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Some Theorems on Kissing Circles and Spheres

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Abstract. When three circles, O_1, O_2, and O_3, are tangent externally to each other, there are only two circles tangent to the original three circles. This is a special case of the Apollonius problem, and such circles are called the inner and outer Soddy circles. Given the outer Soddy circle S, we can construct the new Apollonian circle I_1 that is tangent to S, O_2, and O_3. By the same method, we can construct new circles I_2 tangent to S, O_3, and O_1, and I_3 tangent to S, O_1, and O_2. These seven tangent circles are a subset of an Apollonian packing of circles. In this article, we describe a new inscribed circle tangent to the three pairs of common external tangents of diagonally placed circles, $\{O_1, I_1\}$, $\{O_2, I_2\}$, and $\{O_3, I_3\}$. Furthermore, we found that when two externally tangent triangles of the three circles $\{O_1, O_2, O_3\}$ and $\{I_1, I_2, I_3\}$ are constructed, the three diagonally joined lines of the two triangles are concurrent. These theorems are further generalized to the three-dimensional case on nine tangent spheres. Focusing on visual representations, we established these theorems only by a synthetic method throughout this article.

Key Words: tangent circles and spheres, inversions of circles and spheres

MSC 2010: 51M04, 51N10

1. Introduction

Finding elegant relations among points, lines, and circles are essential in geometry. Already, many geometrical theorems with concurrent lines and collinear points, such as Pascal's theorem and Brianchon's theorem (see, for example, [1, 2, 3], have been discovered. In addition, relations between the radii on mutually tangent circles are also fascinating to many geometricians. Descartes gave the formula known as Descartes theorem for finding the radius of a fourth circle tangent to three given kissing circles (Pedoe [4, p. 157]). Soddy rediscovered this formula and extended it to tangent spheres [5]. Although geometricians have long been interested in systems of tangent circles and spheres, such as Apollonian packing of circles

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(Wells [3]) and mutually contacting spheres (Coxeter [6]), the inscribed circles, spheres, and concurrent lines behind these configurations have not been sufficiently studied, except in a few works (for example, Eppstein [7]). In the present paper, we describe finding inscribed circles and concurrent points for a configuration of seven tangent circles. The theorems were also extended to the three-dimensional space by a synthetic method.

2. Two-dimensional theorems on seven kissing circles

Two theorems are obtained for two-dimensional kissing circles.

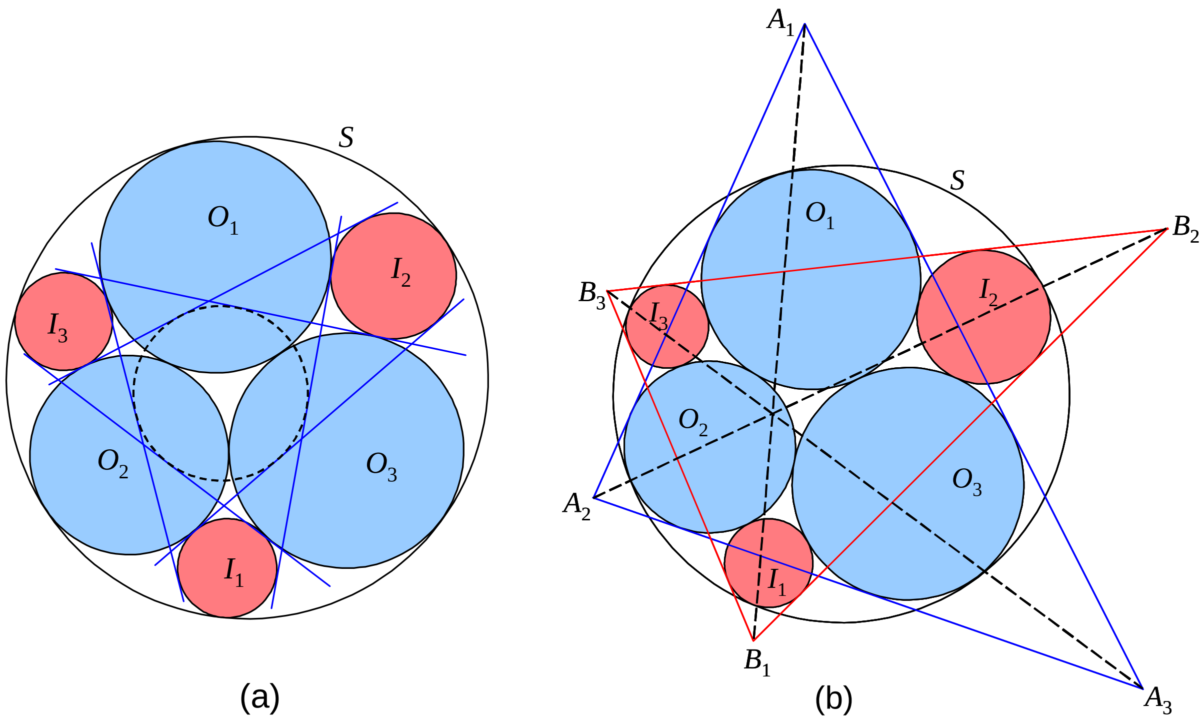

Figure 1: (a) Circle inscribed in externally tangent lines. (b) Concurrent theorem.

