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# Light Ray Trajectories and Projective Correspondences

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**Abstract.** This paper considers trajectories of light rays from the position of projective geometry. On the basis of the well-known Law of Reflection the trajectory of a light ray between two intersecting planes $\alpha$ and $\alpha_1$ is examined. The process of construction of reflected rays from given planes leads to a certain 3D construction $\Lambda$. In this construction we receive a spatial broken line made up of repeatedly reflected rays, the vertices of which generate corresponding fields of points in the planes $\alpha$ and $\alpha_1$. As a result of these constructions we have additional fields of points on two auxiliary planes x and y. It demonstrates that in certain constructions that are particular cases of projective models of a construction $\Lambda$ the fields of points mentioned above are projective. We examine the construction of trajectory of a light ray consisting of four segments in a diamond. The process of light reflection creates a collineation between two fields of points on a diamond facet. Double points of this collineation indicate the presence of closed light contours inside the diamond.
A computer program, developed by the authors, enables users to perform the following operations: change the form of a diamond, select a facet of a diamond, select a point (on a facet) of an incident light ray, change the orientation of a light ray and observe the trajectory of a light ray inside a diamond. The program also computes the intensity of an exiting light stream. This criterion enables one to compare various forms of diamonds and search for the best among them.

*Key Words:* reflected and refracted rays, collineation, double points, graphical interface

*MSC 2010:* 51N05

## 1. Introduction

This paper is a continuation of the previous papers [5], [6] and is devoted to modeling the trajectory of a light ray in a 3D crystal, particularly in a diamond. Some of the works in this

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field are listed in the references, for example, [14, 8, 2, 10, 9, 4]. The authors have elaborated algorithms and programs to calculate the intensity of the light stream exiting the upper parts *(crown)* and the bottom parts *(pavilion)* of a diamond. On the basis of this program the user can choose the most preferable form of a diamond. The criterion for this selection is the intensity of the reflected light ray. This program is based on the well-known laws of optics and grapho-analytical methods. The authors widely use methods of projective geometry. This enables the user to find interesting facts connected to the geometry of reflected light rays.

## 2. Geometry of the reflected ray

Let's recall the well-known *Law of Reflection:*

1. The incident ray $l_1$ and the reflected ray $l_2$ lie in a plane $T$ which includes also the normal $N$ to the reflecting surface $\Sigma$.
2. The ray of light $l_1$ is reflected by the surface $\Sigma$ at the same angle $\Theta_1$ (in absolute value).

The angle $\Theta_1$ is measured between the ray and the normal $N$ (Fig. 1). We are speaking now only about the geometry of the light ray, just as a straight line. Further in this article planes will be regarded as transparent and all lines of construction are visible. We would like to stress that the sketches (Figs. 1–6) are used only to explain the geometric constructions and to prove a few theorems.

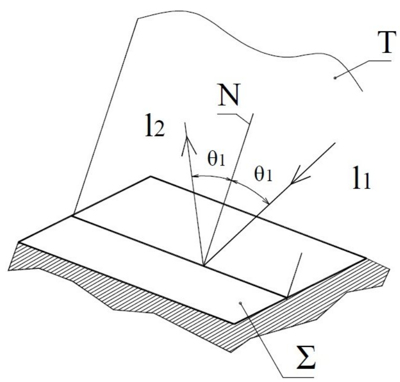


Figure 1: Law of Reflection

Let's construct the trajectory of a light ray between two intersecting planes $\alpha$ and $\alpha_1$ on the basis of the above-mentioned Law of Reflection. Let's remark that a ray trajectory between two intersecting straight lines and inside a plane contour was examined in [5].

Let $LA$ be an arbitrary ray incident in the plane $\alpha$ at the point $A$ (Fig. 2). The point $L$ belongs to the plane $\alpha_1$. This ray will be successively reflected in the planes $\alpha$ and $\alpha_1$ and will form some spatial broken line. Let's construct it:

In the beginning we construct the point x$L$, symmetric to $L$ with respect to plane $\alpha$. Naturally, the points $L$ and x$L$ lie on the normal $n_1$ to the plane $\alpha$. The straight line x$L\,A$ will intersect the plane $\alpha_1$ at the point $A_1$. Then $AA_1$ is the ray after reflection in the plane $\alpha$. After that we construct the point y$A$, symmetric to $A$ with respect to plane $\alpha_1$. The points $A$ and y$A$ lie on the normal $n_2$ to the plane $\alpha_1$. The straight line y$A\,A_1$ will intersect the plane $\alpha$ at the point $B$. So, $A_1B$ is the ray after reflection in the plane $\alpha_1$.

The further constructions are carried out similarly. All the auxiliary points x$L$, x$A_1$, … and y$A$, y$B$, … of this construction lie correspondingly in the planes x and y, as a result of the symmetries with respect to planes $\alpha$ and $\alpha_1$. This results in a construction, which we will call $\Lambda$, and in a spatial broken line made up of reflected rays. In this construction the straight lines $A\,\mathrm{y}A$, $B\,\mathrm{y}B$, ... are parallel; they are the rays of a ray pencil having the improper center $S_2^\infty$. Likewise, the straight lines $L\,\mathrm{x}L$, $A_1\,\mathrm{x}A_1$, ... are the rays of the pencil with the improper center $S_1^\infty$. We can see that the construction $\Lambda$ is a particular case of the more generalized construction $\Omega$, when the centers $S_1^\infty$ and $S_2^\infty$ are replaced by regular points $S_1$ and $S_2$ in the Euclidean space (Fig. 3).

