same because it is twice the loxodromic angle.

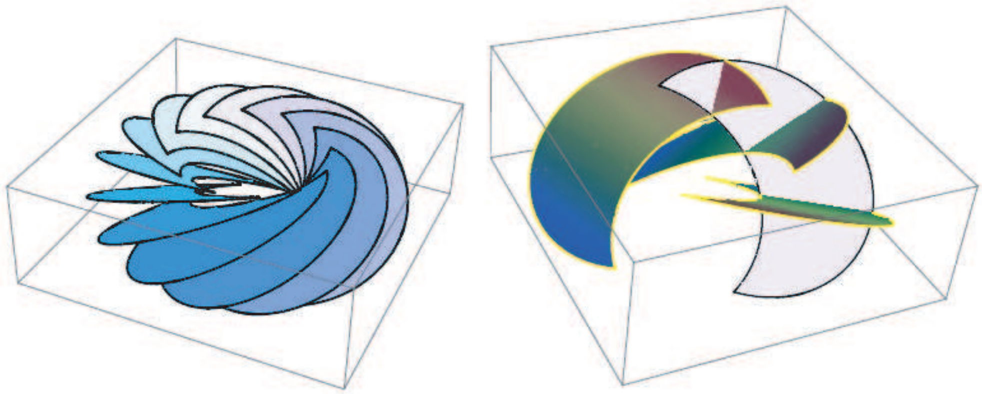


Figure 3: Left: Rotation of one of the two moon-shaped pieces; Right: Intersections of the right moon-shaped piece with three copies of the left moon-shaped piece after rotation

Therefore, to construct a real model of Ψ, we have to compute the intersection segments. Since the Villarceau circles are included in a plane, the intersection segments are the piece inside the Villarceau circles of the intersection line between a plane and the rotation of that plane.

Let $\mathcal{P}_t$ be the image of the plane $\mathcal{P}_0$ after a rotation about the z-axis through the angle t, $t \in [0, 2\pi]$. The normal vector to the plane $\mathcal{P}_t$ is $N_t := (\sin\varphi\sin t,\ -\sin\varphi\cos t,\ \cos\varphi)$, where φ is the loxodromic angle. Notice that $\mathcal{P}_0$ is the initial double tangent plane, this is, the plane passing through the origin and with normal vector $N_0 := (0, -\sin\varphi, \cos\varphi)$.

Therefore, let us compute first, in general, the intersection line between the planes $\mathcal{P}_0$ and $\mathcal{P}_t$: The intersection of the planes P_0 and P_t is the straight line passing through the origin and in direction of the cross product $N_0 \wedge N_t$. An easy computation shows that

$$N_0 \wedge N_t = \lambda\left(-\sin\frac{t}{2}\,,\ \cos\frac{t}{2}\,,\ \tan\varphi\cos\frac{t}{2}\right),$$

for $\lambda = \sin(2\varphi)\sin\frac{t}{2}$ and where φ denotes the loxodromic angle obeying $\sin\varphi = \frac{b}{a}\,.$

The normalization of this vector gives

$$\overrightarrow{\mathbf{v}_t} := \frac{1}{\sqrt{1+\tan^2\varphi\cos^2\frac{t}{2}}}\left(-\sin\frac{t}{2},\ \cos\frac{t}{2},\ \tan\varphi\cos\frac{t}{2}\right).$$

Notice that $\overrightarrow{\mathbf{v}_\pi} = (-1,0,0)$, thus, for $t=\pi$, the intersection line is the x-axis.

Let θ_t be the angle between $\overrightarrow{\mathbf{v}_t}$ and $\overrightarrow{\mathbf{v}_\pi}$. Therefore

$$\cos(\theta_t) = \frac{\sin\frac{t}{2}}{\sqrt{1+\tan^2\varphi\cos^2\frac{t}{2}}}.$$

For $a=2b$ (therefore, the loxodromic angle is $\varphi = \frac{\pi}{6}$) and $t_k := \frac{k\pi}{6}$, $k=0,1,\ldots,12$, the oriented angles of the intersection lines are

$$-90^\circ,\ -76.9^\circ,\ -63.4^\circ,\ -49.1^\circ,\ -33.7^\circ,\ -17.2^\circ,\ 0^\circ,\ 17.2^\circ,\ 33.7^\circ,\ 49.1^\circ,\ 63.4^\circ,\ 76.9^\circ,\ 90^\circ.$$

4. The template

All we have to do is to design the moon-shaped template. In Fig. 4 a template is shown for a torus with $a=2b$ made of 24 pieces, half of them of the left kind. The straight segments indicate the intersections between the different pieces.