Theorem 1. *Let O_1, O_2, and O_3 be three circles externally tangent to each other and S be the outer Soddy circle tangent to O_1, O_2, and O_3 (see Fig. 1a.). Construct new circles I_1 tangent to O_2, O_3, and S, I_2 tangent to O_3, O_1, and S, and I_3 tangent to O_1, O_2, and S. I_1, I_2, and I_3 are different from O_1, O_2, and O_3, respectively. Denote by $\{O_1I_1\}$ the common external tangents between opposite circles O_1 and I_1, and define $\{O_2I_2\}$ and $\{O_3I_3\}$ similarly. Then there exists a unique circle inscribed in $\{O_1I_1\}$, $\{O_2I_2\}$, and $\{O_3I_3\}$.*

Theorem 2. *For the configuration of Theorem 1, let $A_1\,A_2\,A_3$ be a triangle whose sides are given by the outer common external tangents of O_1, O_2, and O_3, and let $B_1\,B_2\,B_3$ be a triangle whose sides are given by the outer common external tangents of I_1, I_2, and I_3, as shown in Fig. 1b.*
Then the three lines joining the opposite vertices of the two triangles, $A_1\,B_1$, $A_2\,B_2$, and $A_3\,B_3$, are concurrent.

Before showing the proofs of Theorems 1 and 2, we introduce a lemma.

Lemma 1. *Let S_1, S_2, and S_3 be three fixed spheres mutually tangent to each other such that S_1 and S_2 are contained in S_3 (see Fig. 2a). Then*

1. *the 1-parameter family of spheres F_1 tangent to S_1, S_2, and S_3 touch a pair of distinct common planes π_1 and π_2,*
2. *the curve of tangency of F_1 and S_3 is a circle on the surface of S_3.*

Note that the enveloping surface of F_1 is a Dupin cyclide.

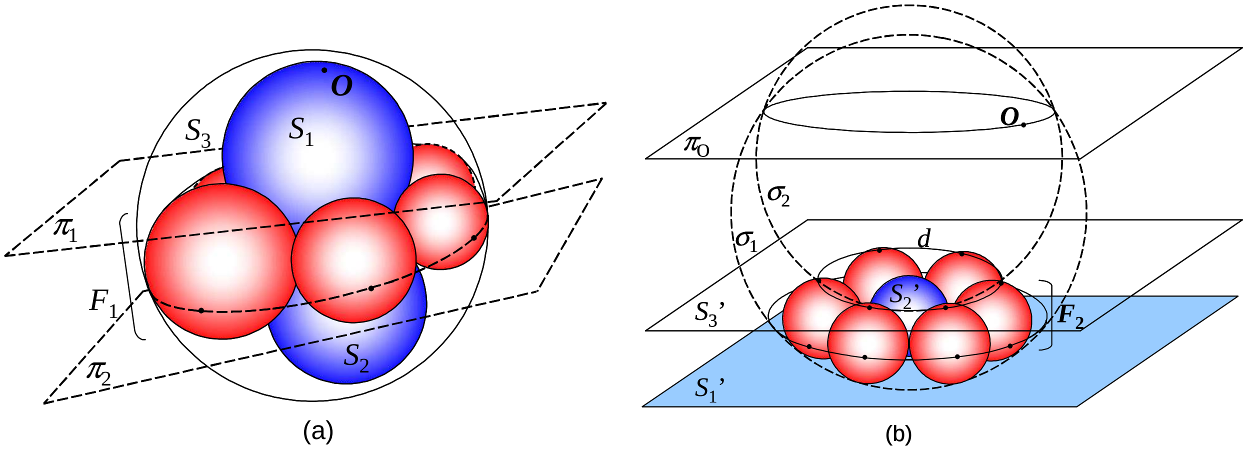

Figure 2: (a) 1-parameter family of spheres tangent to a given set of three tangent spheres, S_1, S_2, and S_3. (b) Image of the 1-parameter family of spheres under inversion.

Proof of Lemma 1: Let O be the point of tangency of S_1 and S_3, and let S_O be a sphere of inversion with center O (Fig. 2a). Let i be an inversion with respect to S_O. Then $S_1' = i(S_1)$ and $S_3' = i(S_3)$ are the planes parallel to the common tangent plane π_0 of S_1 and S_3 at O, as shown in Fig. 2b. Furthermore, $S_2' = i(S_2)$ is a sphere tangent to planes S_1' and S_3'. It follows that there exists a 1-parameter family of spheres F_2 tangent to S_1', S_2', and S_3'. Note that $F_2 = i(F_1)$, and that the spheres of F_2 have centers lying on a circle and a common radius. Moreover, from the symmetry of F_2, we know that the there exist two distinct sphere surfaces σ_1 and σ_2 that are tangent to F_2 and pass through O. It follows that $\pi_1 = i(\sigma_1)$ and $\pi_2 = i(\sigma_2)$ are the distinct planes that are tangent to F_1 since $F_1 = i(F_2)$. This completes the proof of Lemma 1.1.

Let d be the curve of tangency of F_2 and S_3'. From the symmetry of F_2, d must be a circle on S_3'. Therefore, the curve of tangency of F_1 and S_3, that is $i(d)$, is a circle which lies on surface S_3. This completes the proof of Lemma 1.2. □

Proof of Theorem 1: Assume that all the circles and lines in Fig. 1a (Theorem 1) are on a plane π, and let S_{O1}, S_{O2}, S_{O3}, S_{I1}, S_{I2}, S_{I3}, and S_S be the spheres with centers on π containing circles O_1, O_2, O_3, I_1, I_2, I_3, and S, respectively. Let us consider the 1-parameter family G_{12} of spheres tangent to S_{O1}, S_{O2}, and S_S. Similarly, let G_{23} be the family of spheres tangent to S_{O2}, S_{O3}, and S_S, and G_{31} be that tangent to S_{O3}, S_{O1}, and S_S.
Now, let a new sphere S_{O4} be tangent to S_{O1}, S_{O2}, S_{O3}, and S_S. Then we know that G_{12}, G_{23}, and G_{31} share S_{O4}. Applying Lemma 1.1 to G_{12}, G_{23}, and G_{31}, we see that each of these has two distinct planes tangent at the two sides. Furthermore, these six tangent planes are perpendicular to π, since G_{12}, G_{23}, and G_{31} are symmetric with respect to π.

