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On Thébault's Problem 3887

Zvonko Čerin

Kopernikova 7, 10020 Zagreb, Hrvatska, Europa
email: cerin@math.hr

Abstract. The famous Sawayama-Thébault configuration of the triangle ABC depends on a variable point D on its sideline BC and consists of eight circles touching the lines AD and BC and its circumcircle. These circles are best considered in four pairs that are related to the four circles touching the sidelines BC, CA and AB (the incircle and the three excircles). We use analytic geometry to determine the coordinates of the centers P, Q, S, T, U, V, X, and Y of the eight Sawayama-Thébault circles with respect to a parametrization of the triangle ABC with inradius r and cotangents f and g of the angles $\frac{B}{2}$ and $\frac{C}{2}$. The position of the point D is described by the cotangent k of half of the angle between the lines AD and BC. The coordinates of many points in this configuration are rational functions in r, f, g and k that makes most computations simple especially when done by a computer. In this approach, the proof of the original Thébault's problem about the incenter I dividing the segment QP in the ratio k^2 is straightforward. Some other interesting properties of this gem of triangle geometry are explored by analytic methods.

Key Words: triangle, line, concurrent lines, orthopole, Simson-Wallace line, locus, power

MSC 2010: 51N20, 51M04

1. Introduction

In [32], the authors say that the following result is usually called *Thébault's theorem* (see the portion of the Fig. 1 above the line BC). For a point X and a positive real number y, let $k(X, y)$ denote the circle k with the center at X and the radius y.

Theorem 1. *Let $u(I, r)$ be the incircle of a triangle $\triangle ABC$ and D any point on the line BC. Let $k_1(P, r_1)$ and $k_2(Q, r_2)$ be two circles touching the lines AD and BC and the circumcircle $o(O, R)$ of ABC. Then the three centers P, Q and I are collinear and the following relations hold:*

$$PI : IQ = \tau^2, \tag{1}$$

$$r_1 + r_2\,\tau^2 = r(1 + \tau^2), \tag{2}$$

where $2\,\theta = \angle ADB$ and $\tau = \tan\,\theta$.

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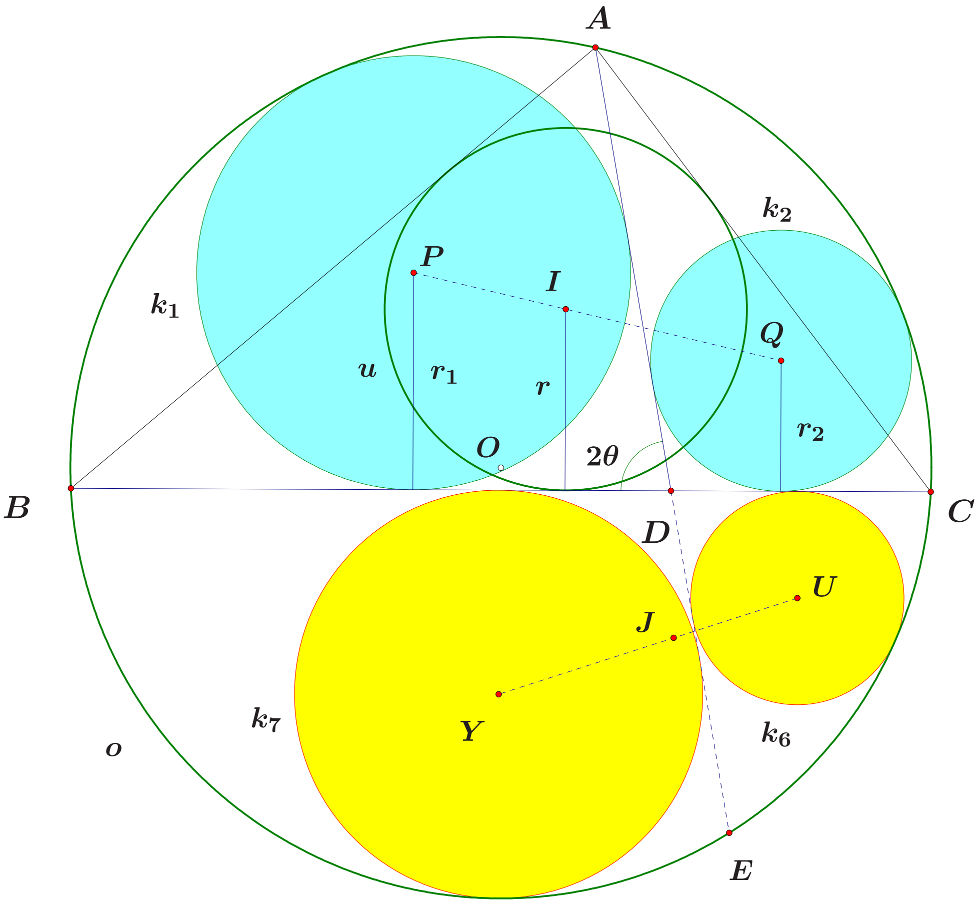

Figure 1: Thébault's theorem

The primary goal of this paper is to give correct versions of the above "theorem". Its formulation needs greater precision because the requirement "touching the lines AD and BC and the circumcircle $o(O, R)$" is not restrictive enough. This is obvious from the part of Fig. 1 under the line BC since the centers Y, U and I are not collinear. On the other hand, the relation (2) does not hold for all positions of the point D on the line BC.

The exciting history of Thébault's theorem is nicely presented in [1], [3], [13], [21], [25] and [32]. Now we only mention briefly key developments in chronological order:

In 1900 and 1905 Sawayama (see [15, p. 142] and [23]) first considered circles touching the circumcircle, the sideline and a variable line through the opposite vertex of a triangle. In 1938 Thébault formulated the above Theorem 1 as the American Mathematical Monthly Problem 3887 with the wrong relation (2) (corrected in [8]). It does appear also in Ogilvy's book [16] from 1962 (and in its translation into German [17] in 1969).

The first solutions by Streefkerk [28], Veldkamp [33], Dijkstra-Kluyver [4] and Dijkstra-Kluyver and Streefkerk [5] are from 1972 and 1973 (all in Dutch). In 1988 and 1989 this was recalled in [7] and in a solution [34] by Veldkamp of [9] for the English speaking world.

In this larger world, the perception was that the first complicated solution [29] of the Problem 3887 (on 24 pages) is by Taylor in 1983. However, already in 1975 a much simpler solution [8] by English was submitted to the American Mathematical Monthly only to be published in 2003.

The following two solutions by Turnwald [31] in 1986 and by Stärk [27] in 1989 are in German. The more recent solutions and further improvements are by Chou [2] in 1988, Demir and Tezer [3] and Lu and Jingzhong [14] in 1991, Rigby [21] in 1995, Shail [25] and Dutta [6] in 2001, Gueron [10] in 2002, Kodokostas [12] in 2004, Veljan and Volenec [32] in 2008, and Ostermann and Wanner [18] in 2010. There were also some other contributions that are partially given in the references.

