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Axiomatizing Perpendicularity and Parallelism

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Abstract. We establish a framework in which one can study perpendicularity and parallelism axiomatically. The lines of the Euclidean plane provide an admissible model of our axiom system but various other models exist, too. Focusing only on these two relations, our approach is more elementary and, thus, more suitable for the teaching of deductive geometry in the upper secondary schools and in mathematics teacher education than other existing axiomatizations of the Euclidean plane geometry.

Key Words: Euclidean geometry, mathematics education, parallelism, perpendicularity

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1. Introduction

Nowadays hardly anyone considers bringing geometry education in school back on the axiomatic basis. However, the need for introducing students to deductive thinking and reasoning has not vanished in time. On the contrary, these skills have become even more important since the appearance of information and communications technology. This encourages us to seek such sections of geometry — or new perspectives to it — that were suitable for enhancing deductive thinking in mathematics education in the upper secondary schools and in mathematics teacher education.

Intuitively, perpendicularity is such an elementary concept that one is tempted to think that it could be defined with a few simple axioms. Indeed, such attempts exist in the literature. Already Euclid's fourth postulate states that all right angles are equal. In Hilbert's approach [5], the same claim — and, implicitly, perpendicularity, too — is established on the theory of the congruence of angles.

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More recent textbooks by Bachmann [1] and Ewald [4] introduce — with a few minor relative variations — a modern axiomatic description of perpendicularity that originates from Thomsen's [10] idea of studying elementary geometry in terms of group-calculus and the concept of reflection which seems to be due to Hjelmslev [6]. The main idea is to let the symbol for a point A or for a line a be used also for the corresponding involutory isometry. From this perspective, A is the reflection in a point (or a half-turn around a point), and a is the reflection in a line. Further, for given points A and B, AB is the translation along the connecting line through twice the distance between them. If a and b are two intersecting lines, ab is the rotation around their intersection point through twice the angle between them. The point A and line a are incident if the corresponding transformations are commutative. The same property applied to two lines makes them perpendicular. In this case, if the lines are a and b, $ab\,(= ba)$ is their intersection point. A thorough review of this approach and [1] is given in [3] .

The approach outlined above assumes three incidence axioms, three axioms on perpendicularity itself (e.g., symmetry) and two for performing reasonable operations on perpendicular lines in the context of isometric planar motions. In principle, this axiom system could be introduced already in the upper secondary school but its educational value may be small because it aims primarily at axiomatizing the plane geometry completely. Therefore, the starting point for the whole manoeuvre may be, in practice, too hard to be discerned for most high school students. Moreover, the originality of this axiomatic approach begins to vanish as the study goes beyond affine geometry. For example, [4] relies heavily on standard analytic tools based on the concepts of coordinate, inner product and metric.

By limiting our focus we may win not only in the simplicity of the resulting axiom system but also in finding new models. To that end, a potential source of problems is the fact that perpendicularity is not transitive. Although in the Euclidean plane we may bring in axioms like

$$a \perp b \,\wedge\, b \perp c \implies a \not\perp c \quad \text{and} \quad a \perp b \,\wedge\, b \perp c \,\wedge\, c \perp d \implies a \perp d,$$

already in the Euclidean space and on spheres, both $a \perp b \wedge b \perp c$ and $a \perp b \wedge b \perp c \wedge c \perp d$ can imply anything. Nevertheless, we will start from a such axiomatic description of perpendicularity that is general enough to cover, at least, the geometry of straight lines of the Euclidean plane as a model and then survey how far our approach may extend. Interestingly enough, various admissible models can be built on, e.g., number sets.

In Sections 2 and 3, we will state four axioms to characterize (planar) perpendicularity and then study perpendicular complements, a kind of analogies to orthogonal components in the inner product spaces. In Section 4, we will add two axioms to introduce parallelism subordinate to perpendicularity. We will see that such parallelism always exists and is unique. This result does not assume introducing the concept of reflection.

In Section 5, we will sketch how to develop our ideas to work in the Euclidean spaces of dimension greater than two. Finally, we will study a few examples in Section 6 and make some concluding remarks in Section 7.