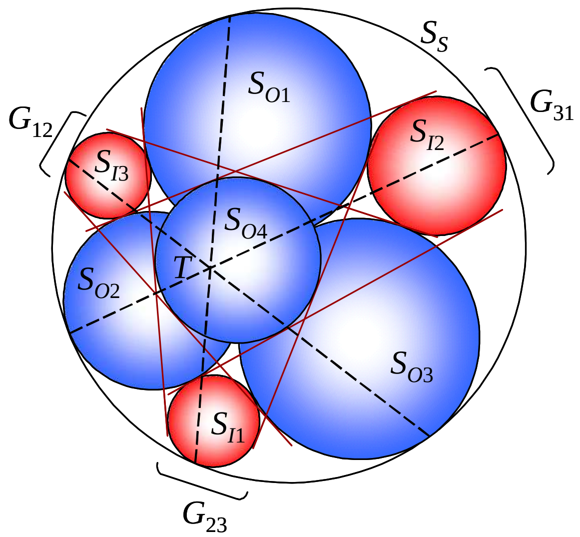

Figure 3: G_{12}, G_{23}, and G_{31} share a common sphere, S_{O4}.

If we consider the orthographic-projection of Fig. 3 into π, it follows that there exists a unique circle inscribed in the common external tangents $\{O_1I_1\}$, $\{O_2I_2\}$, and $\{O_3I_3\}$. This completes the proof of Theorem 1. □

Remark 1. Referring to Figs. 3 and 4a, let $\{C_1, C_2, C_3, D_1, D_2, D_3\}$ be the points of tangency of $\{O_1, O_2, O_3, I_1, I_2, I_3\}$ and S, respectively. Applying Lemma 1.2 to G_{12}, G_{23}, and G_{31}, and S_S, we know that the three curves of tangency of $\{ G_{12}, G_{23}, G_{31} \}$ and S_S are all circles on surface S_S. Thus, since G_{12}, G_{23}, and G_{31} share a common sphere S_{O4} that is also tangent to S_S, these circles intersect at a point T, where T is a point of tangency of S_{O4} and S_S. Note that these circles of tangencies are all symmetric with respect to π since G_{12}, G_{23}, and G_{31} are symmetric with respect to π. Therefore, if we consider the orthographic projection into π (Fig. 3), we have that C_1D_1, C_2D_2, and C_3D_3 are concurrent at a point M, where M is the orthographical projection of point T.

Remark 2. Theorem 1 remains true by replacing the outer Soddy circle by the inner Soddy circle. The proof is almost same like for Theorem 1.

Finally, we will prove Theorem 2. The proof is based on two facts:

(a) Referring to Fig. 4a, let l_{C1}, l_{C2}, l_{C3}, l_{D1}, l_{D2}, and l_{D3} be tangent lines of S at C_1, C_2, C_3, D_1, D_2, and D_3, respectively, and let X, Y, and Z be the intersections of two lines, $l_{C1} \cap l_{D1}$, $l_{C2} \cap l_{D2}$, and $l_{C3} \cap l_{D3}$, respectively. Then, X, Y, and Z are collinear.

(b) Let us denote as a_{12} and b_{12} the lines A_1A_2 and B_1B_2, respectively, in Fig. 1b, and define a_{23}, b_{23}, a_{31}, and b_{31} similarly. Then, $X = a_{23} \cap b_{23}$, $Y = a_{31} \cap b_{31}$, and $Z = a_{12} \cap b_{12}$.

Proof of Theorem 2:
(a) X, Y, and Z can be considered as vertices of externally tangent cones of S_S at circles of tangencies of G_{23} and S_S, G_{31} and S_S, and G_{12} and S_S, respectively (see Fig 4a). In addition, these circles of tangencies intersect at point T. Thus, assuming that α is a plane tangent to S_S at T, we see that α is tangent to three tangent cones at each of the generators. It follows that X, Y, and Z are collinear since these points lie on the intersection of α and π.