Following these 111 years of efforts to understand the *Sawayama-Thébault configuration* it is fair to say that the Problem 3887 is an unusual result in elementary geometry that was more often considered within the analytic geometry rather than in the synthetic geometry. The synthetic approach is traditionally considered as more valuable while the inferior analytic method is always a kind of brute force with lengthly computations.

In this paper we also follow the analytic method but thanks to a right selection of parameters the majority of our calculations and expressions remain quite simple. We need the following auxiliary notation to achieve this goal. Let

$$\begin{array}{lllll} d = f-g, & z = f+g, & \zeta = fg, & h = \zeta-1, & \bar{h} = \zeta+1, \\ f_\pm = f\pm k, & g_\pm = k\pm g, & f^\pm = f^2\pm 1, & g^\pm = g^2\pm 1, & \varphi_\pm = f\,k\pm 1, \\ \psi_\pm = g\,k\pm 1, & K = k^2+1, & L = k^2-1, & & \end{array}$$

and let $\lambda(a, b)$ replace $(\lambda\, a, \lambda\, b)$.

Let ABC be a triangle in the plane. Let $\beta = \angle CBA$ and $\gamma = \angle ACB$. Let $f = \cot\frac{\beta}{2}$ and $g = \cot\frac{\gamma}{2}$ and let $u(I, r)$ be the incircle of the triangle ABC. We shall use the rectangular coordinate system that has point B as the origin and point C on the positive part of the x-axis while point A is above it. For a point P, let x_P and y_P denote its x- and y-coordinate with respect to this system. Then the vertices A, B and C of the triangle ABC have the coordinates

$$\frac{rg}{h}\left(f^-,\, 2f\right), \quad (0,\, 0) \;\text{ and }\; (r\,z,\, 0),$$

where the positive real numbers r, f and g satisfy $h > 0$. The position of a variable point D on the line BC is determined by the positive real number $k = \cot\frac{\delta}{2}$, where δ is the angle between the lines AD and BC. Hence, $D = D_k = \left(\frac{r\,g\,f_+\,\varphi_-}{h\,k},\, 0\right)$.

2. Thébault's theorem

We shall first determine the coordinates of the centers of the *Sawayama-Thébault circles* (see Theorem 2). With this important information the proof of the (complete) Thébault theorem (see Theorems 3, 4 and 5 and Fig. 2) is indeed very simple and straightforward. Of course, our approach is similar to [3] and [25]. However, our choice of the parametrization gives simpler expressions and allows a more extensive study of the Sawayama-Thébault configuration.

Theorem 2. *The points P, Q, S, T, U, V, X, and Y with respective coordinates*

$$\frac{r\varphi_-}{k}\left(1,\, \frac{\psi_+}{hk}\right), \quad r f_+\left(1,\, -\frac{g_-}{h}\right), \quad \frac{rgf_+}{k}\left(1,\, \frac{fg_-}{hk}\right), \quad -rg\varphi_-\left(1,\, \frac{f\psi_+}{h}\right),$$
$$\frac{rg\varphi_-}{hk}\left(z,\, \frac{g_-}{k}\right), \quad \frac{rgf_+}{h}\left(z,\, \psi_+\right), \quad \frac{rf_+}{hk}\left(-z,\, \frac{f\psi_+}{k}\right), \quad \frac{r\varphi_-}{h}\left(z,\, fg_-\right)$$

are the centers and $r_1 = |y_P|, \ldots, r_8 = |y_Y|$ are the radii of the eight circles k_i $(i = 1, \ldots, 8)$ that touch the lines BC and AD and the circumcircle $o(O, R)$.

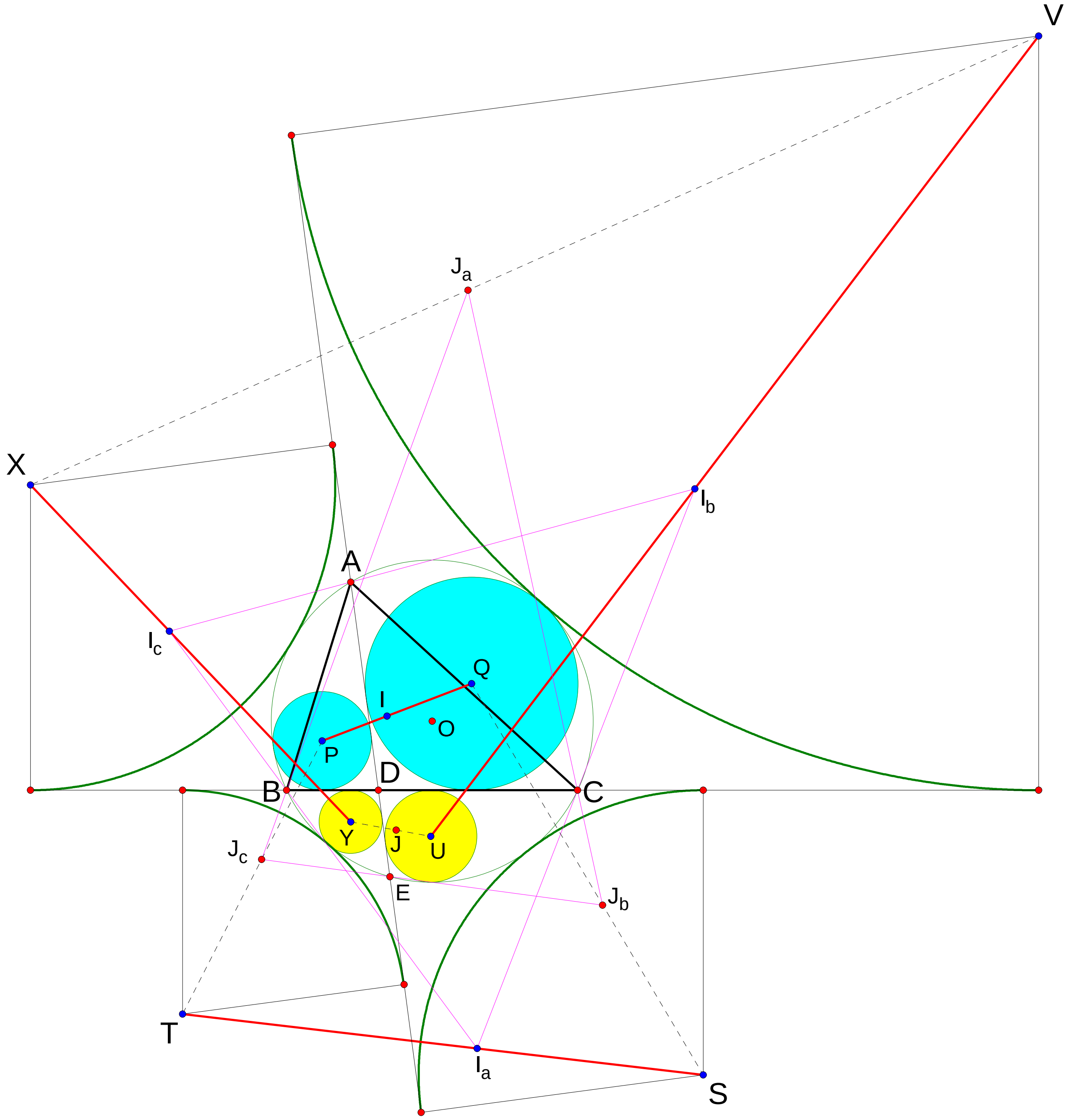

Figure 2: Theorems 3 and 4 together

Proof: Let $P(p,q)$ be the center of the circle that touches the lines BC and AD and the circle o. Then