This paper is related to a project which aims at building bridges between geometry education and the training of researchers in mathematics by pointing out topics where it is possible to gain new insight into geometry basing only on elementary knowledge, see also [8]. Therefore, we have recorded the proofs of the most essential results but also given some theorems and advanced topics to be discussed and verified in classroom. The present paper may be useful also in an introductory discrete mathematics course at tertiary level since it provides nontrivial material for first year students to exercise mathematical reasoning and to familiarize themselves with the theory of relations. In our view, standard examples in such courses are often rather artificial and, thus, quite non-inspirational.

2. Planar perpendicularity

Throughout this paper, X is a nonempty set and $\perp$ is a binary relation there (except in Section 5). Clearly, $\perp$ must be irreflexive and symmetric in order to represent perpendicularity in a proper way. So, we assume first that

(A1) $\forall a \in X\colon\ a \not\perp a$,

(A2) $\forall a, b \in X\colon\ a \perp b \implies b \perp a$.

Further, perpendicularity is not transitive but we require a somewhat similar property:

(A3) $\forall a, b, c, d \in X\colon\ a \perp b \perp c \perp d \implies a \perp d$.

In A3, the left-hand side of the implication stands for $a \perp b \ \wedge\ b \perp c \ \wedge\ c \perp d$.

Finally, it is useful to suppose that each element has one perpendicular element at least:

(A4) $\forall a \in X\colon\ \exists\, b \in X\colon\ a \perp b$.

Alternatively, sometimes it is enough that, at the minimum, one element has this property:

(A4') $\exists\, a, b \in X\colon\ a \perp b$.

If $\perp$ satisfies A1 – A4, we call it *planar perpendicularity.* Namely, if X is the set of lines in the Euclidean plane, then the Euclidean perpendicularity satisfies these axioms. However, there are also various other models for this axiom system; some of them will be discussed in Section 6.

On the other hand, A3 fails in the Euclidean spaces of dimension greater than two. Can we solve this problem by replacing it with the following, seemingly more general, assumption?

(A3') There is a number $n \geq 3$ such that, for all $a_1, \dots, a_{n+1} \in X$,

$$a_1 \perp a_2 \perp \dots \perp a_n \perp a_{n+1} \implies a_1 \perp a_{n+1}.$$

We observe first that, if A1, A2 and A4' hold, then n must be odd. In order to see this, suppose that $n = 2m$ is even and let $a, b \in X$ satisfying $a \perp b$. Then A2 implies that $b \perp a$. Hence $a \perp b \perp a \perp \dots \perp a \perp b \perp a$, where a runs $m + 1$ times and b m times. But this and A3' yield that $a \perp a$ contradicting A1.

As a matter of fact, the following theorem shows that A3' brings nothing new in our axiom system. In Section 5, we will meet another system that is more suitable for studying perpendicularity in space.

Theorem 1. *Assume* A2 *and let* $k \geq 3$ *be odd. Then* $\perp$ *satisfies* A3' *for* $n = k$ *if and only if it satisfies* A3 (*i.e.,* A3' *for* $n = 3$).

Proof: First assume that A3' holds for $k = 5$. To show A3, suppose that $a \perp b \perp c \perp d$. Then $a \perp b \perp a \perp b \perp c \perp d$ by A2, and so $a \perp d$ by A3'.
Conversely, assume now that A3 holds. If $a \perp b \perp c \perp d \perp e \perp f$, then, in particular, $a \perp b \perp c \perp d$, which implies $a \perp d$ by A3 and, consequently, $a \perp d \perp e \perp f$. Again by A3, we obtain $a \perp f$ and, at the same time, A3' for $k = 5$.
Using a simple induction argument it is easy to see that the claim holds also for general k. □

Corollary 1. *Assume* A1 – A3 *and* A4', *and let* $k \geq 2$. *Then, for all* $a_1, \ldots, a_{k+1} \in X$,

$$a_1 \perp a_2 \perp \cdots \perp a_k \perp a_{k+1} \implies a_1 \perp a_{k+1} \tag{1}$$

if and only if k *is odd.*

Proof: We have already seen that, if k is even, then (1) fails. If k is odd, then (1) follows from Theorem 1. □