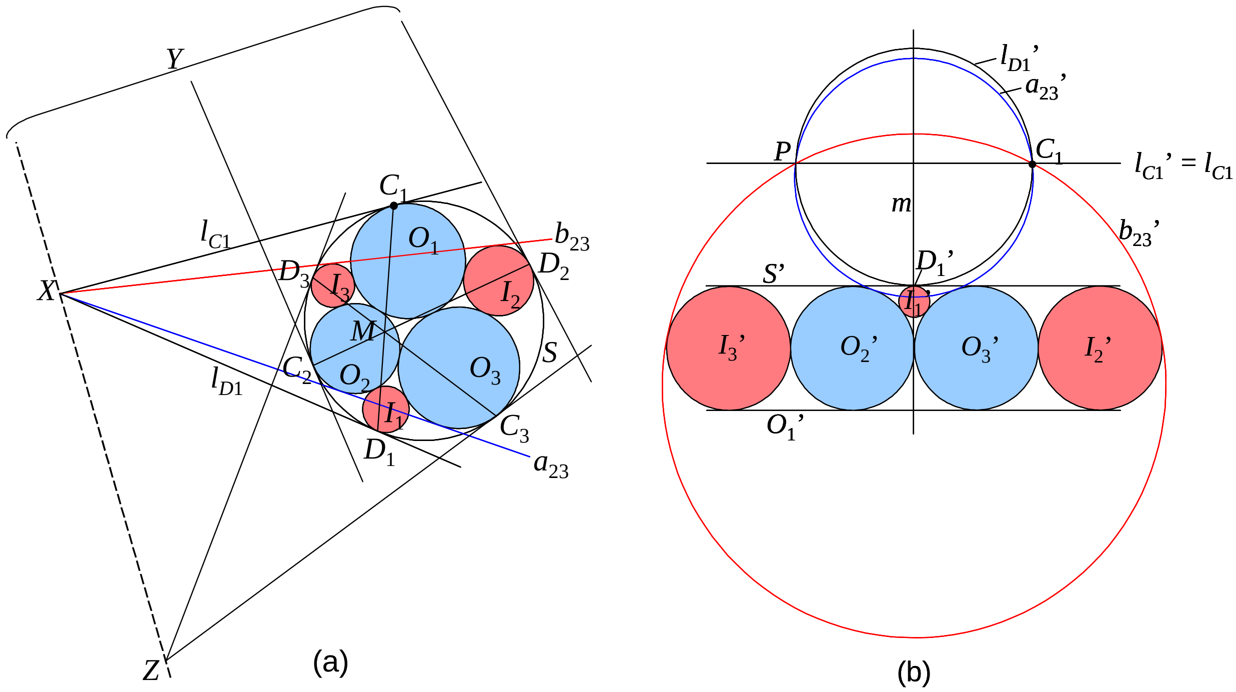

Figure 4: (a) X, Y, and Z are collinear. (b) Image of (a) under inversion.

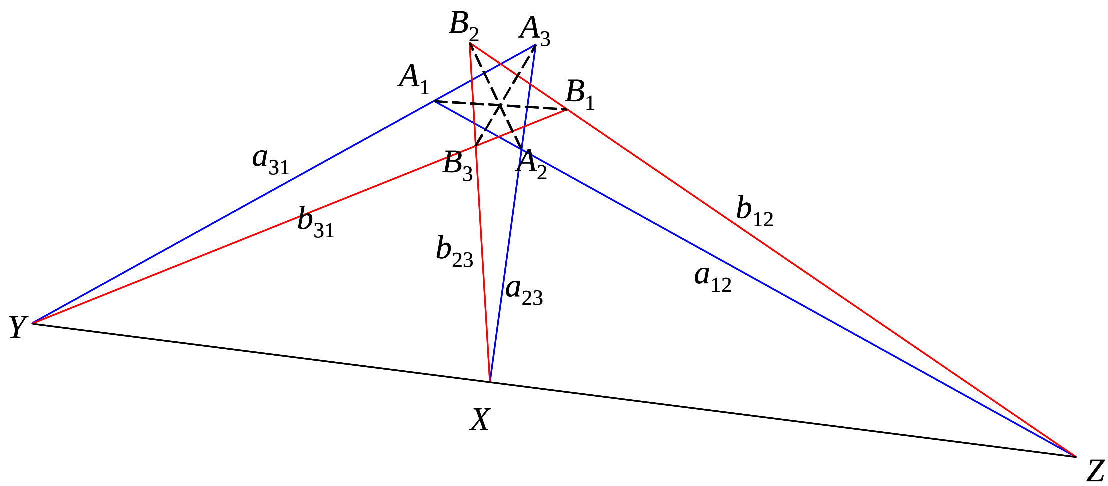

Figure 5: Triangles $A_1A_2A_3$ and $B_1B_2B_3$ have a center of perspective.

(b) Let f be an inversion with center C_1. Then $S' = f(S)$ and $O_1' = f(O_1)$ are lines parallel to the common tangent line l_{C1} of O_1 and S at C_1, and $O_2' = f(O_2)$ and $O_3' = f(O_3)$ are the circles mutually tangent to each other and also tangent to S' and O_1', as seen in Fig. 4b. Furthermore, $I_3' = f(I_3)$ is the circle tangent to S', O_2', and O_1', $I_2' = f(I_2)$ is the circle tangent to S', O_1', and O_3', and $I_1' = f(I_1)$ is the circle tangent to S', O_2', and O_3'. Moreover, $l_{D1}' = f(l_{D1})$ is the circle tangent to I_1' and S' at $D_1' = f(D_1)$ passing through C_1, $a_{23}' = f(a_{23})$ is the circle tangent to O_2' and O_3' passing through C_1, and $b_{23}' = f(b_{23})$ is the circle tangent to I_2' and I_3' passing through C_1.

Let m be the line perpendicular to S' through D_1'. From the symmetry with respect to m, we know that l_{D1}', a_{23}', and b_{23}' pass through a unique point P on l_{C1}' such that P is a symmetrical point of C_1 with respect to m. Therefore, we see that l_{C1}, l_{D1}, a_{23}, and b_{23} pass through a

unique point at $f(P)$. Since $X = l_{C1} \cap l_{D1}$, it follows that $X = f(P) = a_{23} \cap b_{23}$. Similarly, we know that $Y = a_{31} \cap b_{31}$ and $Z = a_{12} \cap b_{12}$.
From facts (a) and (b), and applying Desargues' theorem to X, Y, Z, A_1, A_2, A_3, B_1, B_2, and B_3, shown in Fig. 5, we know that the two triangles $A_1A_2A_3$ and $B_1B_2B_3$ have a center of perspective. This completes the proof of Theorem 2. □

Remark 3. It should be noticed that all the externally tangents of $\{O_2, O_3\}$ and $\{I_2, I_3\}$ intersect at point X. Similarly, all the externally tangents of $\{O_1, O_2\}$ and $\{I_1, I_2\}$ intersect at point Z, and externally tangents of $\{O_3, O_1\}$ and $\{I_3, I_1\}$ intersect at point Y. This directly indicates that Theorem 2 remains true for any of the $2^3 = 8$ triangles formed by any combination of outer tangents, and there exit $8^2 = 64$ centers of perspective given by any combination of two triangles. In general, these centers of perspective are different from point M given in Remark 1.