$$|PP''| = |q|, \tag{3}$$

and

$$|PO|^2 = (R \pm q)^2, \tag{4}$$

where P'' is the orthogonal projection of the point P on the line AD. If

$$\mathfrak{u} = Lp - 2\,k\,q, \quad \mathfrak{v} = Lq + 2\,k\,p, \quad \mathfrak{w} = h\,K^2,$$

then

$$x_{P''} = \frac{4\,r\,g\,k\,f_+\,\varphi_- + hL\mathfrak{u}}{\mathfrak{w}} \quad \text{and} \quad y_{P''} = \frac{2\,r\,g\,L f_+\,\varphi_- - 2\,h\,k\mathfrak{u}}{\mathfrak{w}}, \quad \text{hence } |PP''| = \left|\frac{h\mathfrak{v} - 2\,r\,g\,f_+\,\varphi_-}{\mathfrak{w}}\right|.$$

On the other hand, $R = \frac{rf^+g^+}{4\,h}$ and O has the coordinates $\frac{r}{4h}\,(2z,\ z^2 - h^2)$. It is now easy to see (perhaps with a little help from Maple V) that the above eight cases of pairs $(p,\,q)$ are all solutions of the equations (3) and (4). □

While it is easy to find the coordinates of the centers $P, \ldots, Y$ of the eight Sawayama-Thébault circles and their radii $|y_P|, \ldots, |y_Y|$, it is difficult to describe them precisely by purely geometric means because when the point D changes position on the line BC these circles are changing considerably so that it is hard to tell one from the other. For the points P, Q, S and T this was done in [3, Section 3] by the use of the oriented configurations.

For a real number $\lambda \neq -1$ and different points M and N, the λ-*point* of the segment MN is a unique point F on the line MN such that the ratio of oriented distances $|MF|$ and $|FN|$ is equal to λ. We can extend this definition to the case when $M = N$ taking that the λ-point is the point M for every real number $\lambda \neq -1$. Recall that the coordinates of the λ-point are $\left(\frac{x_M + \lambda\,x_N}{\lambda + 1},\ \frac{y_M + \lambda\,y_N}{\lambda + 1}\right)$.

Let $k_a(I_a,\,r_a)$, $k_b(I_b,\,r_b)$ and $k_c(I_c,\,r_c)$ be the excircles of the triangle ABC. Then I, I_a, I_b and I_c have the coordinates

$$r(f,\,1), \quad rg(1,\,-f), \quad \frac{rgz}{h}\,(f,\,1) \quad \text{and} \quad \frac{rz}{h}\,(-1,\,f)\,.$$

Also,

$$r_a = r f g, \quad r_b = \frac{rgz}{h} \quad \text{and} \quad r_c = \frac{rfz}{h}.$$

The part of the following result for the segment QP is the correct form of *Thébault's theorem* while the part for the segment TS is the correct form of *Thébault's external theorem* (see [32, Remark 2]). In [25], Shail calls Theorem 3 *the full Thébault theorem*.

Theorem 3. *The points I, I_a, I_b and I_c are the k^2-points of the segments QP, TS, VU and YX.*

Proof: From

$$\frac{x_Q + k^2\,x_P}{K} = \frac{r\,f_+ + k^2\,\frac{r\varphi_-}{k}}{K} = rf = x_I \quad \text{and} \quad \frac{y_Q + k^2\,y_P}{K} = \frac{-\frac{rf_+g_-}{h} + k^2\,\frac{r\varphi_-\,\psi_+}{hk^2}}{K} = r = y_I$$

follows that I is the k^2-point of the segment QP. The other cases have similar proofs. □

Corollary 1. *The abscises and the ordinates of the centers of Sawayama-Thébault circles satisfy*

$$x_Q + k^2\,x_P = Krf, \qquad x_T + k^2\,x_S = Krg, \tag{5}$$

$$x_V + k^2\,x_U = Kr_bf, \qquad f(x_Y + k^2\,x_X) = -Kr_c\,, \tag{6}$$

$$y_Q + k^2\,y_P = Kr, \qquad y_T + k^2\,y_S = -Kr_a\,, \tag{7}$$

$$y_V + k^2\,y_U = Kr_b\,, \qquad y_Y + k^2\,y_X = Kr_c\,. \tag{8}$$

Note that only when the point D is on the segment BC it holds $y_P = r_1$, $y_Q = r_2$, $y_S = -r_3$, and $y_T = -r_4$ so that from (7) we get (2) since $k = 1/\tau$. The second relation in (7) gives us the analogous formula $r_3 + r_4\,\tau^2 = r_a(1+\tau^2)$ for the Thébault's external theorem.

On the other hand, when the point D is on the left from the point B, the ordinate y_P of the center P is negative so that the relation (7) gives $r_2 - k^2\,r_1 = (1+k^2)r$. Moreover, when the point D is on the right from the point C, the ordinate y_Q is negative so that the relation (7) implies the third part $k^2\,r_1 - r_2 = (1+k^2)r$ of the correct version of the formula (2).

As was already noticed in [27], the eight Sawayama-Thébault circles are also connected with the triangle EBC, where the point E is the second intersection (besides the point A) of the line AD and the circumcircle o. Its coordinates are $\frac{r\,f_+\,\varphi_-}{h\,K^2}\left(\psi_+^2 - g_-^2,\ 2\,\psi_+\,g_-\right)$. One can easily find that its incenter J and the excenters J_b, J_c and J_e have the coordinates

$$\frac{r\,z\,\varphi_-}{h\,K}\,(\psi_+,\ g_-)\,,\quad \frac{r\,f_+}{K}\,(\psi_+,\ g_-)\,,\quad \frac{r\,\varphi_-}{K}\,(g_-,\ -\psi_+)\,,\ \text{and}\ \frac{r\,z\,f_+}{h\,K}\,(-g_-,\ \psi_+)\,.$$

It is important to note here that as the parameter k changes the actual role of these points changes so that from the excenters they can become other excenters or the incenter and vice versa.

