Journal for Geometry and Graphics
Volume 16 (2012), No. 1, 1–11.

On Kiepert Conics in the Hyperbolic Plane

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Abstract. The *Kiepert hyperbola* and the *Kiepert parabola* of a triangle in the Euclidean plane are the background of this paper. Its main issue is the question whether a similar phenomenon can be found in the *hyperbolic plane*. The considerations are set in the *disk model* of hyperbolic geometry where classical projective reasoning can also be employed.

Key Words: Elementary hyperbolic geometry, Cayley-Klein geometry, triangle geometry, Kiepert conics, hyperbolic isogonal transformation.

MSC 2010: 51M09, 51N15, 51F99

1. Introduction

We consider a geometry that obeys the axioms of Euclid except for the well-known parallel postulate. The parallel postulate of Euclidean geometry is replaced by the statement: 'To any given line g and point P not on g there are at least two distinct lines through P which do not intersect g.' This axiom gets us to *hyperbolic geometry*. As a model of that geometry we use the *disk model* in the projective plane.

We start with the real projective plane equipped with a Euclidean structure. The real projective plane — in turn — is embedded into the complex projective plane. Next, we elect a circle called the absolute conic m. The interior points of m are (ordinary) points, the (open) chords of m are the lines in the *disk model of hyperbolic geometry*. The points on the absolute conic m itself are called *ideal points* whereas the points outside m are referred to as *ultra-ideal points* of the model. So, a hyperbolic line in our geometry is basically a segment; we identify it with the coinciding line in the projective plane. Depending on whether two lines intersect at an ordinary point, an ideal point, or an ultra-ideal point, we call the lines intersecting, *asymptotically parallel*, or *ultra-parallel*, respectively. This way we can formulate theorems of hyperbolic geometry in the language of projective geometry and vice versa. E.g., the ultra-ideal point in which ultra-parallel lines intersect is the pole of their common perpendicular with regard to the absolute conic m (cp. [4, 1, 2, 8, 11]).

Kiepert conics and related issues are the germ of our investigation of a triangle in hyperbolic geometry. Section 2 contains a short roundup of properties of a triangle and its *Kiepert*

ISSN 1433-8157/\$ 2.50

hyperbola and *Kiepert parabola* in *Euclidean geometry*. The *hyperbolic isogonal transformation* (in short: *h-isogonal transformation*) with respect to a hyperbolic triangle is addressed in Section 3. In Section 4 we construct the analogue to the Kiepert hyperbola, called *first Kiepert conic* in the hyperbolic plane and reflect on some of their properties. Section 5 resorts to a theorem on perspective triangles in the projective plane. This theorem is the background of Section 6 where we prove the existence of the *second Kiepert conic* in the hyperbolic plane which is the analogue to the Kiepert parabola. Finally, a summary and an outlook on further investigations are given in Section 7.

2. Kiepert hyperbola and Kiepert parabola in the Euclidean plane

Let ABC be an arbitrary triangle and $A'BC$, $AB'C$, and ABC' isosceles triangles with some base angle ρ, attached externally (or internally) to the edges AB, BC, and CA. The points A', B', C' form a new triangle called *Kiepert triangle* to the base angle ρ. The triangle ABC and each Kiepert triangle $A'B'C'$ are perspective from some centre $K(\rho)$ which is the intersection point of AA', BB' and CC'. If the base angle ρ varies between $-\pi/2$ and $\pi/2$ the locus of $K(\rho)$ is a rectangular hyperbola, called *Kiepert hyperbola* k (cp. [3, 5], see Fig. 1).