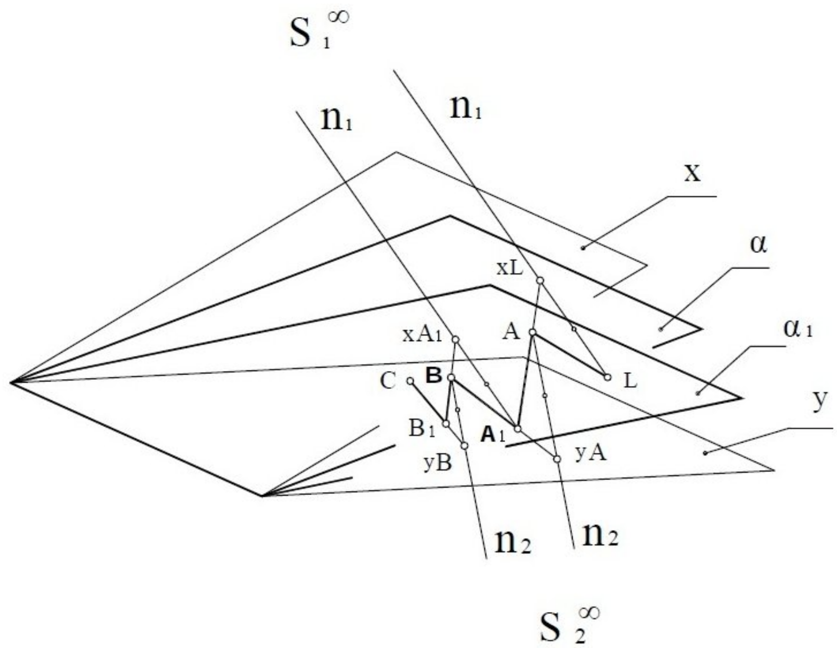


Figure 2: The spatial broken light line between the two planes $\alpha$ and $\alpha_1$

## 3. "The light ray trajectory" in the construction $\Omega$ with two centers $S_1$ and $S_2$

Below we will explain why the expression "the light ray trajectory" is enclosed in quotation marks.

Let's show that the construction based on four planes $\alpha$, $\alpha_1$, x, y, intersecting at one line, and on two centers $S_1$ and $S_2$ determines "the trajectory of the light ray" with the vertices in the planes $\alpha$ and $\alpha_1$ (Fig. 3).

Let $LA$ be a "ray" exiting from some point $L$, which belongs to the plane $\alpha_1$, which meets the plane $\alpha$ at $A$. In the beginning we construct the point x$L$ as the intersection of the straight line $S_1L$ and the plane x. This construction can be written in the symbolic form:

1. $\mathrm{x}L = S_1\,L \bigcap \mathrm{x}$. The further constructions is as follows:
2. $A_1 = \mathrm{x}L\,A \bigcap \alpha_1$.
3. $\mathrm{y}A = S_2\,A \bigcap \mathrm{y}$.
4. $B = \mathrm{y}A\,A_1 \bigcap \alpha$.
5. $\mathrm{x}A_1 = S_1\,A_1 \bigcap \mathrm{x}$.
6. $B_1 = \mathrm{x}A_1\,B \bigcap \alpha_1$.

The points $(C, C_1)$, $(D, D_1)$ and so on are constructed similarly. As a result, we receive a spatial broken line the vertices of which make corresponding fields of points $A, B, C, D, \dots$ and $A_1, B_1, C_1, D_1, \dots$ in the planes $\alpha$ and $\alpha_1$, respectively.

What can be said about these fields? Let's look at this problem from the standpoint of projective geometry.

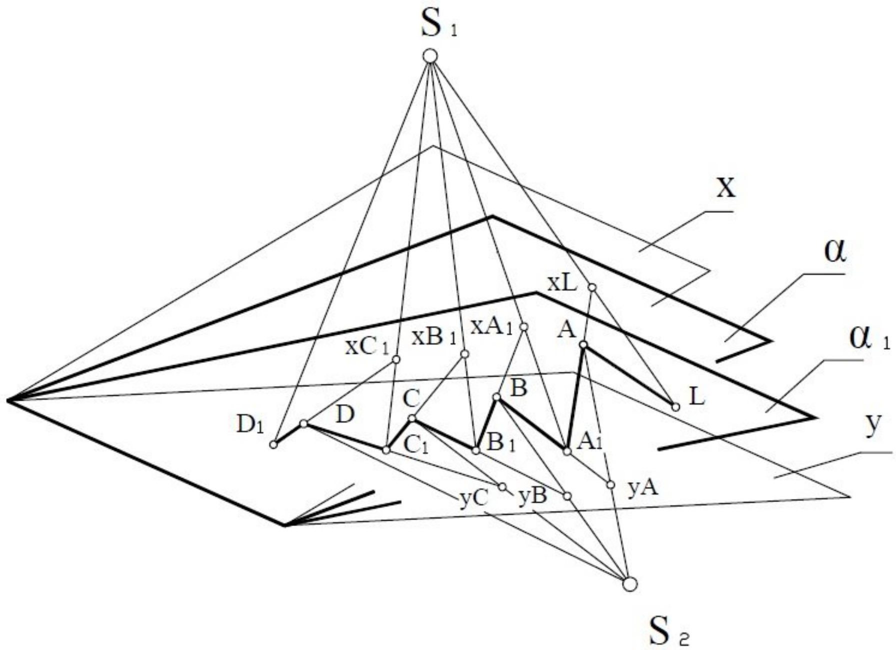


Figure 3: The "spatial broken light line" in the construction $\Omega$ with two centers, $S_1$ and $S_2$

The construction $\Omega$ will be perceived as a projective model of the construction $\Lambda$. In this construction the points x$L$, x$A_1$, ... will not be symmetrical to the respective points $L$, $A_1$ relative to the plane $\alpha$. This is explained by the fact that in projective models the metric is not preserved, for example, the equality of segments and angles. Thus, the notion of "reflected rays" loses its physical meaning, but instead, acquires the geometric one as an element of projective model. For this reason, furtheron the expressions "reflected ray", "light ray trajectory" or "spatial broken light line" will be enclosed in quotation marks.