3. Perpendicular complement

We will first consider complementary sets induced by a general binary relation and then focus on such sets induced by the perpendicularity relation. To that end, let $\sharp$ be a binary relation in X. If $\emptyset \neq A \subseteq X$, we define the *$\sharp$-complement* of A being

$$A^{\sharp} = \{y \in X \mid y \sharp A\}.$$

Here $y \sharp A$ means that $y \sharp x$ for all $x \in A$. We also define $\emptyset^{\sharp} = X$. If $\sharp$ is a planar perpendicularity, i.e., $\sharp = \perp$, we call $A^{\perp}$ the *(planar) perpendicular complement* of A.

Theorem 2. *For all* $A, B \subseteq X$,

(i) $A \subseteq B \implies B^{\sharp} \subseteq A^{\sharp}$,

(ii) $(A \cup B)^{\sharp} = A^{\sharp} \cap B^{\sharp}$,

(iii) $(A \cap B)^{\sharp} \supseteq A^{\sharp} \cup B^{\sharp}$,

(iv) $A^{\sharp\sharp} \supseteq A$ *if and only if* $\sharp$ *is symmetric.*

Proof: (i): Trivial.

(ii): $x \in A^{\sharp} \cap B^{\sharp} \iff x \sharp A \wedge x \sharp B \iff x \sharp A \cup B \iff x \in (A \cup B)^{\sharp}$.

(iii): $x \in A^{\sharp} \cup B^{\sharp} \iff x \sharp A \vee x \sharp B \implies x \sharp A \cap B \iff x \in (A \cap B)^{\sharp}$.

(iv): Suppose first that $\sharp$ is symmetric and let $x \in A$. Then $y \sharp x$ for all $y \in A^{\sharp}$. Hence, by symmetry, $x \sharp y$ for all $y \in A^{\sharp}$, i.e., $x \in A^{\sharp\sharp}$. In other words, $A^{\sharp\sharp} \supseteq A$ holds. Suppose next that there are $a, b \in X$ so that $b \sharp a$ and $a \not\sharp b$. If we set $A = \{a\}$, then $b \in A^{\sharp}$ and $a \notin A^{\sharp\sharp}$ violating the inclusion $A^{\sharp\sharp} \supseteq A$. □

Assuming now that $\sharp$ is a planar perpendicularity in X, can we sharpen Theorem 2 ? This raises another question: Do perpendicular complements form a partition of X? The answer to both questions is positive.

Theorem 3. *Perpendicular complements of all nonempty proper subsets of* X *form a partition of* X.

Proof: Let Σ denote the family of all nonempty proper subsets of X. Let $x \in X$. By A4, there exists $A \in \Sigma$ such that $x \in A^{\perp}$. Thus x belongs, at least, to one perpendicular complement of a set of Σ. In order to show that there is one such perpendicular complement at most, we prove that if $A^{\perp} \cap B^{\perp} \neq \emptyset$, then $A^{\perp} = B^{\perp}$. By symmetry, it is enough to show that $x \in A^{\perp} \implies x \in B^{\perp}$. Let $x \in A^{\perp}$, i.e., $x \perp a$ for all $a \in A$. It suffices to confirm that $x \perp b$ for all $b \in B$. To that end, fix $a \in A$ and $b \in B$. By the assumption, there is $y \in A^{\perp} \cap B^{\perp}$ and, consequently, $y \perp a$ and $y \perp b$. But now $x \perp a \perp y \perp b$ by A2, and A3 implies then that $x \perp b$. Since $b \in B$ can be arbitrary, the claim follows. □

Corollary 2. *Let $\perp$ be a planar perpendicularity in X and $\emptyset \neq A, B \subset X$ so that $A^\perp, B^\perp \neq \emptyset$. Then*

(i) $A \subseteq B \implies A^\perp = B^\perp$,

(ii) *If $A^\perp = B^\perp$, then $(A \cup B)^\perp = A^\perp = B^\perp$. Otherwise $(A \cup B)^\perp = \emptyset$.*