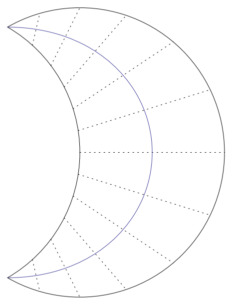


Figure 4: The template for a torus with $a=2b$ and 24 moon-shaped pieces

Notice that all the pieces are congruent, but recall that the intersection lines of half of them must be cut from one of the boundary curves to the central curve. The slots for the other set of pieces goes from the other boundary curve to the central curve. Therefore, we have to determine the central curve of the moon-shaped template.

For each angle $t\in[-\frac{\pi}{2},\frac{\pi}{2}]$, the straight line passing through the origin and defining and angle t with the positive x-axis intersect the two arcs of the Villarceau circles, $(x\pm b)^2+y^2=a^2$, at two points $c_1(t)$ and $c_2(t)$, whose explicit expressions are:

$$\begin{aligned} c_1(t) &= \left(b\cos t+\sqrt{a^2-b^2\sin^2 t}\right)(\cos t,\ \sin t),\\ c_2(t) &= \left(-b\cos t+\sqrt{a^2-b^2\sin^2 t}\right)(\cos t,\ \sin t). \end{aligned}$$

Notice that the scalar product $c_1(t)\cdot c_2(t)$ is equal to a^2-b^2. This means that the image of the curve c_1 by the inversion with respect to the circle centered at the origin and of radius $\sqrt{a^2-b^2}$ is the curve c_2, and reciprocally. This is a geometric property inherited from the

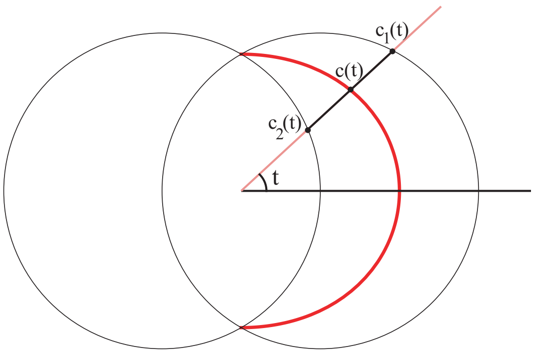


Figure 5: The central curve of the moon-shaped piece

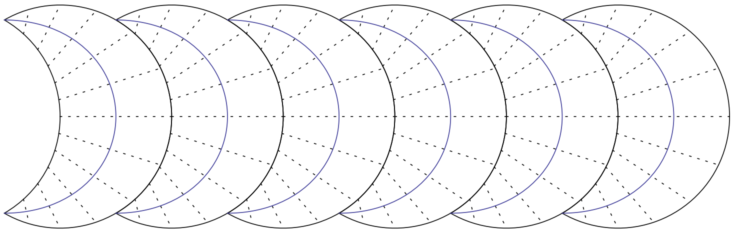


Figure 6: A multiple template for the same torus

Villarceau circles: if we consider the sphere centered at the origin and passing through the bitangency points defining the Villarceau circles then, the inversion with respect to the sphere maps one family of circles onto the other. Combining this with the previous fact about the curves c_1 and c_2, we have proved that each moon-shaped piece of our construction is *self-inverse* with respect to the sphere which is concentric with the torus and passing through the corners of the moon-shaped piece.

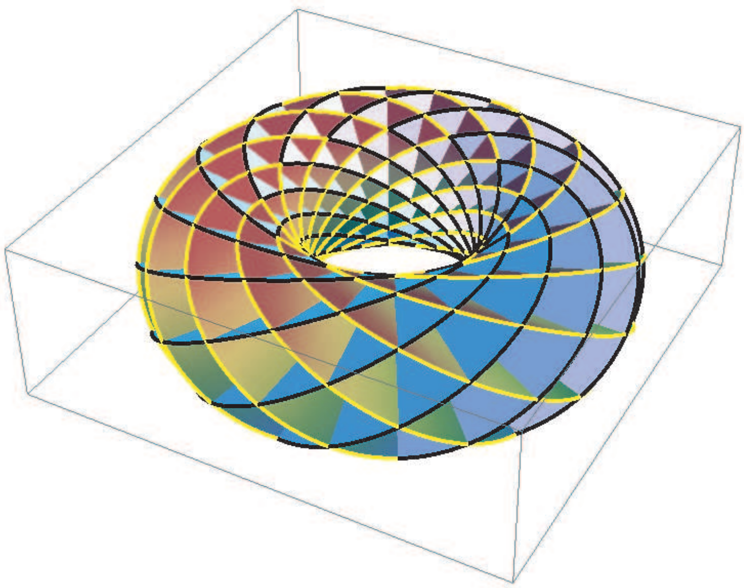


Figure 7: The virtual model

The central curve is the curve of the medial points between $c_1(t)$ and $c_2(t)$