3. Three-dimensional theorems on nine kissing spheres

Two theorems are obtained on the three-dimensional kissing spheres.

Theorem 3. *Let S_{O1}, S_{O2}, S_{O3}, and S_{O4} be four spheres externally tangent to each other and S be an outer sphere tangent to S_{O1}, S_{O2}, S_{O3}, and S_{O4} (see Fig. 6a). Construct new spheres S_{I1} tangent to S_{O2}, S_{O3}, S_{O4}, and S, S_{I2} tangent to S_{O3}, S_{O4}, S_{O1}, and S, S_{I3} tangent to S_{O4}, S_{O1}, S_{O2}, and S, and S_{I4} tangent to S_{O1}, S_{O2}, S_{O3}, and S. Denote by $\{S_{O1}S_{I1}\}$ the common external tangent cone of opposite spheres S_{O1} and S_{I1}, and define $\{S_{O2}S_{I2}\}$, $\{S_{O3}S_{I3}\}$, and $\{S_{O4}S_{I4}\}$ similarly.*
Then there exists a unique sphere inscribed in $\{S_{O1}S_{I1}\}$, $\{S_{O2}S_{I2}\}$, $\{S_{O3}S_{I3}\}$, and $\{S_{O4}S_{I4}\}$.

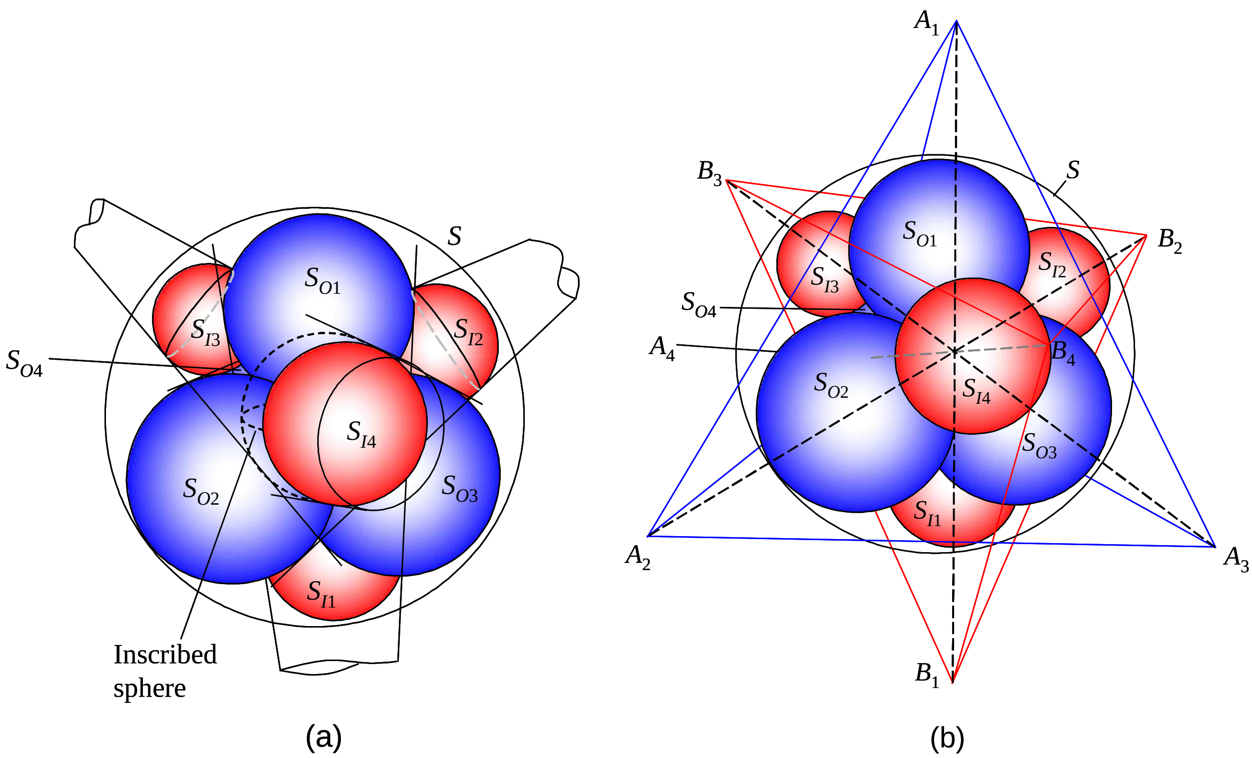

Figure 6: (a) Sphere inscribed in externally tangent cones. (b) Concurrent theorem.

Theorem 4. *For the configuration of Theorem 3, let $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ be tetrahedra whose surfaces are given by the outer common external tangent planes of the spheres $\{S_{O1}, S_{O2}, S_{O3}, S_{O4}\}$ and $\{S_{I1}, S_{I2}, S_{I3}, S_{I4}\}$, respectively, as shown in Fig. 6b.*
Then the four lines joining the opposite vertices of the two tetrahedra, A_1B_1, A_2B_2, A_3B_3, and A_4B_4, are concurrent.