(iii) *If $A \cap B = \emptyset$, then $(A \cap B)^\perp = X$. Otherwise $(A \cap B)^\perp = A^\perp = B^\perp$.*

What about the converse of Theorem 3? Given a partition Π of X, is there a planar perpendicularity $\perp$ such that Π is the family of nonempty perpendicular complements of nonempty proper subsets of X? Not necessarily. For a counterexample, let $X = \{a, b, c\}$ and $\Pi = \{\{a\}, \{b\}, \{c\}\}$ and suppose that such $\perp$ exists. Then $a \perp b$ or $a \perp c$ by A4. Without loss of generality, we may assume that $a \perp b$. Now $\{a\}^\perp = \{b\}$ and $\{b\}^\perp = \{a\}$ but $\{c\}^\perp = \emptyset$ contradicting A4. The reason why this counterexample works is that Π has an odd number of elements. Otherwise the converse holds.

Theorem 4. *Let $\Pi = \{A_i \mid i \in I\}$ be a partition of X, where the number of elements of I is even or infinite. Then there exists planar perpendicularity $\perp$ such that Π is the family of nonempty perpendicular complements of nonempty proper subsets of X.*

Proof: By the assumption, I can be partitioned into equipotent sets I_1 and I_2. Let $\sigma\colon I_1 \to I_2$ be a bijection. We define $\perp$ now by setting

$$x \perp y \iff \begin{cases} x \in A_i \ \wedge \ y \in A_{\sigma(i)} & \text{whenever } i \in I_1, \\ x \in A_i \ \wedge \ y \in A_{\sigma^{-1}(i)} & \text{whenever } i \in I_2. \end{cases}$$

Then $\perp$ is planar perpendicularity. The validity of A1 and A4 follows from the fact that Π is a partition and that of A2 directly from the definition of $\perp$. To verify A3, notice that $A_i^\perp = A_{\sigma(i)}$ if $i \in I_1$, and $A_i^\perp = A_{\sigma^{-1}(i)}$ if $i \in I_2$; therefore, $A_i^{\perp\perp} = A_i$ for every $i \in I$. Hence, if $a \perp b \perp c \perp d$, then d is in the perpendicular complement of the set where a belongs to, i.e., $a \perp d$. So, Π is the desired family of perpendicular complements. $\square$

We complete this section with a few observations. First, the item (i) in Corollary 2 implies that already the perpendicular complements of the unit sets of X establish the partition of X. Second, given the partition $\{A_i\}$, $i \in I$, induced by the planar perpendicularity $\perp$, for each $i \in I$, there is $j \in I \setminus \{i\}$ such that $A_i^\perp = A_j$ and $A_j^\perp = A_i$. This claim follows from the fact that, by Theorem 3, every A_i is the perpendicular complement of some $B \subset X$ and from the fact that $B^{\perp\perp\perp} = B^\perp$ by A3. In other words, $A_i^{\perp\perp} = A_i$. For $n = 2$, the identity holds and, for $n \geq 3$, it fails if we partition the set of all Euclidean lines in $\mathbb{R}^n$ using the ordinary perpendicularity. This fact is another reason why we call perpendicularity satisfying A1–A4 planar.

4. Planar parallelism

The perpendicularity relation constructed in Theorem 4 is such that, for every $a, b, c \in X$, the condition $a \perp b \perp c$ implies that a and c belong to the same $A_i \in \Pi$. In the Euclidean plane, $a \perp b \perp c$ implies that a and c are parallel. These facts encourage us to enrich our axiom system with parallelism. More precisely, let $\perp$ be a planar perpendicularity in X and let $\parallel$ be another binary relation in X. We interlink $\parallel$ with $\perp$ by assuming that

(A5) $\forall a, b \in X\colon\ a \parallel b \implies \exists\, c \in X\colon a \perp c \perp b$,

(A6) $\forall a, b, c \in X\colon\ a \perp b \perp c \implies a \parallel c$.