In Figs. 4 and 5 we examine a construction of a "broken light line" in a different design $\Omega_1$, where the straight lines $A$x$A$, $A_2$x$A_2$, etc. (Fig. 5) form a pencil of lines with the center $O$. These straight lines are analogous to the straight lines $A$x$L$, $B$x$A_1$, etc. shown in Fig. 3.

In Section 5 below, we will describe in detail the construction of the "broken light line" in the design $\Omega_1$, which is, basically, a special case of $\Omega$. Henceforth, we will not refer to the construction $\Omega$ and we will introduce different designations for certain points in order to ease reading of the designs. As we will see below, the design $\Omega_1$ induces a formulation of interesting and important problems, associated with the light ray trajectories in a real crystal. In connection with the latter, let us first examine this design.

## 4. The collinear fields in the construction $\Omega_1$ with the three centers $S_1$, $S_2$ and O

Let four planes $\alpha$, $\alpha_1$, x, y intersecting in the straight line $v$ and three centers $S_1$, $S_2$, $O$ be given (Fig. 4). The centers $S_1$, $S_2$ and $O$ are arbitrary points in space, not belonging to the

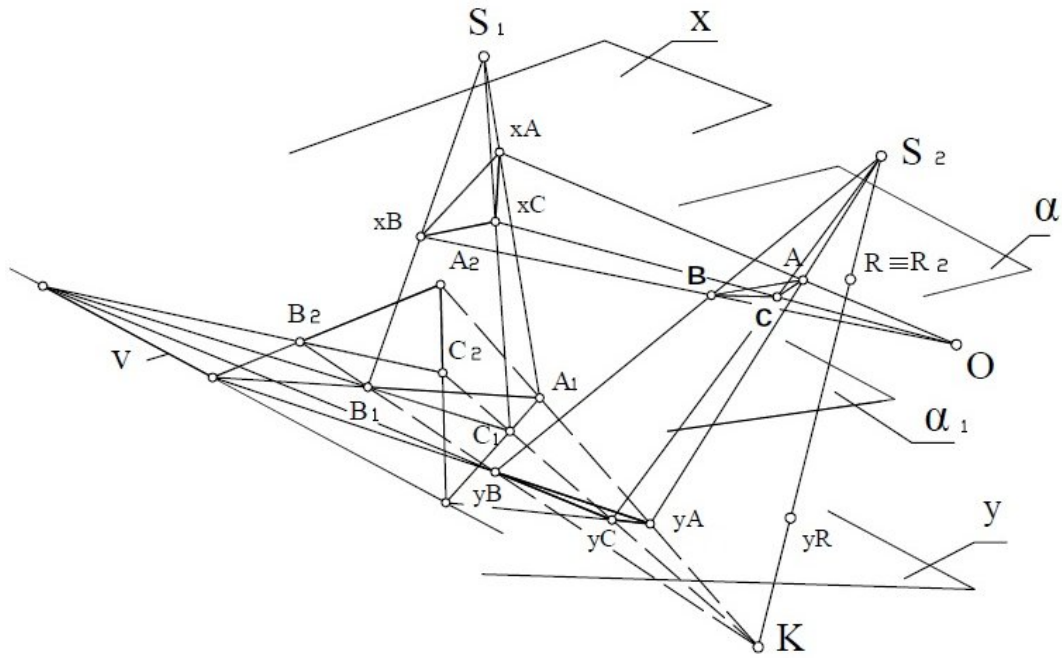


Figure 4: The collinear fields in construction $\Omega_1$ with three centers $S_1$, $S_2$ and $O$

mentioned planes. Let's call the following construction $\Omega_1$.

We will construct corresponding point fields in the planes $\alpha$ and $\alpha_1$ in a way, which somehow differs from the procedure explained before.

We assume that in the plane $\alpha$ there is a set of points $A, B, C, \ldots$ (for the sake of convenience when reading the sketch, the points $A, B, C$ are connected by straight lines). Let's select one point, for example $A$, and make the following constructions: through center $O$ and point $A$ we draw a straight line intersecting the plane x at point x$A$. In symbolic form we can write:

1. $\mathrm{x}A = \ O\,A \ \bigcap \ \mathrm{x}$. The further constructions will be written as follows:
2. $A_1 \ = S_1\,\mathrm{x}A \bigcap \ \alpha_1$, where the straight line $S_1\,\mathrm{x}A$ passes through the points $S_1$ and x$A$.
3. $\mathrm{y}A = \ \ S_2\,A \ \bigcap \ \mathrm{y}$.
4. $A_2 \ = \mathrm{y}A\,A_1 \bigcap \ \alpha$, where the straight line $\mathrm{y}A\,A_1$ passes through the points y$A$ and $A_1$.

Thus, for each point $A \in \alpha$ we have received the corresponding points $A_1 \in \alpha_1$ and $A_2 \in \alpha$. Identical constructions will be carried out for the other points $B, C, \ldots$ in $\alpha$ which gives the respectively corresponding points $B_1, B_2$, $C_1, C_2$, and so on.

If we connect the points $A, A_1, A_2$ by segments, then $AA_1$ will be the incident light ray and $A_1A_2$ will be the reflected one. (These segments are not shown in Fig. 4, because we didn't want to complicate the drawing). A detailed construction of light rays will be shown in Fig. 5, as we have already mentioned above.

We will prove below that the considered constructions lead to important theorems of projective geometry which enable us to look at a trajectory of the light ray in a new fashion.

It is well known from classical projective geometry that a sequence (chain) of perspective collineations sets up a projective correspondence between point fields of the preceding link and the subsequent link (for example, the first link and the last link). In other words, the product of collineations is again a collineation (see, e.g., [1, 3, 7, 13, 15, 12]).

Theorems 1 and 2, listed below, are a direct consequence of such a product. However, we give proofs of these theorems for two reasons. The first reason is that the theorems apply to the specific design $\Omega_1$; the second reason is that it is necessary to show, that such a sequence (chain) of perspective collineations exists in $\Omega_1$.