Theorem 4. *The four points J, J_b, J_c, and J_e are the k^2-points of the segments YU, QS, TP, and VX.*

Proof: Since

$$\frac{x_Y + k^2 x_U}{K} = \frac{rz\varphi_-}{hK} + \frac{rzgk\varphi_-}{hK} = \frac{rz\varphi_-\psi_+}{hK} = x_J$$

and

$$\frac{y_Y + k^2 y_U}{K} = \frac{rf\varphi_- g_-}{hK} + \frac{rg\varphi_- g_-}{hK} = \frac{rz\varphi_- g_-}{hK} = y_J,$$

it follows that J is the k^2-point of the segment YU. The other cases have similar proofs. □

The approach in [27] also suggests that the other two triangles ABE and ACE and their incenters and the excenters should play a similar role. We denote those centers by $\mathfrak{I}$, $\mathfrak{I}_a$, $\mathfrak{I}_b$, $\mathfrak{I}_e$, and $\mathfrak{J}$, $\mathfrak{J}_a$, $\mathfrak{J}_c$, $\mathfrak{J}_e$. Their respective coordinates are

$$\begin{array}{cc}
\frac{r\,\varphi_-}{h\,K}\,(h\,k+z,\ z\,k-h)\,, & -\frac{r\,g\,\varphi_-}{h\,K}\,(h-z\,k,\ h\,k+z)\,,\\
\frac{rf_+\,g}{h\,K}\,(h\,k+z,\ z\,k-h)\,, & \frac{rf_+}{h\,K}\,(h-z\,k,\ h\,k+z)\,,\\
\frac{r}{h\,K}\,(\zeta\,z\,k^2 - g^+\,k + f\,h,\ g_-(h\,k-z))\,, & \frac{r}{h\,K}\,(g\,h\,k^2 - f^2\,g^+\,k - z,\ f\,g_-(z\,k+h))\,,\\
-\frac{r}{h\,K}\,(z\,k^2 + f^2 g^+\,k - g\,h,\ f\,\psi_+(h\,k-z))\,, & \frac{r}{h\,K}\,(f\,h\,k^2 + g^+\,k + \zeta\,z,\ \psi_+(z\,k+h))\,.
\end{array}$$

Theorem 5.

(i) *The points $\mathfrak{I}$, $\mathfrak{I}_a$, $\mathfrak{I}_b$ and $\mathfrak{I}_e$ are the k^2-points of the segments YP, TU, VS and QX.*

(ii) *The points $\mathfrak{J}$, $\mathfrak{J}_a$, $\mathfrak{J}_c$ and $\mathfrak{J}_e$ are the k^2-points of the segments QU, YS, TX and VP.*

Proof: Since

$$\frac{x_Y + k^2\,x_P}{K} = \frac{rz\varphi_-}{hK} + \frac{r\,\varphi_-\,k}{K} = \frac{r\,\varphi_-(h\,k+z)}{hK} = x_{\mathfrak{I}}$$

and

$$\frac{y_Y + k^2\,y_P}{K} = \frac{r\,f\,\varphi_-\,g_-}{hK} + \frac{r\,\varphi_-\,\psi_+}{hK} = \frac{r\,\varphi_-(z\,k-h)}{hK} = y_{\mathfrak{I}},$$

it follows that $\mathfrak{I}$ is the k^2-point of the segment YP. The other cases have similar proofs. □

Now we could say that the Theorems 3, 4 and 5 together represent the *complete Thébault theorem.*

The rather simple coordinates of the incenters and the excenters of the triangles ABC, BCE, ABE and ACE allow us to prove easily the following results that Johnson in [11, p. 193] calls the *"Japanese Theorem"* (see also [20]).

Theorem 6.

(i) *The following quadrangles $I\mathfrak{I}J\mathfrak{J}$, $I_a\mathfrak{I}_bJ_e\mathfrak{J}_c$, $I_b\mathfrak{I}_aJ_c\mathfrak{J}_e$ and $I_c\mathfrak{I}_eJ_b\mathfrak{J}_a$ are rectangles.*

(ii) *Their areas satisfy: $|I\mathfrak{I}J\mathfrak{J}|\,|I_a\mathfrak{I}_bJ_e\mathfrak{J}_c| = |I_b\mathfrak{I}_aJ_c\mathfrak{J}_e|\,|I_c\mathfrak{I}_eJ_b\mathfrak{J}_a|$.*

(iii) *Their centers are vertices of a parallelogram with the center at the circumcenter O of the triangle ABC.*

Proof: Since the lines $I\mathfrak{I}$ and $J\mathfrak{J}$ have the equations $k\,x - y = r\,\varphi_-$ and $k\,x - y = \frac{r\,g\,z\,\varphi_-}{h}$, we infer that they are parallel. Similarly one can prove that the lines IJ and $\mathfrak{I}\mathfrak{J}$ are also parallel. If follows that $I\mathfrak{I}J\mathfrak{J}$ is a parallelogram. On the other hand, since the lines $I\mathfrak{I}$ and $I\mathfrak{J}$ have the equations $k\,x - y = r\,\varphi_-$ and $x + k\,y = r\,f_+$, we conclude that they are perpendicular and $I\mathfrak{I}J\mathfrak{J}$ is a rectangle.

Since the area of a rectangle is the product of the lengths of its adjacent sides, we see that $|I\mathfrak{I}J\mathfrak{J}| = \frac{r^2\,f^+\,g^+\,|g_-\,\varphi_-|}{h^2\,K}$. Similarly,

$$|I_a\mathfrak{I}_bJ_e\mathfrak{J}_c| = \frac{r^2\,\zeta\,f^+\,g^+\,f_+\,\psi_+}{h^2\,K}, \quad |I_b\mathfrak{I}_aJ_c\mathfrak{J}_e| = \frac{r^2\,g\,f^+\,g^+\,\psi_+\,|\varphi_-|}{h^2\,K},$$

and $|I_c\mathfrak{I}_eJ_b\mathfrak{J}_a| = \frac{r^2\,f\,f^+\,g^+\,f_+\,|g_-|}{h^2\,K}$. The identity in (ii) is now obvious.

Finally, it is easy to check that the circumcenter O is the midpoint of the two segments $G_{I\mathfrak{I}J\mathfrak{J}}\,G_{I_a\mathfrak{I}_bJ_e\mathfrak{J}_c}$ and $G_{I_b\mathfrak{I}_aJ_c\mathfrak{J}_e}\,G_{I_c\mathfrak{I}_eJ_b\mathfrak{J}_a}$ joining the centers (i.e., the centroids) of these rectangles. □

Note that the inradii j, $\mathfrak{r}$ and $\mathfrak{j}$ and the exradii j_b, j_c, j_e, $\mathfrak{r}_a$, $\mathfrak{r}_b$, $\mathfrak{r}_e$, $\mathfrak{j}_a$, $\mathfrak{j}_c$ and $\mathfrak{j}_e$ of the triangles BCE, ABE and ACE are the absolute values of the quotients

$$\frac{r\,g_-\,z\,\varphi_-}{h\,K}, \quad \frac{r\,\varphi_-(\bar{h} - d\,k)}{h\,K}, \quad \frac{r\,g_-(\bar{h}\,k + d)}{h\,K}, \quad \frac{r\,f_+\,g_-}{K}, \quad \frac{r\,\varphi_-\,\psi_+}{K}, \quad \frac{r\,f_+\,\psi_+\,z}{h\,K},$$

$$\frac{r\,\varphi_-\,g(\bar{h}\,k + d)}{h\,K}, \quad \frac{r\,f_+\,g(\bar{h} - d\,k)}{h\,K}, \quad \frac{r\,f_+(\bar{h}\,k + d)}{h\,K}, \quad \frac{r\,f\,g_-(\bar{h} - d\,k)}{h\,K}, \quad \frac{r\,f\,\psi_+(\bar{h}\,k + d)}{h\,K}, \quad \frac{r\,\psi_+(\bar{h} - d\,k)}{h\,K}.$$

Now, at least under the assumption that D is on the segment BC, we can easily check the following identities:

$$r + j = \mathfrak{r} + \mathfrak{j}, \quad r_a + j_e = \mathfrak{r}_b + \mathfrak{j}_c, \quad r_b + j_c = \mathfrak{r}_a + \mathfrak{j}_e, \quad r_c + j_b = \mathfrak{r}_e + \mathfrak{j}_a.$$

The first is the relation (2.2) in [20].