We call $\parallel$ *planar parallelism subordinate to* $\perp$. Clearly, the axioms A1 – A6 are compatible with the ordinary perpendicularity and parallelism in the Euclidean plane. The next theorems confirm that A1 – A6 are, indeed, sufficient to induce the fundamental properties that the perpendicular and parallel lines have in the Euclidean plane.

Theorem 5. *The relation $\parallel$ is an equivalence relation.*

Proof: Reflexivity: Let $a \in X$. By A4, there is $b \in X$ such that $a \perp b$. Since $b \perp a$ by A2, we have $a \perp b \perp a$ and, hence, $a \parallel a$ by A6.

Symmetry: Suppose that $a \parallel b$. Axiom A5 implies now that there is $c \in X$ such that $a \perp c \perp b$. Further, A2 yields that $b \perp c \perp a$. Therefore, $b \parallel a$ by A6.

Transitivity: Let $a \parallel b$ and $b \parallel c$. By A5, there are $d, e \in X$ such that $a \perp d \perp b$ and $b \perp e \perp c$. Thus $a \perp d \perp b \perp e$ and so, by A3, $a \perp e$. Recalling that $e \perp c$, we have $a \perp e \perp c$ and A6 yields that $a \parallel c$. □

Theorem 6. *For all $a, b \in X$, $a \parallel b \implies a \not\perp b$.*

Proof: Let $a, b \in X$ satisfying $a \parallel b$ and $a \perp b$. Axiom A5 says that there exists $c \in X$ such that $a \perp c \perp b$. Since $b \perp a$ by A2, we have $b \perp a \perp c \perp b$. Now A3 implies that $b \perp b$ contradicting A1. □

The proofs of the next two theorems are similar to those of Theorems 5 and 6; they are suitably easy to be discussed in classroom. We also leave the verification of Theorem 9 to the reader. This result is complementary and analogous to Corollary 1.

Theorem 7. *For all $a, b, c \in X$,*

(i) $a \parallel b \wedge b \perp c \implies a \perp c$,

(ii) $a \perp b \wedge b \not\perp c \implies a \nparallel c$.

Theorem 8. *For all $a, b, c \in X$,*

(i) $a \nparallel b \wedge b \perp c \implies a \not\perp c$,

(ii) $a \parallel b \wedge b \not\perp c \implies a \not\perp c$.

Theorem 9. *Let $k \geq 2$. Then, for all $a_1, \dots, a_{k+1} \in X$,*

$$a_1 \perp a_2 \perp \cdots \perp a_k \perp a_{k+1} \implies a_1 \parallel a_k$$

if and only if k is even.

Given a planar perpendicularity, is there always a planar parallelism subordinate to it? The answer turns out to be positive. Moreover, such a parallelism is unique.

Theorem 10. *Let $\perp$ be planar perpendicularity. Then the relation*

$$a \parallel b \iff \{a\}^{\perp} = \{b\}^{\perp}, \tag{2}$$

and only it, is a planar parallelism subordinate to $\perp$.

Proof: First, we claim that (2) implies A5 and A6. To verify A5, assume that $a \parallel b$. Since $\{a\}^\perp = \{b\}^\perp \neq \emptyset$ by A4, there exists $c \in \{a\}^\perp = \{b\}^\perp$, i.e., $a \perp c \perp b$. To verify A6, assume that $a \perp b \perp c$. Then $b \in \{a\}^\perp$ and $b \in \{c\}^\perp$; consequently, $\{a\}^\perp \cap \{c\}^\perp \neq \emptyset$. If $\{a\}^\perp \neq \{c\}^\perp$, then $\{a, c\}^\perp = \emptyset$ by item (ii) of Corollary 2 which contradicts with $a \perp b \perp c$. Therefore, $\{a\}^\perp = \{c\}^\perp$, i.e., $a \parallel c$.