**Theorem 1.** *The fields of points $A, B, C, \ldots$ and $A_2, B_2, C_2, \ldots$, both located in the plane $\alpha$, are collinear.*

*Proof:* Consider the following sequences of the perspective correspondences with centers $O$ and $S_1$ (Figs. 4 and 5):

$$\alpha(A, B, C) \overline{\wedge}\, \mathrm{x}\,(\mathrm{x}A,\ \mathrm{x}B,\ \mathrm{x}C, \ldots) \overline{\wedge}\, \alpha_1(A_1, B_1, C_1) \tag{1}$$

and (2) with center $S_2$:

$$\alpha(A, B, C, \ldots) \overline{\wedge}\, \mathrm{y}\,(\mathrm{y}A,\ \mathrm{y}B,\ \mathrm{y}C, \ldots) \tag{2}$$

From (1) and (2) it follows:

$$\alpha_1(A_1, B_1, C_1) \wedge\ \mathrm{y}\,(\mathrm{y}A,\ \mathrm{y}B,\ \mathrm{y}C, \ldots) \tag{3}$$

In accordance with Fig. 4 the common points of the mutually collinear fields $\alpha_1(A_1, B_1, C_1, \ldots)$ and $\mathrm{y}\,(\mathrm{y}A,\ \mathrm{y}B,\ \mathrm{y}C, \ldots)$ in the planes $\alpha_1$ and y lie on the line $v$ or intersection. In the collineation (3) the common points correspond to themselves. In this case we obtain a *perspective* collineation. The latter is verified by the fact that the corresponding triangles $A_1, B_1, C_1$ and $\mathrm{y}A, \mathrm{y}B, \mathrm{y}C$ satisfy Desargues' theorem in the following sense: corresponding sides of triangles $A_1, B_1, C_1$ and $\mathrm{y}A, \mathrm{y}B, \mathrm{y}C$ intersect at three points on the same straight line $v$. Consequently, the straight lines joining the corresponding vertices $(A_1,\ \mathrm{y}A)$, $(B_1,\ \mathrm{y}B)$, $(C_1,\ \mathrm{y}C)$ pass through the same point $K$. These lines are displayed dotted in (Fig. 4).

Consequently, the fields $\alpha(A_2, B_2, C_2, \ldots)$, $\alpha_1(A_1, B_1, C_1, \ldots)$ and $\mathrm{y}\,(\mathrm{y}A,\ \mathrm{y}B,\ \mathrm{y}C, \ldots)$ are sections of the same bundle of straight lines with center $K$.

$$\alpha(A_2, B_2, C_2, \ldots) \overline{\wedge}\, \alpha_1(A_1, B_1, C_1, \ldots) \overline{\wedge}\, \mathrm{y}\,(\mathrm{y}A,\ \mathrm{y}B,\ \mathrm{y}C, \ldots) \tag{4}$$

From (2) and (4) we conclude:

$$\alpha(A, B, C, \ldots) \wedge\ \alpha(A_2, B_2, C_2, \ldots) \tag{5}$$

Thus Theorem 1 is proved. □

We have shown, that in the given construction $\Omega_1$ the point field $\alpha(A_2, B_2, C_2, \ldots)$ is perspective to the point field $\mathrm{y}\,(\mathrm{y}A,\ \mathrm{y}B,\ \mathrm{y}C, \ldots)$ with the center $K$. On the other hand the point field $\mathrm{y}\,(\mathrm{y}A,\ \mathrm{y}B,\ \mathrm{y}C, \ldots)$ is perspective to the point field $\alpha(A, B, C, \ldots)$ with the center $S_2$ in accordance with the construction. The product of these two perspective collineations is the collineation (5).

It is known that a collineation, in the general case, has no more than three double points $P$, $Q$ and $R$ and no more than three double lines which are sides of the triangle $PQR$. Their construction is described in detail in the literature, for example, in [1]. However, in the construction $\Omega_1$ we are facing a different case.

Let's imagine that we have a point of the field $\alpha(A, B, C, \ldots)$ on the straight line $v$. Then, its corresponding point in the field $\alpha(A_2, B_2, C_2, \ldots)$, according to the construction, coincides with the first one. This means that the whole straight line $v$ consists of double points of the mentioned fields. In this case the collineation (5) is a homology with double line $v$ and homology center $R$. All the straight lines connecting pairs of corresponding points $(A, A_2)$, $(B, B_2)$, $(C, C_2)$, $\dots$ pass through this center $R$. In Fig. 4 these straight lines are not shown to simplify the sketch.

For the fields $\alpha(A, B, C, \dots)$ and $\alpha(A_2, B_2, C_2, \dots)$ also Desargues' theorem holds true. It's not difficult to see from the construction of the "reflected ray" that the homology center $R$ lies on the straight line $S_2K$ and that $R$ as a double point coincides with its image $R_2$ under the collineation (5).

Let's see the points $\alpha(A_2, B_2, C_2, \dots)$ as the points of the first field $\alpha(A, B, C, \dots)$ and perform once more the construction presented above. We shall receive a field $\alpha(A_4, B_4, C_4, \dots)$ projective to the field $\alpha(A_2, B_2, C_2, \dots)$. From point $A_2$ we receive the points $A_3$ and $A_4$ in the respective planes $\alpha_1$ and $\alpha$, and so on.

Shortly, in the plane $\alpha$ we receive a sequence of the fields $(A, B, C, \dots)$, $(A_2, B_2, C_2, \dots)$, $(A_4, B_4, C_4, \dots)$, in which each is projective to the following one. In the plane $\alpha_1$ we have the fields $(A_1, B_1, C_1, \dots)$, $(A_3, B_3, C_3, \dots)$, $(A_5, B_5, C_5, \dots$. If we consider all the points $A, B, C, \dots,\ A_2, B_2, C_2, \dots,\ A_4, B_4, C_4, \dots$ as points of the first field in the plane $\alpha$, then the second field will include the points $A_2, B_2, C_2, \dots,\ A_4, B_4, C_4, \dots,\ A_6, B_6, C_6, \dots$ in the same plane.