Proof of Theorem 3: Let O_1, O_2, O_3, O_4, I_1, I_2, I_3, and I_4 be the centers of S_{O1}, S_{O2}, S_{O3}, S_{O4}, S_{I1}, S_{I2}, S_{I3}, and S_{I4}, respectively (see Fig. 6a). We know that S_{O3}, S_{O4}, S_{I3}, and S_{I4} are the spheres tangent to S_{O1}, S_{O2}, and S.
Applying Lemma 1.1 to these spheres, we then know that S_{O3}, S_{O4}, S_{I3}, and S_{I4} are tangent to two distinct common planes, π_1 and π_2. Hence, there exists a unique intersection point P_{34} of the two lines O_3I_3 and O_4I_4 since the four centers O_3, O_4, I_3, and I_4 lie on the median plane of π_1 and π_2. Therefore, if S_{P34} is the sphere tangent to π_1 and π_2 with center P_{34}, then we know that S_{P34} is a unique sphere inscribed in tangent cones $\{S_{O3}S_{I3}\}$ and $\{S_{O4}S_{I4}\}$.
Similarly, there exists a unique intersection point P_{41} of the two lines O_4I_4 and O_1I_1 and the sphere S_{P41} inscribed in the tangent cones $\{S_{O4}S_{I4}\}$ and $\{S_{O1}S_{I1}\}$, where P_{41} is the center of S_{P41}, and there exists a unique intersection point P_{13} of the two lines O_1I_1 and O_3I_3 and the sphere S_{P13} inscribed in the tangent cones $\{S_{O1}S_{I1}\}$ and $\{S_{O3}S_{I3}\}$, where P_{13} is the center of S_{P13}.
Now let us assume that P_{34} $(O_3I_3 \cap O_4I_4)$ and P_{41} $(O_4I_4 \cap O_1I_1)$ are distinct points. If so, then the lines O_1I_1 and O_3I_3 do not intersect. However, this cannot be true from the above discussion since O_1I_1 and O_3I_3 intersect at P_{13}. Hence, P_{34} and P_{41} must be identical, i.e., $P_{34} = P_{41} = P_{13}$, and we have that three lines O_1I_1, O_3I_3, and O_4I_4 are concurrent.
Considering the permutation symmetry, we know that

(a) all the lines joining the opposite centers of inner spheres, O_1I_1, O_2I_2, O_3I_3, and O_4I_4 intersect at a unique point. Furthermore, from the above discussion, we also know that

(b) there exists a unique common sphere S_{Pjk} with center P_{jk} inscribed in the two diagonally joined tangent cones of $\{S_{Oj}S_{Ij}\}$ and $\{S_{Ok}S_{Ik}\}$, where P_{jk} is a intersecting point of O_jI_j and O_kI_k for j, $k = 1, 2, 3, 4$ and $j \neq k$.

From (a) and (b) follows that there exists a unique sphere inscribed in the tangent cones $\{S_{O1}S_{I1}\}$, $\{S_{O2}S_{I2}\}$, $\{S_{O3}S_{I3}\}$, and $\{S_{O4}S_{I4}\}$. This completes the proof of Theorem 3. □

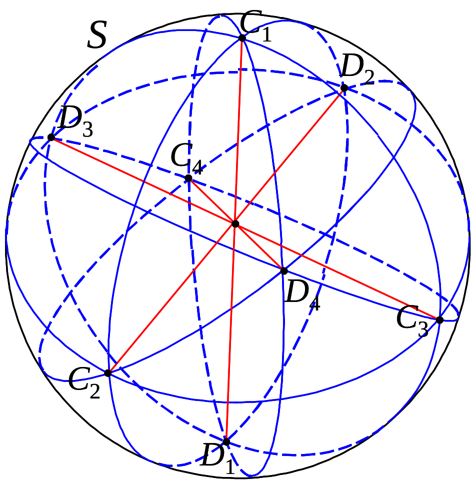

Figure 7: Tangent points of inner spheres and S.

Remark 4. Let $\{C_1, C_2, C_3, C_4, D_1, D_2, D_3, D_4\}$ be the points of tangency of $\{S_{O1}, S_{O2}, S_{O3}, S_{O4}, S_{I1}, S_{I2}, S_{I3}, S_{I4}\}$ and S, respectively (see Fig. 7). Applying Lemma 1.2 to S_{O1}, S_{O2}, and S, we know that the four points C_3, C_4, D_3, and D_4 lie on a circle on S. It follows that lines C_3D_3 and C_4D_4 intersect at a point Q_{34}.
Similarly, the sets of four points $\{C_1, C_2, D_1, D_2\}$, $\{C_2, C_3, D_2, D_3\}$, $\{C_2, C_4, D_2, D_4\}$, $\{C_3, C_1, D_3, D_1\}$, and $\{C_4, C_1, D_4, D_1\}$ also lie on circles on S, and the pairs of lines $\{C_1D_1, C_2D_2\}$, $\{C_2D_2, C_3D_3\}$, $\{C_2D_2, C_4D_4\}$, $\{C_3D_3, C_1D_1\}$, and $\{C_4D_4, C_1D_1\}$, intersect at points Q_{12}, Q_{23}, Q_{24}, Q_{31}, and Q_{41}, respectively. As in the proof of Theorem 3, let us assume that Q_{34} and Q_{41} are distinct points. Then the lines C_3D_3 and C_1D_1 do not intersect. However, this cannot be true from the above discussion since there exists Q_{31} which is the intersection of C_3D_3 and C_1D_1.
Hence, Q_{34} and Q_{41} must be the same point, i.e., $Q_{34} = Q_{41} = Q_{31}$, and we have that three lines C_1D_1, C_3D_3, and C_4D_4 are concurrent. Considering the permutation symmetry, we conclude that all four lines, C_1D_1, C_2D_2, C_3D_3, and C_4D_4, intersect at a unique point.