3. Some conics as loci and envelopes

In order to find the locus of the *Sawayama-Thébault center* P, let us eliminate the parameter k from the equations $x_P = x$ and $y_P = y$. We get the equation $y = \frac{x(rz - x)}{rh}$ of the parabola

μ with the circumcenter O as the focus and the horizontal line ε above the line BC at the distance R (the circumradius) as the directrix. Repeating this for the centers Q, S and T will always produce the same parabola μ. On the other hand, doing this for the centers U, V, X and Y, will give the equation $y = \frac{hx(x-rz)}{rz^2}$ of the parabola ν also with the circumcenter O as focus and the horizontal line ε^* below the line BC at the distance R as directrix.

Corollary 2. *The points P, Q, S and T are on the parabola μ and the points U, V, X and Y are on the parabola ν.*

The parabolas μ and ν intersect only in the points B and C and they enclose the region with the area $\frac{2}{3}\,aR$.

When the point D moves on the line BC, the many lines joining pairs of Sawayama-Thébault centers provide families of lines that envelop some interesting conics of the triangle ABC. For example, one interpretation of the Theorem 3 is that the lines PQ, ST, UV and XY envelop the points I, I_a, I_b and I_c (considered as degenerated ellipses), respectively. On the other hand, it was noted in [3], the lines PS, QT, UX, and VY envelop the parabola λ of focus A and directrix BC having the equation

$$y = \frac{h}{4r\zeta}\,x^2 - \frac{f^-}{2f}\,x + \frac{rg(f^+)^2}{4fh}\,.$$

The parabolas λ, μ and ν are closely related in many respects: They have parallel directrices and axes and the distance between the foci of λ and μ and between the foci of λ and ν is equal to the distance between their directrices. It is not difficult to see that λ and μ touch in the $\frac{(b+c)^2-a^2}{a^2}$-point T_μ of the segment AO and that λ and ν touch in the $\frac{(b-c)^2-a^2}{a^2}$-point T_ν of the segment AO (when $b \neq c$).

When $b \neq c$, the lines PT and QS envelop the same hyperbola η with the equation $\zeta(2x-rz)^2 - (hy-2r\zeta)^2 = r^2 d^2 \zeta$ ([3, Remark 7]).

The lines UY and VX envelop the same ellipse χ with the equation

$$h^2\zeta(2x-rz)^2 + z^2(hy-2r\zeta)^2 = r^2\bar{h}^2 z^2\zeta.$$

It can be shown that χ is symmetric with respect to the perpendicular bisector of BC, tangent to ν at B and C, tangent to lines $T_\nu I_b$ and $T_\nu I_c$ and to the perpendiculars to BC through I_b and I_c.

4. The line AD tangent to the circumcircle

We shall see that some positions of the point D on the line BC are particularly important. In the following two results we identify what happens when the line AD is tangent to the circumcircle o at the point A. In this exceptional case many points of the configuration coincide. Of course, this can happen only when the angles B and C are different.

Let $P_o, \ldots, Y_o$ denote the points in which the Sawayama-Thébault circles touch the circumcircle o. Their respective coordinates are

$$\frac{r\varphi_-}{P_1}\,(P_2,\ 2h\psi_+)\,,\quad \frac{rf_+}{Q_1}\,(Q_2,\ 2hg_-)\,,\quad \frac{rgf_+}{S_1}\,(S_2,\ 2hfg_-)\,,\quad \frac{rg\varphi_-}{T_1}\,(T_2,\ -2hf\psi_+)\,,$$
$$\frac{rgz\varphi_-}{hU_1}\,(U_2,\ 2zg_-)\,,\quad \frac{rgzf_+}{hV_1}\,(V_2,\ 2z\psi_+)\,,\quad \frac{rzf_+}{hX_1}\,(X_2,\ 2zf\psi_+)\,,\quad \frac{rz\varphi_-}{hY_1}\,(Y_2,\ 2zfg_-)$$

with

$$\begin{array}{lll} P_1 = (h^2+d^2)k^2 - 4dk + 4, & Q_1 = 4k(k+d) + h^2 + d^2, & S_1 = (h^2+d^2)k^2 - 4\zeta(dk\zeta), \\ T_1 = 4\zeta k(\zeta k + d) + h^2 + d^2, & U_1 = (\bar{h}^2 + z^2)k^2 - 4g(\bar{h}k - g), & V_1 = 4gk(gk + \bar{h}) + \bar{h}^2 + z^2, \\ X_1 = (\bar{h}^2 + z^2)k^2 + 4f(\bar{h}k + f), & Y_1 = 4fk(fk - \bar{h}) + \bar{h}^2 + z^2 & \end{array}$$

and

$$\begin{array}{lll} P_2 = (h^2 + dz)k - 2z, & Q_2 = 2zk + h^2 + dz, & S_2 = (h^2 - dz)k + 2z\zeta, \\ T_2 = 2z\zeta k - h^2 + dz, & U_2 = (\zeta^2 + z^2 - 1)k - 2gh, & V_2 = 2ghk + \zeta^2 + z^2 - 1, \\ X_2 = (\zeta^2 - z^2 - 1)k + 2fh, & Y_2 = 2fhk - \zeta^2 + z^2 + 1\,. & \end{array}$$

For eight points $G_1, \ldots, G_8$, let $D(G_1, \ldots, G_8)$ be the determinant

$$\begin{vmatrix} x_{G_1} & y_{G_1} & x_{G_2} & y_{G_2} \\ x_{G_3} & y_{G_3} & x_{G_4} & y_{G_4} \\ x_{G_5} & y_{G_5} & x_{G_6} & y_{G_6} \\ x_{G_7} & y_{G_7} & x_{G_8} & y_{G_8} \end{vmatrix}.$$