These two fields will be in projective correspondence (homology) (5) with the center $R$. The field of points $\alpha(A_2, B_2, C_2, \dots,\ A_4, B_4, C_4, \dots,\ A_6, B_6, C_6, \dots)$ will also be in perspective collineation with the field of points $\alpha_1(A_1, B_1, C_1, \dots,\ A_3, B_3, C_3, \dots,\ A_5, B_5, C_5, \dots$ with the center $K$, accordingly to the construction $\Omega_1$. If these points are connected by segments in a certain order, we shall have a "broken light line" consisting of "incident" and "reflected" rays. We shall discuss this in the next section.

## 5. The "broken light line" in the construction $\Omega_1$

We refer to the construction $\Omega_1$ again, to explain the construction of the "broken light line" (Fig. 5): Let's assume that the ray $AA_1$ exiting from point $A$ meets the plane $\alpha_1$. In accordance with our construction (see Fig. 4), after reflection in the plane $\alpha_1$ the ray $A_1A_2$ will become the "reflected" one. This was previously mentioned in Section 4. The points $A, A_1$, $A_2$ can be seen in Fig. 4.

Now, we replace point $A$ by point $A_2$ and perform the constructions presented in Figs. 4 and 5.

$$
\begin{aligned}
\mathrm{x}A_2 &= \ O\,A_2 \ \bigcap \ \mathrm{x} \\
A_3 &= S_1\,\mathrm{x}A_2 \bigcap \ \alpha_1 \\
\mathrm{y}A_2 &= \ S_2\,A_2 \ \bigcap \ \mathrm{y} \\
A_4 &= \mathrm{y}A_2\,A_3 \bigcap \ \alpha
\end{aligned}
$$

Now we replace point $A_2$ by point $A_4$ and perform the constructions presented above, and so on. As a result, a broken line $A, A_1, A_2, A_3, A_4, \dots$ (sequence of "incident" and "reflected" rays) is constructed. What can be said about this line?

Earlier, we showed that the points $A$, $A_2$, $A_4$, $\dots$ corresponding in the homology (5) lie on the same line passing through the homology center $R$. According to constructions (Figs. 4 and 5), the points $A_1$, $A_3$, $A_5$, $\dots$ also lie on a common line. Hence, the entire broken line $A, A_1, A_2, A_3, A_4, \dots$ will lie in some plane $\delta$. This plane is spanned by the point $A$ and the straight line $S_2K$. Below, we will show that the vertices of this broken line, taken in a certain order, form projective rows on the straight line $WR$ (Fig. 5).

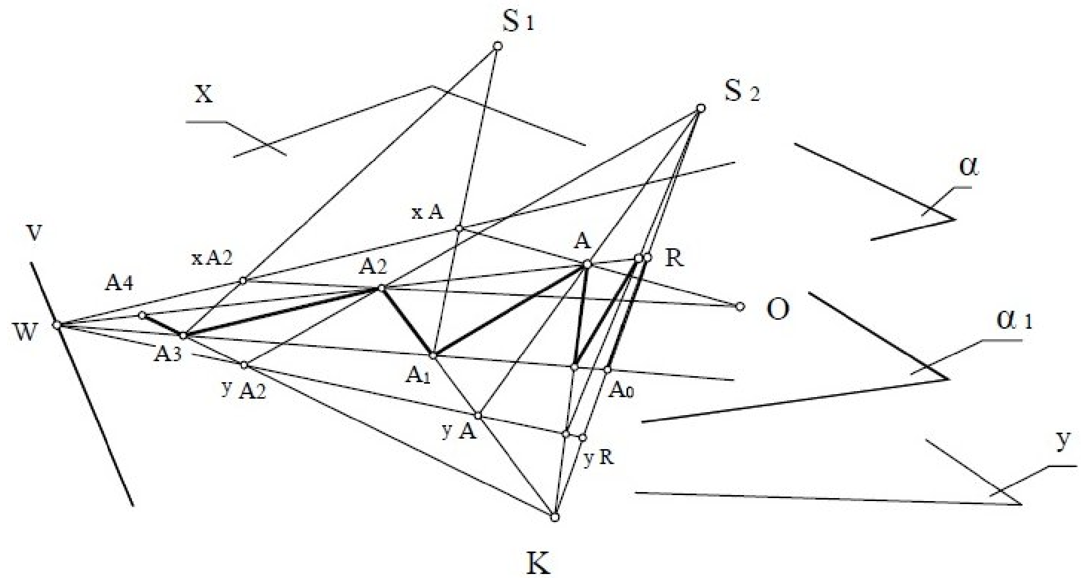


Figure 5: Sequence of "incident and reflected rays" in the construction $\Omega_1$

**Theorem 2.** *In the construction $\Omega_1$ the vertices of "the light broken line" are in projective correspondence in which every point of the first row $WR\,(A, A_2, A_4, \dots)$ has a corresponding successive point in the second row $WR\,(A_2, A_4, A_6, \dots)$ on the same straight line $WR$.*

The proof of this theorem is also evident, since it stems from the product of two perspective collineations already mentioned in Section 4.