$$c(t) = \frac{c_1(t) + c_2(t)}{2} = \sqrt{a^2 - b^2 \sin^2 t}\,\big(\cos t,\, \sin t\big), \qquad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

To optimize the construction process and to avoid material waste one can use a multiple template like in Fig. 6 .

Notice that it is impossible to build this model of ring torus with rigid materials. The reason is the following: The cuts of each piece are not mutually parallel, they are diameters of a circle. This fact implies that when we have the model almost built, we must separate the pieces to introduce a new one. This is possible with flexible material like paper, but not with wood or similar.

5. Final construction

Figure 7 shows a virtual model made thanks to Mathematica. Notice that any two intersecting disks are symmetric with respect to the incident meridian plane. Finally, Figures 8 and 9 are photos of a real materialization, but just with 20 pieces.

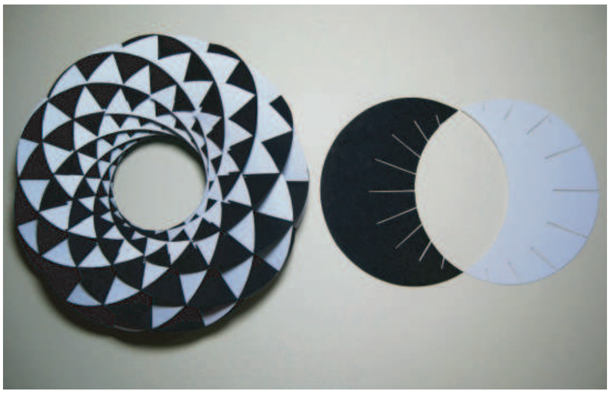


Figure 8: The real model and two of the moon-shaped pieces used to built it

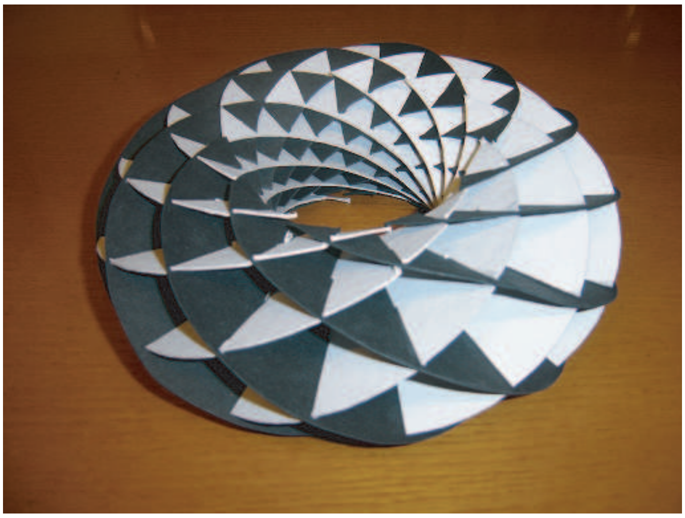


Figure 9: Side view of the same real model

This model has been made with paper of 600 gr., but it is also possible to build a ring torus with paper of 180 gr. or less.

Acknowledgments

We thank the referees for the valuable comments and suggestions which have improved this paper.

References

[1] H. Martyn Cundy, A.P. Rollett: *Mathematical models*. Tarquin Publications, Norfolk/England 1981.

[2] A. Hirsch: *Extension of the 'Villarceau-Section' to Surfaces of Revolution with a Generating Conic*. J. Geometry Graphics **6**, 121–132 (2002).

[3] J. Sharp: *Surfaces: Explorations With Sliceforms*. Tarquin Publications, Norfolk 2004.

[4] J. Sharp: *Sliceforms: Mathematical Models from Paper Sections*. Tarquin Publications, Norfolk 1995.

[5] H. Stachel: *Remarks on A. Hirsch's Paper concerning Villarceau Sections*. J. Geometry Graphics **6**, 133–139 (2002).

Received December 15, 2010; final form March 21, 2011