Second, we claim that A5 and A6 imply (2). If $a \parallel b$, then $\{a\}^\perp \cap \{b\}^\perp \neq \emptyset$ by A5. A similar reasoning as above gives $\{a\}^\perp = \{b\}^\perp$. Conversely, if $\{a\}^\perp = \{b\}^\perp$ ($\neq \emptyset$ by A4), then again a similar reasoning as above verifies that $a \parallel b$ by A6. □

5. Spatial perpendicularity and parallelism

Irreflexivity and symmetry are general and uniform properties of perpendicularity; shifting from plane to space does not cause any difficulties. However, we noticed already in the introduction that perpendicularity as a binary relation is not generally transitive in space. A solution to this problem might be that we focused in space only on such restricted models where, for example,

$$a \perp b \ \wedge \ b \perp c \implies a \perp c \ \vee \ a \parallel c \quad \text{or} \quad a \perp b \ \wedge \ b \perp c \ \wedge \ c \perp d \implies a \perp d \ \vee \ a \parallel d.$$

Another way to overcome the mentioned problem is that we consider, compatibly with the n-dimensional Euclidean space, perpendicularity as a relation of n mutual elements. So, in the rest of this section, we assume that $\perp$ is an n-ary relation in X. For the reader's convenience, we fix $n = 3$ but the generalization of the following approach is quite obvious.

Instead of A1 − A4, we assume now that the following properties hold. The right-hand side of the implication in B1 reads that a, b and c are all unequal.

(B1) $\forall a, b, c \in X\colon \perp(a, b, c) \implies a \neq b \neq c$.

(B2) $\forall a, b, c, x, y, z \in X\colon \perp(a, b, c) \implies \perp(x, y, z)$ if $\{x, y, z\} = \{a, b, c\}$.

(B3) $\forall a, b, c, d, e, f \in X\colon \perp(a, b, c) \wedge \perp(b, c, d) \wedge \perp(d, e, f) \implies \perp(a, e, f)$.

(B4) $\forall a \in X\colon \exists\, b, c \in X\colon \perp(a, b, c)$.

Clearly, the axioms B1, B2 and B4 correspond to A1, A2 and A4, respectively. The axiom B3 is a little more complicated to perceive but the original idea behind it becomes clear as one thinks the mentioned triples via the model of perpendicular lines in $\mathbb{R}^3$.

If $\perp$ satisfies B1 − B4, we call it *(three-dimensional) spatial perpendicularity.* We define perpendicularity between two individual elements by

$$a \perp b \iff \exists c \in X : \perp(a, b, c).$$

Let $A \subseteq X$ be nonempty. If we defined the perpendicular complement of A similarly as in Section 3 by setting

$$A^\perp = \{y \in X \mid y \perp A\},$$

then Theorem 3 would not remain valid. (For a counterexample, consider lines in the three-dimensional Euclidean space.) Instead of that, it is more reasonable to specify that $A^\perp$ is in $X \times X$. So, given $a, b, c \in X$, we define

$$(a, b) \perp c \iff c \perp (a, b) \iff \perp(a, b, c)$$

and

$$A^\perp = \{(y,z) \in X \times X \mid (y,z) \perp A\}.$$

Here $(y,z) \perp A$ means that $(y,z) \perp x$ for all $x \in A$. We also set $\emptyset^\perp = X \times X$. Now, by B2 and B3, we gain, for example, the following result.

$$a \perp (b,c) \perp d \perp (e,f) \implies a \perp (e,f).$$

In order to make the notation $A^{\perp\perp}$ sensible, we further define that, for $B \subset X \times X$, $B \neq \emptyset$,

$$B^\perp = \{x \in X \mid x \perp B\}.$$

Here $x \perp B$ means that $x \perp (y,z)$ for all $(y,z) \in B$. Whenever $\emptyset$ is regarded as a subset of $X \times X$, we define $\emptyset^\perp = X$.

Can we extend also the parallelism subordinate to perpendicularity from the planar case to higher dimensions? Well, assume first that $\perp$ satisfies B1 – B4. Then the natural replacements for A5 and A6 follow as we define *(three-dimensional) spatial parallelism subordinate to* $\perp$ as a relation $\parallel$ in X satisfying

(B5) $\forall a, b \in X\colon\ a \parallel b \implies \exists\, (c,d) \in X \times X\colon\ a \perp (c,d) \perp b,$

(B6) $\forall a, b, c, d \in X\colon\ a \perp (b,c) \perp d \implies a \parallel d.$

Now, one can find for every result of Sections 2 – 4 an analogy that holds in the context of spatial perpendicularity and parallelism. We encourage the reader to survey them.