*Proof:* Let's consider the row of points $A$, $A_2$, $A_4$, … on the straight line $WR$. These points generate the pencil of lines with center $S_2$. The rays of this pencil intersect the plane y at points y$A$, y$A_2$, y$A_4$, …In accordance with the constructions in (Fig. 5) the following sequences are true:

$$WR\,(A, A_2, A_4, \dots) \overline{\overline{\wedge}}\, \mathrm{y}\,(\mathrm{y}A, \mathrm{y}A_2, \mathrm{y}A_4, \dots) \tag{6}$$

with center $S_2$, and

$$\mathrm{y}\,(\mathrm{y}A, \mathrm{y}A_2, \mathrm{y}A_4, \dots) \overline{\overline{\wedge}}\; WR\,(A_2, A_4, A_6, \dots) \tag{7}$$

with center $K$. The common point of these rows is the point $W$. In the projective correspondence (7) this common point corresponds to itself. Hence, these rows are perspective. In Fig. 5 we can see that the rays $A_2\,\mathrm{y}A$, $(A_4\,\mathrm{y}A_2)$, … form a pencil of lines with center $K$.

From (6) and (7) we conclude

$$WR\,(A, A_2, A_4, \dots) \overline{\wedge}\;\; WR\,(A_2, A_4, A_6, \dots) \tag{8}$$

In other words, two rows of points $A, A_2, A_4, \dots$ and $A_2, A_4, A_6, \dots$ are projective on the straight line $WR$. Thus Theorem 2 is proved. □

We have shown that on line $WR$ there is a projective correspondence (8), which has two double points. The first double point of this correspondence is $W$, which is the intersection of the plane $S_2KA$ with the straight line $v$ (axis of homology). The second double point is $R$, which belongs to straight line $S_2K$.

This double point is, in some "physical sense", a special, very interesting point of the "broken light line". Suppose that a ray exits from such a double point $R$. Having been reflected in the plane $\alpha_1$ at the point $A_0$, it returns to the same point $R$. In other words, having reached this point, the light ray will travel infinitely along the same segment $RA_0$. In Fig. 5 we can see how the broken line tends to the segment $RA_0$ on the right. Similar to the double point, the segment $RA_0$ can be called a *double segment*. This segment forms an immovable, as if frozen, element of the "broken light line".

Now, one more question arises: can there exist "a spatial closed broken light line" induced by double points?

In order to answer this question, let's consider a crystal in the form of a pyramid $ABCD$ (Fig. 6). We will consider one of the possible variants of reflection of rays from facets inside this crystal. For convenience, let's denote facets of the pyramid as follows: $ABC$ as $\alpha$, $ACD$ as $\beta$, $BCD$ as $\gamma$.

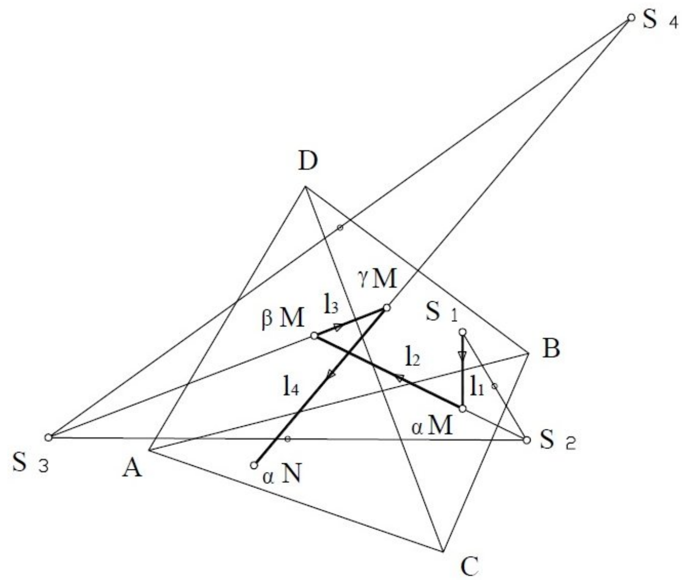


Figure 6: Collineation on facet $ABC$, generated by reflected rays

All these constructions generate the following sequence of perspective correspondences:

$$\begin{aligned}&\alpha\,(\alpha M_1, \alpha M_2, \alpha M_3, \dots)\,\overline{\barwedge}\,\beta\,(\beta M_1, \beta M_2, \beta M_3, \dots)\,\overline{\barwedge}\,\gamma\,(\gamma M_1, \gamma M_2, \gamma M_3, \dots)\\ &\overline{\barwedge}\,\alpha\,(\alpha N_1, \alpha N_2, \alpha N_3, \dots)\end{aligned} \tag{9}$$

As a result, we receive on the facet $\alpha$ the following collineation between two planar fields:

$$\alpha\,(\alpha M_1, \alpha M_2, \alpha M_3, \dots)\barwedge\,\alpha\,(\alpha N_1, \alpha N_2, \alpha N_3, \dots) \tag{10}$$

In Section 4 we have already mentioned that a collineation, in a general case, has no more than three double points. The double points of the collineation (10) in the plane of crystal's facet $\alpha$ can be either inside or outside the facet. Let's imagine that at least one of these points is located inside the facet $\alpha$.

If a certain ray $l_1$ from the bundle $S_1$ passes through this double point, it will return after reflection in the crystal's facet to the same point as ray $l_4$. This fact tells us that the double points of the collineation (10), under certain conditions, can induce the formation of *closed light contours* inside the crystal. It is possible to formulate the problem in a different way:

One white light ray enters at point $S_1$. After refraction at point $S_1$ we receive a bundle of rays $l_1$ of different colors inside the crystal (this is a phenomenon of dispersion). From this bundle, a ray of a certain color which will pass through the double point of the collineation in $\alpha$ will return to the same point.

## 6. Computer realization of the light ray trajectory

The authors have created a program for constructing the trajectory of a light ray inside the diamond. The plane section of a diamond (profile) is defined by the coordinates of its vertices: 1, 2, 3, 4, 5, and 6 (Figs. 7 and 8). The shape of the diamond is created by rotating this profile around the vertical axis going through the points 3 and 6. The program is managed by means of a graphical interface (Fig. 8).