Theorem 7. *The following statements are equivalent:*
(i) $P = S$, (ii) $V = Y$, (iii) $P_o = A$, (iv) $S_o = A$, (v) $V_o = A$, (vi) $Y_o = A$,
(vii) $I = J_c$, (viii) $I_a = J_b$, (ix) $I_b = J$, (x) $I_c = J_a$, (xi) $\mathfrak{I} = \mathfrak{I}_b$, (xii) $\mathfrak{I} = \mathfrak{I}_a$,
(xiii) $\mathfrak{I} = \mathfrak{I}_e$, (xiv) $\mathfrak{I}_a = \mathfrak{I}_e$, (xv) *the lines* $\mathfrak{I}_b\mathfrak{I}_c$ *and* I_cJ_e *are perpendicular,*
(xvi) $D(I, I_a, I_b, I_c, J, J_e, J_b, J_c) = 0$, (xvii) $\mathfrak{I}_b \in AD$, (xviii) $\mathfrak{I}_e \in AD$,
(xix) *the lines* $\mathfrak{I}_b\mathfrak{I}_e$ *and* AD *are perpendicular,*
(xx) *the lines* I_cJ_a *and* AD *are parallel,* (xxi) *the lines* I_aJ_c *and* AD *are parallel, and*
(xxii) *the angle* B *is smaller than the angle* C *and the lines* AD *and* AO *are perpendicular.*

Proof: Since $|PS|^2 = \frac{r^2K(\bar{h} - dk)^2}{k^4}$, we conclude that $P = S$ if and only if $k = \frac{\bar{h}}{d}$. However, the parameter k is positive, so that $f > g$ (i.e., the angle B is smaller than the angle C) and the point D divides the segment BC in the ratio $-\frac{|AB|^2}{|AC|^2}$ (i.e., the point D is the intersection of the tangent to the circumcircle at the vertex A with the line BC). This shows the equivalence of (i) and (xxii). For the other parts, it suffices to note that the only factor that could be zero in the squares of distances of the points in this part is always the same $\bar{h} - dk$. □

The following companion result has a similar proof. This time the common factor is $d + \bar{h}k$.

Theorem 8. *The following statements are equivalent:*
(i) $Q = T$, (ii) $U = X$, (iii) $Q_o = A$, (iv) $T_o = A$, (v) $U_o = A$, (vi) $X_o = A$,
(vii) $I = J_b$, (viii) $I_a = J_c$, (ix) $I_b = J_a$, (x) $I_c = J$, (xi) $\mathfrak{I} = \mathfrak{I}_a$, (xii) $\mathfrak{I} = \mathfrak{I}_e$,
(xiii) $\mathfrak{I} = \mathfrak{I}_c$, (xiv) $\mathfrak{I}_a = \mathfrak{I}_e$, (xv) *the lines* $\mathfrak{I}_b\mathfrak{I}_c$ *and* I_bJ_e *are perpendicular,*
(xvi) $\mathfrak{I}_e \in AD$, (xvii) $\mathfrak{I}_c \in AD$, (xviii) *the lines* $\mathfrak{I}_e\mathfrak{I}_c$ *and* AD *are perpendicular,*
(xix) *the lines* I_bJ_a *and* AD *are parallel,* (xx) *the lines* I_aJ_b *and* AD *are parallel, and*
(xxi) *the angle* B *is larger than the angle* C *and the lines* AD *and* AO *are perpendicular.*

5. Equal radii r_1 and r_2

In this section we shall explore when the radii r_1 and r_2 of the first and the second Sawayama-Thébault circles are equal. In fact, the problem is to describe the positions of the point D on the line BC when $r_1 = r_2$ holds. It turns out that the equality happens for three values of the parameter k. The simpler value corresponds to the case when $r_1 = r_2 = r$ (see Theorem 9) and the two more complicated values to the case $r_1 = r_2$ and either $r_1 \neq r$ or $r_2 \neq r$ (see Theorem 10). In each situation many other geometric consequences hold. Some are characteristic for the equality of r_1 and r_2 (with r).

Let $k_{I'_a} = \frac{\sqrt{d^2+4}-d}{2}$ be the positive root of the polynomial $p_{I'_a} = L + dk$. Let the perpendicular bisector of the segment BC intersect the circumcircle o in the points Z_1 and Z_2 such that Z_1 is above and Z_2 is below the line BC. Note that Z_1 is the midpoint of I_bI_c and the circle $k_{I_bI_c}$ goes through B, C and J_a. Similarly, Z_2 is the midpoint of J_bJ_c and the circle $k_{J_bJ_c}$ goes through B, C and I_a.

Theorem 9. *The following statements are equivalent:*

(i) *the point D is the orthogonal projection I'_a of the excenter I_a onto the line BC,*

(ii) *the parameter k is $k_{I'_a}$,* (iii) *the lines PQ and BC are parallel,*

(iv) *the lines P_oQ_o and BC are parallel,* (v) *the line AD bisects the segment PQ,*

(vi) *the segments PQ and $P''Q''$ share the midpoints,*

(vii) *the line joining incenter I and midpoint of the segment BC is parallel to the line AD,*

(viii) *the line joining the circumcenter O and the midpoint of either the segment $P'Q'$ or $P''Q''$ is perpendicular to the line PQ,*

(ix) *the midpoint of the segment BC has the same power with respect to the circles k_1 and k_2,*

(x) *the points P_o and Q_o are equidistant from the point Z_1 and/or Z_2, and*

(xi) *the equalities $r_1 = r$ and $r_2 = r$ hold.*

Proof: Since the point I'_a has the coordinates $(rg, 0)$, we get that $|DI'_a|$ is equal $\frac{r\zeta\,|p_{I'_a}|}{hk}$. Hence, (i) and (ii) are equivalent.
The lines PQ and BC are parallel if and only if the points P and Q have equal ordinates. Since $y_P - y_Q = \frac{rKp_{I'_a}}{hk^2}$, we see that (ii) and (iii) are equivalent.
Similarly, since $y_{P_o} - y_{Q_o} = \frac{2rhKf^+g^+p_{I'_a}}{P_1Q_1}$, it follows that (ii) and (iv) are equivalent.
The midpoint of the segment PQ has the coordinates $\frac{r}{2k}\left(L + 2fk,\ -\frac{p_4}{hk}\right)$, where p_4 is defined below. It is on the line AD whose equation is $2kx + Ly = \frac{2rgf_+\varphi_-}{h}$ if and only if $\frac{r^2\zeta K^2 p_{I'_a}}{2h^2k^3} = 0$. Hence, (ii) and (v) are equivalent.
The orthogonal projections P'' and Q'' of P and Q onto the line AD have

$$\frac{r\varphi_-}{hkK}\left(hk^2 + 2gk + \bar{h},\ 2\psi_+k\right) \quad \text{and} \quad \frac{rf_+}{hK}\left(\bar{h}k^2 - 2gk + h,\ -2g_-\right)$$

as coordinates. It follows that the midpoints of the segments PQ and $P''Q''$ are $\frac{rK|p_{I'_a}|}{2hk^2}$ apart. Therefore, (ii) and (vi) are equivalent.