6. Examples and exercises

Example 1

Let us consider $X = \{1,2\}$ and $X = \{1,2,3,4\}$ and write y = Yes and n = No. We define $\perp$ and $\parallel$ as follows.

$\perp$	1	2
1	n	y
2	y	n

$\parallel$	1	2
1	y	n
2	n	y

$\perp$	1	2	3	4
1	n	y	y	n
2	y	n	n	y
3	y	n	n	y
4	n	y	y	n

$\parallel$	1	2	3	4
1	y	n	n	y
2	n	y	y	n
3	n	y	y	n
4	y	n	n	y

It is easy to see in both cases that $\perp$ is a planar perpendicularity and $\parallel$ its subordinate parallelism. The two-element example represents the smallest possible structure with planar perpendicularity, while the other stands for the smallest structure with the following property: For all $a \in X$, there exists $b \in X \setminus \{a\}$ such that $a \parallel b$.
In the former structure, the nonempty perpendicular complements of nonempty proper subsets of X are

$$\{1\}^\perp = \{2\}, \quad \{2\}^\perp = \{1\}$$

and, in the latter,

$$\{1\}^\perp = \{4\}^\perp = \{1,4\}^\perp = \{2,3\}, \quad \{2\}^\perp = \{3\}^\perp = \{2,3\}^\perp = \{1,4\}.$$

Example 2

Let $X = \mathbb{R} \setminus \{0\}$. (The sets $X = \mathbb{Z} \setminus \{0\}$ and $X = \mathbb{Q} \setminus \{0\}$ work as well.) We define now

$$x \perp y \iff xy < 0 \quad \text{and} \quad x \parallel y \iff xy > 0.$$

It is again easy to see that $\perp$ is a planar perpendicularity and $\parallel$ its subordinate parallelism. However, for the reader's convenience, we prove A3 and A6.

Assume that $a \perp b \perp c \perp d$, i.e., $ab, bc, cd < 0$. If $a > 0$, then $b < 0$, $c > 0$ and $d < 0$. Similarly, $a < 0$ implies $d > 0$. In any case, $ad < 0$. In other words, $a \perp d$ and A3 holds.

To show A6, suppose now that $a \perp b \perp c$ or, equivalently, $ab, bc < 0$. If $a > 0$, then $b < 0$ and further $c > 0$. Similarly, if $a < 0$, then $c < 0$. Thus, $ac > 0$ meaning that $a \parallel c$.

The nonempty perpendicular complements of nonempty proper subsets of X are the following: $A^\perp = \mathbb{R}_-$ if $A \subseteq \mathbb{R}_+$, and $A^\perp = \mathbb{R}_+$ if $A \subseteq \mathbb{R}_-$. Here $\mathbb{R}_+$ and $\mathbb{R}_-$ denote the set of positive and negative real numbers, respectively.

Example 3

Let $X = \mathbb{F} \setminus \{-1, 0, 1\}$, where $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{R}$. We set

$$x \perp y \iff |xy| = 1 \quad \text{and} \quad x \parallel y \iff |x| = |y|.$$

Also in this example, it is easy to see that $\perp$ is a planar perpendicularity and $\parallel$ its subordinate parallelism. We verify here only A3.

Assume that $a \perp b \perp c \perp d$, i.e., $|ab| = |bc| = |cd| = 1$. Then $|b| = |a|^{-1}$, $|c| = |b|^{-1} = |a|$ and $|d| = |c|^{-1} = |a|^{-1}$. Consequently, $|ad| = 1$ showing that $a \perp d$ as A3 requires.

Let $\emptyset \neq A \subset X$. Then $A^\perp$ is nonempty if and only if A consists of one element or of two elements with same absolute value. If $A = \{a\}$ or $A = \{a, -a\}$, then $A^\perp = \{1/a, -1/a\}$.