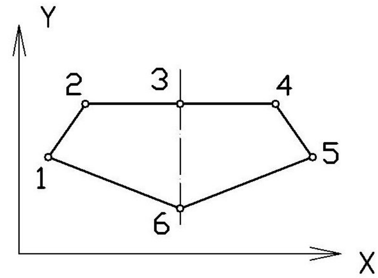


Figure 7: The plane section of a diamond (profile)

On this interface there are the following designations: $\overline{r}$ is the incident and $\overline{r}_1$ the reflected ray at the point $M$ of the upper facet with number 3. $\overline{r}_2$ is the refracted ray, $\overline{r}_3$, $\overline{r}_4$, $\overline{r}_5$ are rays reflected from the facets whose borders are distinguished by means of thicker lines by the program itself. $n1, n2, n3, n4$ are the normals to the planes spanned by the facets.

The light ray $\overline{r}$ can be incident at every chosen upper facet at a point whose position is defined by two parameters: $t$ and $q$. The parameter $t$ defines the displacement of the point $M$ along one side of the triangle and $q$ along the other one. The direction of the incident light ray $\overline{r}$ is given by two parameters: *'slope'* and angle of rotation, which in the graphical interface (Figs. 8, 9 and 10) is called *'rotate'* for brevity.

*'Mode'* is a parameter indicating the choice of facets automatically or in accordance with the user's requirement. *'Scale (A)'* is the scale of the drawing. *'TOP LPW'* is the intensity of the rays exiting from above. The graphical interface allows to search for light rays which generate a projective correspondence in the plane of the chosen facet. The double point of this correspondence, if it exists inside the facet, defines a closed light contour. If the angles of incident light rays are less than critical, the program also draws the rays exiting the crystal and calculates their intensity.

The percentage intensity of reflected light rays is calculated in accordance to the equation (11) of Fresnel:

$$R = \frac{1}{2}\left[\frac{\tan^2(\Theta_i - \Theta_r)}{\tan^2(\Theta_i + \Theta_r)} + \frac{\sin^2(\Theta_i - \Theta_r)}{\sin^2(\Theta_i + \Theta_r)}\right] \tag{11}$$

where $\Theta_i$ is the angle of the incident light ray and $\Theta_r$ the angle of the refracted light ray.

The percentage intensity of the refracted light rays is calculated as a difference of the intensities of the incident and the reflected light rays.

Two examples of diamond shapes with light ray trajectories are shown in Figs. 9 and 10. One light ray meets a triangular facet of the top part of the crown. In this example we show only one incident ray in order to make this figure less complicated. In Fig. 9 we see the shape