The line joining the incenter I and the midpoint of the segment BC has the equation $2x-dy = rz$. It will be parallel to the line AD if and only if $\frac{r^2\zeta p_{I'_a}}{hk} = 0$. This shows the equivalence of (ii) and (vii).
The line PQ has the equation $p_{I'_a}\,x + hk\,y = r\,f_+\,\varphi_-$. The line joining the circumcenter O and the midpoint of the segment $P'Q'$ has the equation

$$2(h^2 - z^2)k\,y - 4\,h\,p_{I'_a}\,y = r(L + 2\,f\,k)(h^2 - z^2).$$

They will be perpendicular if and only if $\frac{r^2\,K\,f^+\,g^+\,p_{I'_a}}{4\,h^2\,k^2} = 0$. The line joining O and the midpoint of the segment $P''Q''$ is more complicated but it will be perpendicular to the line PQ if and only if the same condition holds. This shows the equivalence of (ii) and (viii).
The power $w(A_g, k_2)$ of the midpoint A_g of the segment BC with respect to the circle k_2 is $|A_gQ|^2 - r_2^2$ or $\frac{r^2(d+2k)^2}{4}$. Similarly, $w(A_g, k_1)$ is $\frac{r^2(dk-2)^2}{4k^2}$. Their difference is $\frac{r^2Kp_{I'_a}}{k^2}$. Hence, (ix) and (ii) are equivalent.
The differences of squares $|QZ_1|^2 - |PZ_1|^2$ and $|PZ_2|^2 - |QZ_2|^2$ of distances are equal

$$\frac{r^2\,K\,(f^+)^2\,(g^+)^2\,p_{I'_a}}{[(h^2+d^2)k^2 - 4\,d\,k + 4](4\,k^2 + 4\,d\,k + h^2 + d^2)}.$$

It follows that (x) and (ii) are equivalent.
Finally, since $r_1^2 - r^2 = \frac{r^2Mp_{I'_a}}{h^2k^4}$ and $r_2^2 - r^2 = \frac{r^2Np_{I'_a}}{h^2}$ and the factors $M = (2\zeta-1)k^2 + dk - 1$ and $N = k^2 + dk - 2\zeta + 1$ are not both zero at any real number k, we conclude that (ii) and (xi) are equivalent. □

Let $k_\pm = \frac{\sqrt{2\,N_\pm} \pm M - d}{4}$ be the positive roots of the quartic polynomial $p_4 = L(L+dk) - 2hk^2$, where $M = \sqrt{d^2+8h}$ and $N_\pm = d^2 \mp dM + 4\bar{h}$.

Theorem 10. *The following statements are equivalent:*
(i) *the parameter k is either k_+ or k_-,* (ii) *the lines PQ and AD are parallel,*
(iii) *the line P_oQ_o bisects the segment $P'Q'$,* (iv) *the line PQ bisects the segment $P'Q'$,*
(v) *the segments PQ and $P'Q'$ share the midpoints, and*
(vi) *the lines AD and DI_a are perpendicular.*

Proof: Since

$$p_{I'_a}x + hky = rf_+\varphi_- \;\text{ and }\; 2kx + Ly = \frac{2rgf_+\varphi_-}{h}$$

are the equations of the lines PQ and AD, they will be parallel if and only if $p_4 = 0$. This shows that (i) and (ii) are equivalent.
The orthogonal projections P' and Q' of the centers P and Q onto the line BC (the x-axis) have the abscises $\frac{r\varphi_-}{k}$ and rf_+. It follows that the midpoint of the segment $P'Q'$ lies on the line P_oQ_o (i.e., on the line $2hp_{I'_a}x - [2dL + (z^2 - \bar{h}^2 - 4)k]y = 2rhf_+\varphi_-$), provided

$$p_{I'_a}\left(\frac{rL}{2k} + rf\right) - rf_+\varphi_- = \frac{rp_4}{2k} = 0.$$

Hence, (i) and (iii) are equivalent.
This same calculation applies also in the proof that (i) and (iv) are equivalent because the line PQ has the equation $p_{I'_a}x + hky = rf_+\varphi_-$.

The midpoints of PQ and $P'Q'$ are $\frac{r|p_4|}{2hk^2}$ apart. We easily conclude that (i) and (v) are equivalent.
Finally, since $hkx - p_{I'_a}y = rf_+g\varphi_-$ is the equation of the line DI_a, we get that this line is perpendicular with the line AD if and only if $2hk^2 - p_{I'_a}L = -p_4 = 0$. Hence, the first and the last statements are equivalent. □

Note that the condition (ii) in Theorem 10 implies $r_1 = r_2$. Hence, the correct version of Theorem 4 in [32] is the following result.

Corollary 3. *The following statements are equivalent:*

(i) *the equality $r_1 = r_2$ holds,*

(ii) *the parameter k is either $k_{I'_a}$, k_+ or k_-,*

(iii) *the points P and Q are at equal distance from the midpoint of $P'Q'$ and/or $P''Q''$.*

Proof: Since $r_1 = |y_P|$ and $r_2 = |y_Q|$, it follows that $r_1 = r_2$ if and only if $y_P^2 - y_Q^2 = \frac{r^2Kp_{I'_a}p_4}{h^2k^4} = 0$. Let M' and M'' be the midpoints of $P'Q'$ and $P''Q''$. Then $|QM'|^2 - |PM'|^2 = |QM''|^2 - |PM''|^2 = \frac{r^2Kp_{I'_a}p_4}{h^2k^4}$. Hence, our claim follows from Theorems 9 and 10 because the parameter k is a positive real number. □

6. Lines connecting the touching points $P_o, \ldots, Y_o$

The points where the eight Sawayama-Thébault circles touch the circumcircle have many properties. Some are revealed in the next result.

Let $M_1, \ldots, M_{24}$ denote the intersections of the lines

$$\begin{array}{cccccccc} P_oT_o, & P_oV_o, & P_oQ_o, & S_oT_o, & Q_oU_o, & Q_oS_o, & Q_oS_o, & Q_oX_o, \\ P_oQ_o, & S_oT_o, & P_oY_o, & P_oT_o, & P_oT_o, & P_oV_o, & P_oY_o, & S_oV_o, \\ S_oY_o, & U_oY_o, & Q_oU_o, & P_oY_o, & U_oV_o, & P_oQ_o, & Q_oX_o, & P_oV_o \end{array}$$

with the respective lines

$$\begin{array}{cccccccc} U_oY_o, & S_oY_o, & X_oY_o, & U_oV_o, & T_oX_o, & V_oX_o, & U_oY_o, & T_oU_o, \\ U_oV_o, & X_oY_o, & S_oV_o, & V_oX_o, & Q_oS_o, & Q_oU_o, & Q_oX_o, & T_oU_o, \\ T_oX_o, & V_oX_o, & S_oY_o, & T_oU_o, & X_oY_o, & S_oT_o, & S_oV_o, & T_oX_o. \end{array}$$

Theorem 11. *The point D lies on the following lines: P_oS_o, Q_oT_o, U_oX_o and V_oY_o. The intersections $M_1, \ldots, M_{24}$ are on the lines $\mathfrak{I}\mathfrak{I}_a$, $\mathfrak{I}\mathfrak{I}_b$, $\mathfrak{I}\mathfrak{I}_e$, $\mathfrak{I}_a\mathfrak{I}_b$, $\mathfrak{I}_a\mathfrak{I}_e$, $\mathfrak{I}_b\mathfrak{I}_e$, $\mathfrak{J}\mathfrak{J}_a$, ..., II_a, ..., J_cJ_e, respectively. The points M_2, M_5, M_8, M_{11}, M_{13}, M_{18}, M_{21} and M_{22} are on the line perpendicular to the line DO.*

The point D is collinear with the points M_1, M_6, M_9, M_{10}, M_{14}, M_{17}, M_{20} and M_{23} as well as with the points M_3, M_4, M_7, M_{12}, M_{15}, M_{16}, M_{19} and M_{24}.