Example 4

Let X be the set of all lines in the Euclidean plane. We define now $x \perp y$ if and only if the smallest angle between x and y measures $\pi/4$ and $x \parallel y$ if and only if x and y are perpendicular or parallel in the ordinary sense. Then $\perp$ is planar perpendicularity and $\parallel$ its subordinate parallelism. The validity of axioms A1, A2 and A4 – A6 is easy to see. To verify A3, it suffices to notice that the summing of three angles measuring $\pi/4$ between the perpendicular lines a, b, c and d results in the angle measuring $3\pi/4$, in which case the supplement of the sum measures $\pi/4$, or directly in the angle measuring $\pi/4$.

We complete this section by recording a few questions and topics for further discussions in classroom.

Exercises

1. Why is 0 excluded in Examples 2 – 3 and also ± 1 in Example 3 ?
2. How must we modify Example 3 if $\mathbb{F} = \mathbb{C}$?
3. Why is this question not sensible if we concern Example 2 ?
4. How many elements do we need a) for the smallest three-dimensional structure in $\mathbb{R}^3$, b) for the smallest $(X, \perp, \parallel)$ satisfying B1 – B6 ?
5. How do models satisfying either $a \perp b \wedge b \perp c \implies a \perp c \vee a \parallel c$ or $a \perp b \wedge b \perp c \wedge c \perp d \implies a \perp d \vee a \parallel d$ differ from each other?
6. Give another example of planar perpendicularity among the lines in the Euclidean plane which differs from the usual perpendicularity. How many such relations can one find ?

7. Conclusions and remarks

Is it a reasonable idea to teach axiomatic mathematics to teenagers? At least, in 1960's, many mathematicians believed so. For example, the well-known logician Patrick Suppes spoke eagerly for that. In [9], he demonstrates that it is possible to find interesting axiomatic approaches to algebra, calculus, geometry and logic which are suitable to be discussed already in high school. In our view, his arguments are valid even today.

Nevertheless, the fact remains that, in this millennium, the textbooks for secondary schools that consider any axiomatical systems are very rare. An example of such scarce books is [7] which introduces a quasiaxiomatical method to prove elementary theorems and to study ruler-and-compass constructions. The textbook is intended to be used both in mathematics teacher education and in an advanced course in upper secondary schools. Our own experience from teaching axiomatic thinking by this book is mainly positive.

Another encouraging point of view is that modern educational technology provides powerful tools for the elucidation of mathematical thought. For example, the software *Cabri* can be — and has already been — used to support axiomatic study of Euclidean geometry (see [2]).

In the previous sections, we have outlined a framework for studying perpendicularity strictly deductively but which, at the same time, allows the reader to rest on the concrete ideas of the most natural models, the Euclidean plane and space. The axiomatic approach almost unavoidably builds on the symbolic language, which always challenges students, but a motivated teacher can surely find a reasonable way to introduce the issue already in the upper secondary school or in mathematics teacher education. For example, acknowledging only the definition of perpendicular complements and skipping the rest of Section 3, our approach is simple enough to be adopted within a short time but, simultaneously, general enough to provide some nontrivial and unexpected results and possibilities for discoveries by oneself, too.

Despite concerning perpendicularity only from a quite restricted point of view, this paper may help students and other readers to discern in what way it is a more complicated concept than, e.g., equivalence or order. In applying perpendicularity, the axiom A3 (or B3 or any other similar property) is definitely needed. However, this axiom is meaningful only in certain contexts. The same can be said about A4 – A6 (or B4 – B6); only A1 and A2 are invariant properties of perpendicularity.

Second, our definition of parallelism is based on perpendicularity. An alternative starting point is to define parallelism first and then try to reduce perpendicularity to it. But how do we axiomatize parallelism? As an equivalence relation? If so, how do we then separate parallelism from other equivalences? This is difficult or it may be even impossible since parallelism does not seem to have any other general properties. Consequently, also the concept of parallelism depends essentially on the context. After all, this is not surprising, parallelism and perpendicularity are closely related and, in a sense, complementary concepts.

Third, it may be an eye-opening experience for students to see in Example 4 that the perpendicularity of lines is not absolutely tied to the certain measure of the angle between them.

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