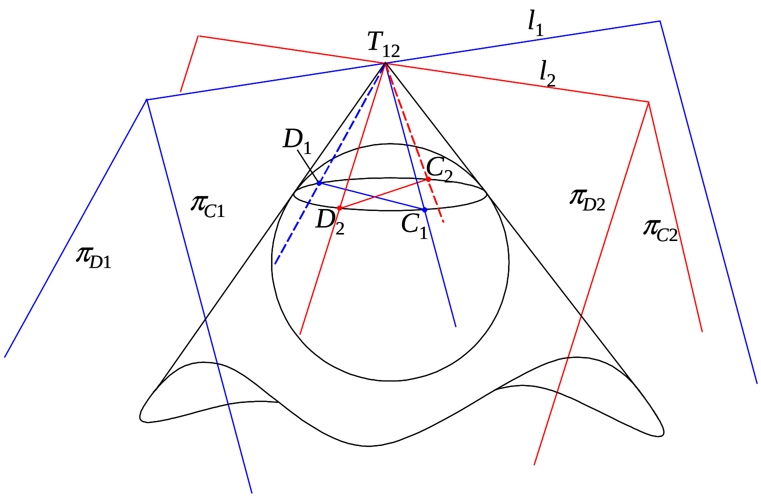

Figure 8: π_{C1}, π_{D1},π_{C2}, π_{D2} pass through T_{12}, the point where l_1 and l_2 intersect.

We will now prove Theorem 4 using the facts (c) and (d) below.

(c) Let π_{C1}, π_{C2}, π_{C3}, π_{C4}, π_{D1}, π_{D2}, π_{D3}, and π_{D4} be tangent planes of S at C_1, C_2, C_3, C_4, D_1, D_2, D_3, and D_4, respectively (Fig. 8), and let lines l_1, l_2, l_3, and l_4 be the intersection lines of pairs of planes, $\pi_{C1} \cap \pi_{D1}$, $\pi_{C2} \cap \pi_{D2}$, $\pi_{C3} \cap \pi_{D3}$, and $\pi_{C4} \cap \pi_{D4}$, respectively. Then l_1, l_2, l_3, and l_4 lie on a unique plane and determine a complete quadrilateral (Fig. 10).

(d) Let us denote as α_{123} and β_{123} the planes $A_1A_2A_3$ and $B_1B_2B_3$, respectively (Fig. 9a), and define α_{234}, α_{341}, α_{412}, β_{234}, β_{341}, and β_{412} similarly. Then $l_1 = \alpha_{234} \cap \beta_{234}$, $l_2 = \alpha_{341} \cap \beta_{341}$, $l_3 = \alpha_{412} \cap \beta_{412}$, and $l_4 = \alpha_{123} \cap \beta_{123}$.

Proof of Theorem 4:
(c) As seen in Remark 4, we know that the four points C_1, D_1, C_2, and D_2 lie on a circle, namely, $C_1D_1C_2D_2$ (see Fig. 8). Hence, the lines C_1D_1 and C_2D_2 intersect at a point. Let T_{12} be a vertex of the tangent cone of S at the circle $C_1D_1C_2D_2$. Vertex S could be at infinity. Then we know that all four planes, π_{C1}, π_{D1}, π_{C2}, and π_{D2}, pass through T_{12}, since all the planes pass through the generator of this tangent cone. It follows that $T_{12} = l_1 \cap l_2$ since $l_1 = \pi_{C1} \cap \pi_{D1}$ and $l_2 = \pi_{C2} \cap \pi_{D2}$.

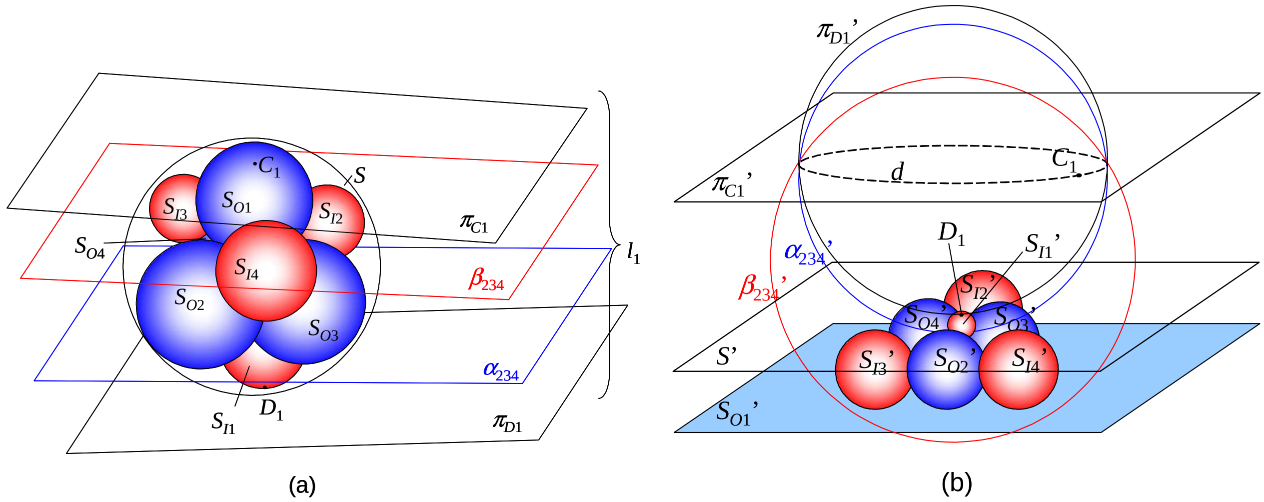

Figure 9: (a) Tangent spheres and planes. (b) Image of (a) under inversion.

Similarly, using analogous definitions to that of T_{12} for T_{23}, T_{24}, T_{34}, T_{31}, and T_{41}, we obtain $T_{23} = l_2 \cap l_3$, $T_{24} = l_2 \cap l_4$, $T_{34} = l_3 \cap l_4$, $T_{31} = l_3 \cap l_1$, and $T_{41} = l_4 \cap l_1$. Since T_{12}, T_{23}, T_{24}, T_{34}, T_{31}, and T_{41} are distinct points, we conclude that, l_1, l_2, l_3, and l_4 lie on a unique plane and determine a complete quadrilateral.