The point A is on the circles $k_{M_1M_6}$, $k_{M_7M_{12}}$ and $k_{M_{13}M_{18}}$, the point B is on the circles $k_{M_2M_5}$, $k_{M_{14}M_{17}}$ and $k_{M_{19}M_{24}}$, the point C is on the circles $k_{M_8M_{11}}$, $k_{M_{15}M_{16}}$ and $k_{M_{20}M_{23}}$, and the point E is on the circles $k_{M_3M_4}$, $k_{M_9M_{10}}$ and $k_{M_{21}M_{22}}$.

Moreover, there are 32 triples of collinear points beginning with $\{M_4, M_2, M_1\}$ and ending with $\{M_{24}, M_{23}, M_{22}\}$ (one from each of the above three groups of eight points).

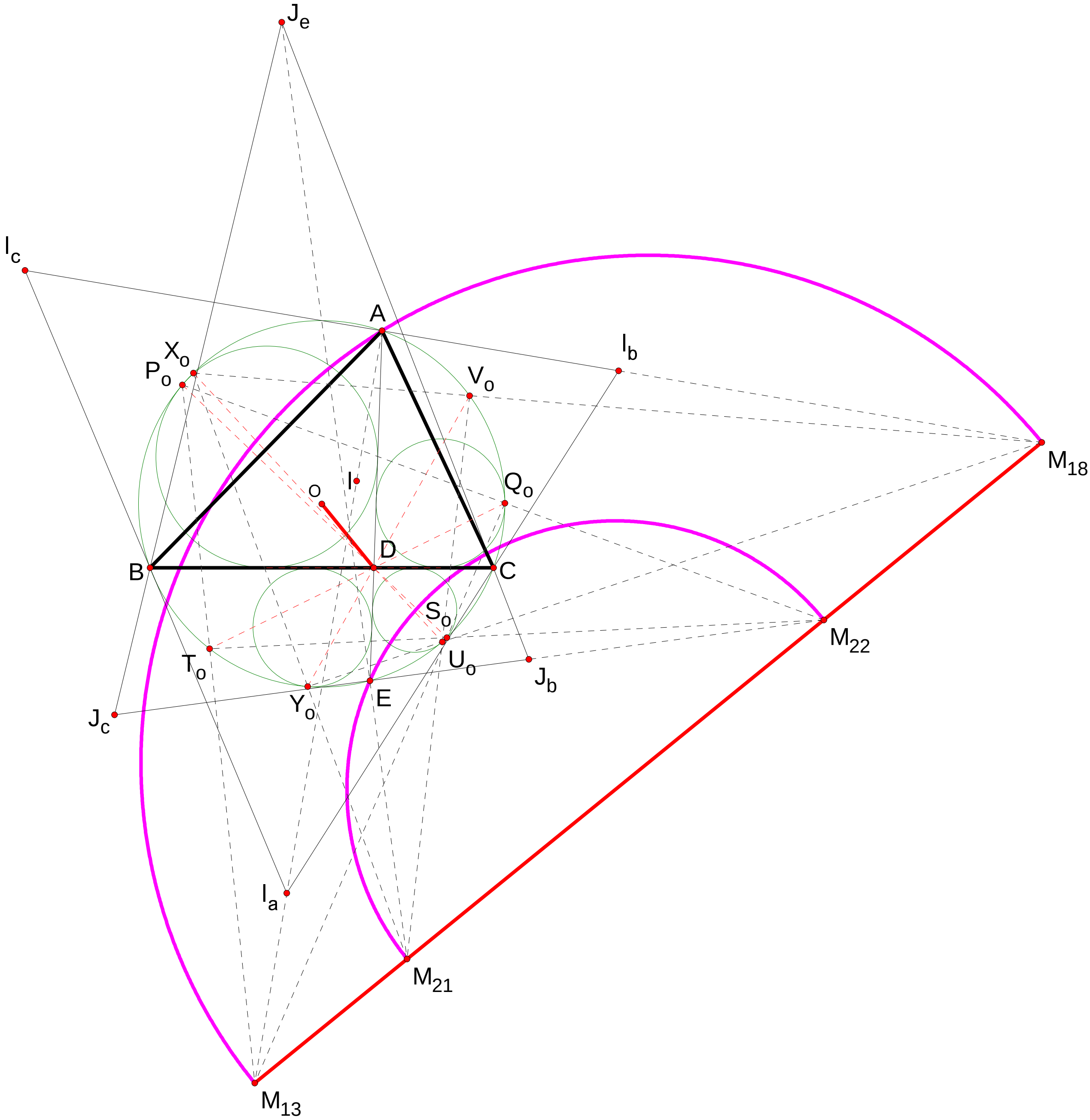

Figure 3: The points M_{13}, M_{18}, M_{22} and M_{21}

Proof: When $\bar{h} \neq dk$, then the line P_oS_o has the equation

$$2h^2kx + [(h^2 + d^2)k^2 - 2d\bar{h}k + 4\zeta]y = 2rghf_+\varphi_-.$$

The coordinates of the point D satisfy this equation. We prove similarly that D also lies on the lines Q_oT_o, U_oX_o and V_oY_o.
The intersection M_{13} has the coordinates $\frac{rg}{M}(N, -2fhs_2)$, where

$$M = 4d\zeta L + \bar{h}k(d^2 + h^2 - 4\zeta) \quad \text{and} \quad N = 2dfzL + k[(f^-)^2g^+ - 4\zeta f^+].$$

It lies on the line II_a with the equation $\bar{h}x - dy = rgf^+$.

Similarly, the point M_{18} has the coordinates $\frac{rgz}{hM}(N,\ 2fzp_2)$, where

$$M = 4\bar{h}\zeta L + dk(d^2 + h^2 + 4\zeta) \quad \text{and} \quad N = 2fh\bar{h}L + k[(f^-)^2 g^+ + 4\zeta f^+].$$

It lies on the line $I_b I_c$ with the equation $dx + \bar{h}y = \frac{rgzf^+}{h}$. The line $M_{13}M_{18}$ is perpendicular to the line DO with the equation

$$k(h^2 - d^2)x - (4\zeta L + 2d\bar{h}k)y = \frac{(h^2 - d^2)rgf_+\varphi_-}{h}.$$

Moreover, the midpoint of $M_{13}M_{18}$ is equidistant from M_{13} and A.
The intersections M_{22} and M_{21} are treated similarly. Of course, they both lie on the line $M_{13}M_{18}$. □

7. Concluding remarks

The longer original version (45 pages) of this paper is available on the author's web page `http://math.hr/~cerin/`. It includes an extensive study of the Sawayama-Thébault configuration. In particular, we explore the equalities $r_3 = r_4$, $r_5 = r_6$ and $r_7 = r_8$ and present various identities for the radii of the Sawayama-Thébault circles.

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