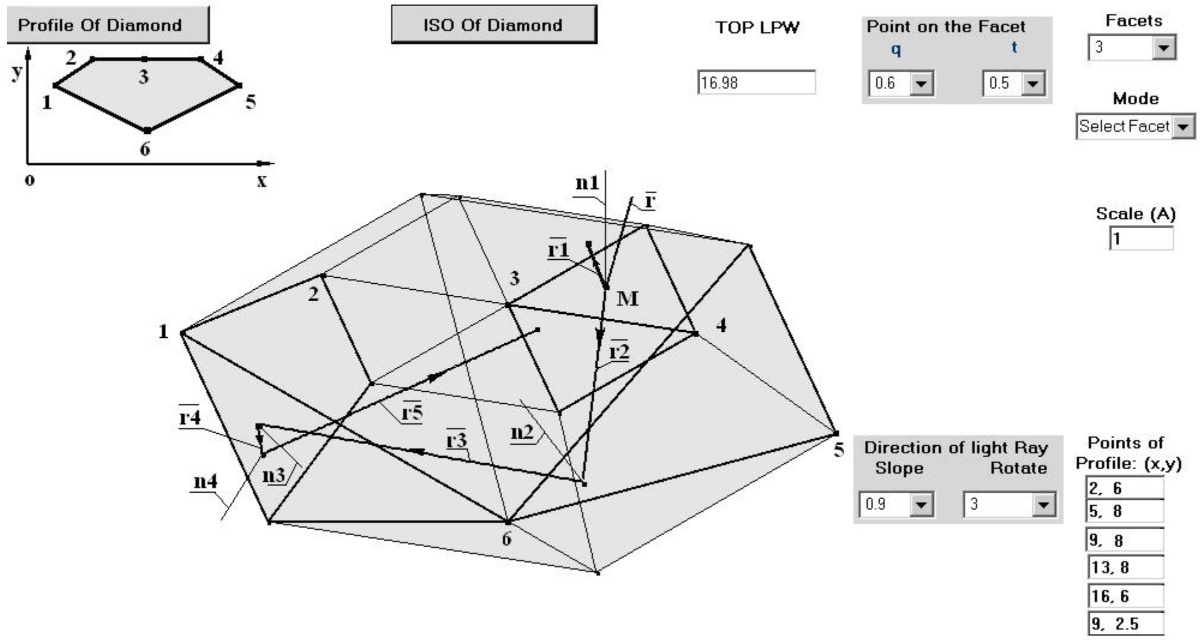


Figure 8: Graphical interface

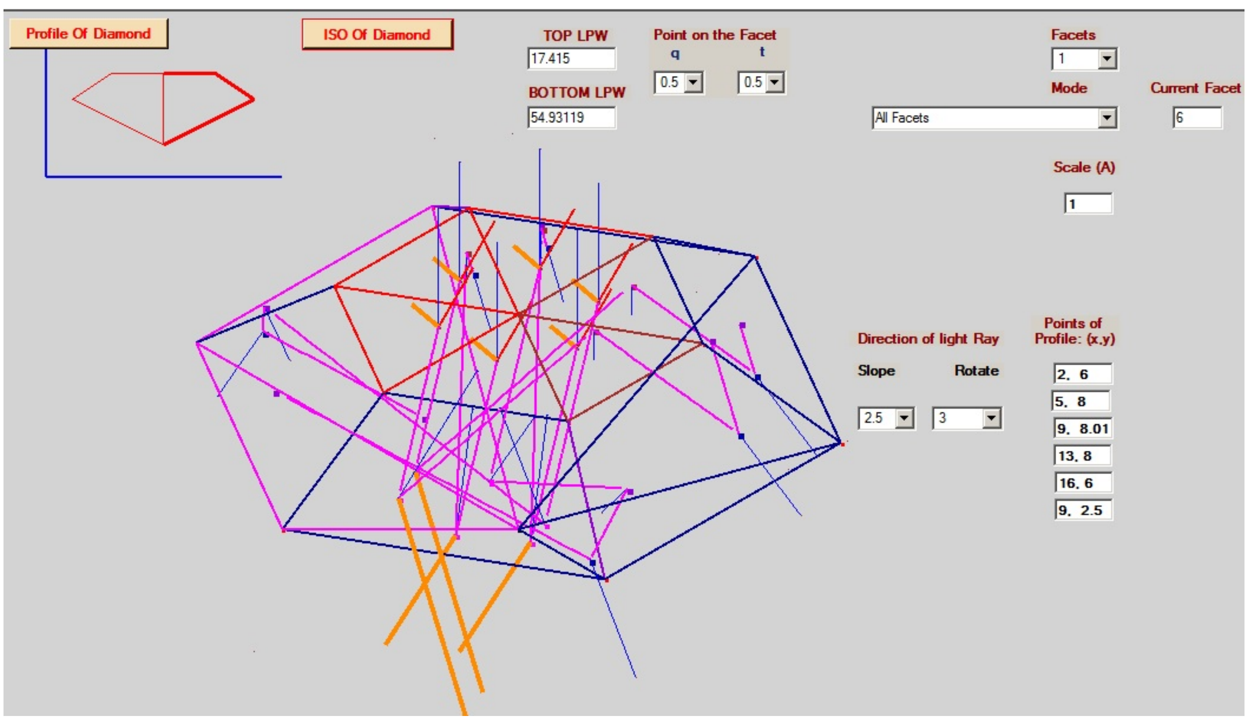


Figure 9: Diamond according to the proportion of M. Tolkowsky

of the diamond according to M. Tolkowsky's proportions [11, 14]. For this shape the top light power is 17.415 and the bottom light power (the intensity of the rays exiting from below) is 54.93119 . In Fig. 10 we have another profile and shape; the coordinates of the vertices 3 and 6 have been changed. For this case the top light power is 99.81763 and the bottom light power is 0. The rays exiting the crown and the pavilion (bottom part) are shown in orange color.

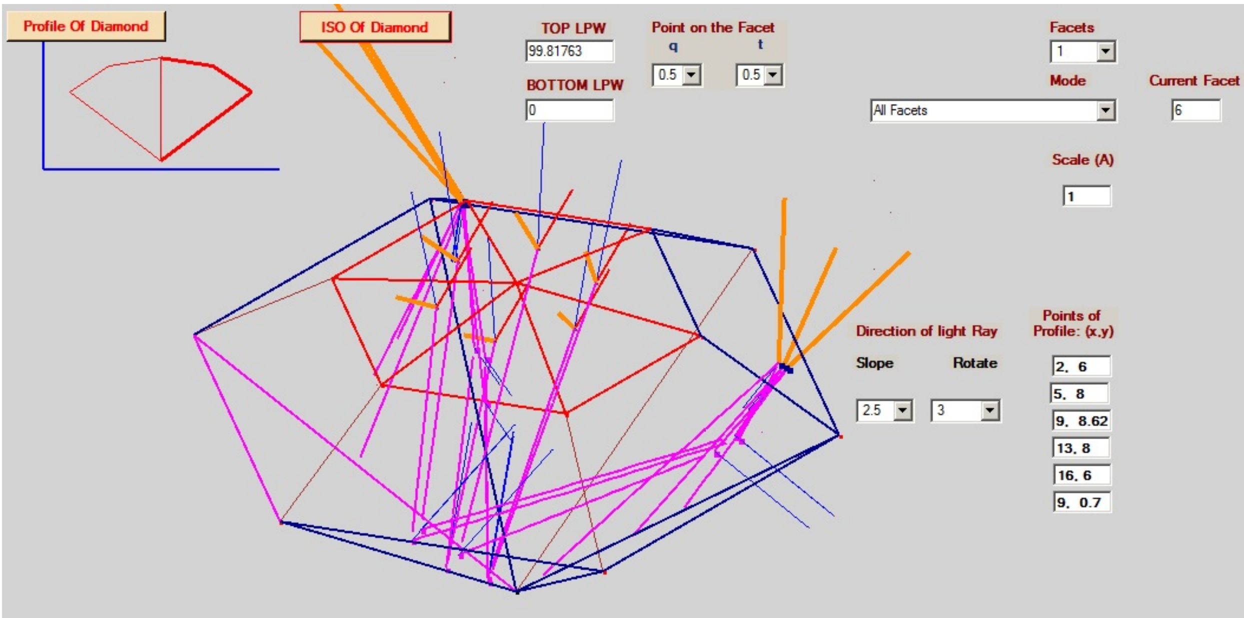


Figure 10: Diamond with new coordinates of vertices 3 and 6

## 7. Conclusion

This work shows that the methods of projective geometry make it possible to obtain certain data about the behavior of a light ray inside a crystal. The double points of projective correspondences indicate the existence of closed light contours.

The examined construction $\Omega_1$ leads to the formulation of new problems related to the existence of special trajectories of a light ray inside a crystal. For example, the problem of finding the point on a crystal surface and the direction of the light ray, exiting at that point, such that the trajectory in the limit approaches some static contour.

The behavior of the desired ray in this example is analogous to the ray, shown in Fig. 5, where the "broken light line" tends to the segment $RA_0$. Thanks to the computer algorithm created by the authors, a user is able to find the crystal shapes with the best light-reflection ability.

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