(d) Referring to Figs. 9a and 9b, let i be inversion with center C_1. Then $S' = i(S)$ and $S'_{O1} = i(S_{O1})$ are the planes parallel to the common tangent plane $\pi'_{C1} = i(\pi_{C1}) = \pi_{C1}$, and $S'_{O2} = i(S_{O2})$, $S'_{O3} = i(S_{O3})$, and $S'_{O4} = i(S_{O4})$ are mutually tangent spheres and are also tangent to S' and S'_{O1}, as shown in Fig. 9b.

Furthermore, $S'_{I2} = i(S_{I2})$, $S'_{I3} = i(S_{I3})$, $S'_{I4} = i(S_{I4})$, and $S'_{I1} = i(S_{I1})$ are spheres tangent to $\{S', S'_{O1}, S'_{O3}, S'_{O4}\}$, $\{S', S'_{O2}, S'_{O4}, S'_{O1}\}$, $\{S', S'_{O1}, S'_{O2}, S'_{O3}\}$, and $\{S', S'_{O2}, S'_{O3}, S'_{O4}\}$, respectively. Let $\alpha'_{234} = i(\alpha_{234})$, $\beta'_{234} = i(\beta_{234})$, and define α'_{123}, α'_{341}, α'_{412}, β'_{123}, β'_{341}, and β'_{412} similarly.

Then, α'_{234} and β'_{234} are the spherical surfaces that pass through C_1 tangent to $\{S'_{O2}, S'_{O3}, S'_{O4}\}$ and $\{S'_{I2}, S'_{I3}, S'_{I4}\}$, respectively. Note that from the symmetry, α'_{234} and β'_{234} contain the same circle, namely d, on $\pi'_{C1} = i(\pi_{C1})$ and pass through C_1. Moreover, $\pi'_{D1} = i(\pi_{D1})$ also contain the circle d. The inversion i with center C_1 maps any circle through C_1 onto a line not passing through C_1. It follows that $\alpha_{234} = i(\alpha'_{234})$ and $\beta_{234} = i(\beta'_{234})$ intersect at line $l_1 = i(d)$. Similarly, $l_2 = \alpha_{341} \cap \beta_{341}$, $l_3 = \alpha_{412} \cap \beta_{412}$, and $l_4 = \alpha_{123} \cap \beta_{123}$.

From facts (c) and (d), we know that $T_{12} = l_1 \cap l_2$, $l_1 = \alpha_{234} \cap \beta_{234}$, and $l_2 = \alpha_{341} \cap \beta_{341}$ (see Fig. 10). It follows that the lines A_3A_4 and B_3B_4 intersect at point T_{12}, since α_{234}, β_{234}, α_{341}, and β_{341} are the planes $A_2A_3A_4$, $B_2B_3B_4$, $A_3A_4A_1$, and $B_3B_4B_1$, respectively. Similarly, A_2A_3 and B_2B_3 intersect at point T_{41}, and A_2A_4 and B_2B_4 intersect at point T_{31}. Hence, the two triangles $A_2A_3A_4$ and $B_2B_3B_4$ have an axis of perspective.

From Desargues' theorem, we know the two triangles $A_2A_3A_4$ and $B_2B_3B_4$ have a center of perspective. Similarly, the pairs of triangles $A_3A_4A_1$ and $B_3B_4B_1$, $A_4A_1A_2$ and $B_4B_1B_2$, and $A_1A_2A_3$ and $B_1B_2B_3$ each have an axis of perspective. It follows that A_1B_1, A_2B_2, A_3B_3, and A_4B_4 are concurrent. This completes the proof of Theorem 4. $\square$

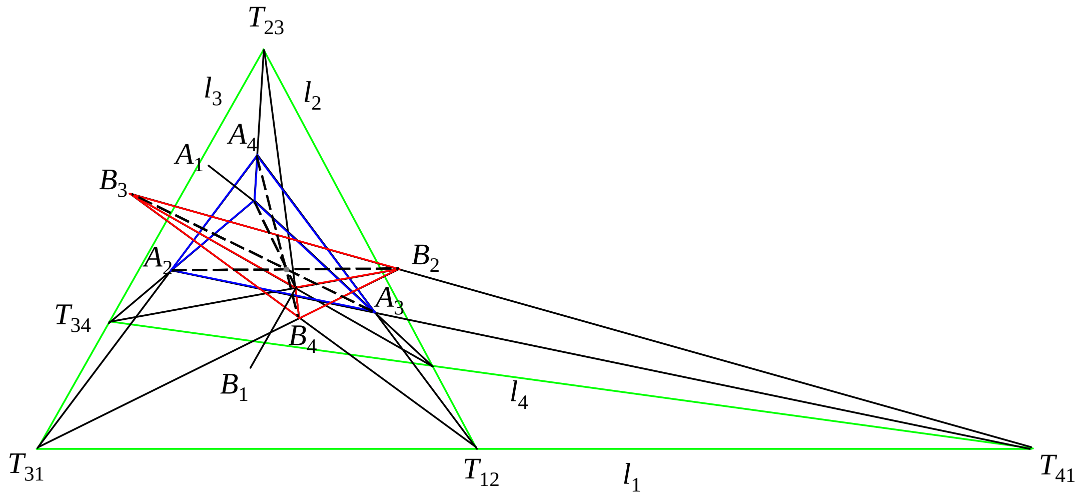

Figure 10: Complete quadrilateral and concurrent lines joining the opposite vertices of the two tetrahedra.

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