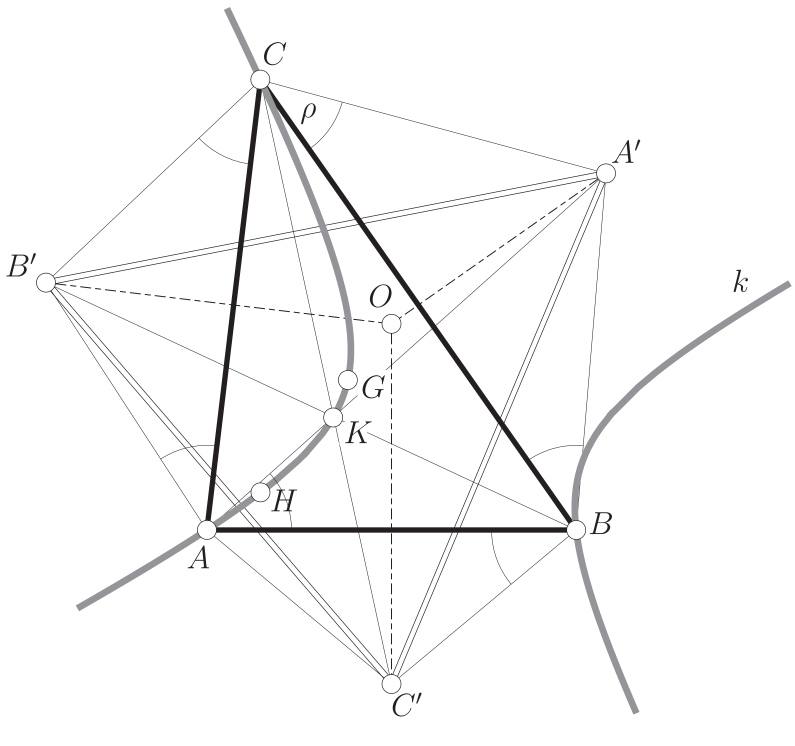

Figure 1: The triangle ABC in the Euclidean plane and each Kiepert triangle $A'B'C'$ are perspective. The respective centres K lie on the Kiepert hyperbola k which also contains the centre of gravity G, the orthocentre H and the vertices of the triangle ABC.

We refer to the well-known Desargues Theorem: As the triangles ABC and $A'B'C'$ are perspective from a centre $K(\rho)$, there must be some Desargues axis $d(\rho)$. The envelope of all these axes $d(\rho)$ is the *Kiepert parabola* p. In the Euclidean plane the Kiepert hyperbola and the Kiepert parabola are closely linked to other remarkable points and lines of the triangle. For instance, the vertices A, B, C, the *centroid* G and the *orthocentre* H are points on the Kiepert hyperbola k. The image of the Kiepert hyperbola k under the isogonal transformation with respect to triangle ABC is the *Brocard axis* k^* of ABC. The *Lemoine line* is among the tangents to the *Kiepert parabola* p. If the isosceles triangles BCA', CAB', and ABC' are even equilateral triangles attached externally to the edges of the given triangle ABC, the centroids of these triangles are vertices of an equilateral triangle. This proposition is generally known as *Napoleon's theorem*.

We can find analogues to most of these statements in the hyperbolic plane. Napoleon's theorem, though, does not have a hyperbolic analogue.

3. Isogonal transformations

3.1. Isogonal transformation in the Euclidean plane

Two lines l, l^* through the vertex of an angle are said to be *isogonal* if they are symmetric with regard to the bisector of that angle. If ABC is a triangle and P is a point in the projective plane, not on the edges of ABC, the connecting lines AP, BP, CP are called *cevians of* P. The *isogonal conjugate to* P is the intersection point P^* of the lines isogonal to the cevians through P. The three isogonal lines are concurrent due to *Ceva's theorem*. The relation of pairs of isogonal conjugate points is an *involutoric quadratic Cremona transformation*, its fundamental points being A, B, and C. The transformation is well-defined for all points off the edges of the triangle ABC.

3.2. Isogonal transformation in the hyperbolic plane

In the hyperbolic plane two lines l, l^* through the vertex of an angle are said to be *hyperbolically isogonal* (in short: *h-isogonal*) if they are symmetric in the hyperbolic plane with regard to the h-bisector of this angle. We give the following

Definition 1. If ABC is a triangle and P is a point in the projective plane, off the edges of the triangle, the *hyperbolically isogonal conjugate to* P (*h-isogonal conjugate*) is the intersection point P^* of the lines h-isogonal to the cevians through P.

In order to show that the definition makes sense we have to prove that the three h-isogonal lines of the cevians through the original point are concurrent. Let ABC be an arbitrary triangle in the hyperbolic plane. The h-incentre I is the point of intersection of the inner h-bisectors of the triangle (see Fig. 2). If U is the centre of the absolute circle m there exists a hyperbolic reflection with some axis s which maps the point I onto $U = I'$. As from now we consider the image $A'B'C'$ with incentre $U = I'$. The h-bisectors of the triangle $A'B'C'$ are diameters of the absolute circle m. The h-reflection and the Euclidean reflection in the angle bisectors are identical. Accordingly, the three h-isogonal lines of the cevians through the point P' are concurrent and the h-isogonal conjugate P'^{**} is well-defined. Applying the above-mentioned h-reflection once more we get the h-isogonal conjugate P^* to the original point P. The previous considerations show that the relation between h-isogonal points is associated with some involutoric quadratic Cremona transformation with fundamental points A, B, and C. As the absolute circle m is a conic not incident with the points A, B, and C, the image of m is a curve of order 4 and a hyperbolic point will not necessarily be mapped onto a hyperbolic point. Thus, it will only be reasonable to speak of an *h-isogonal transformation* if the range of the map is extended to the underlying projective plane. Under such assumptions the transformation is well-defined for all points which are not incident with one of the triangle edges.

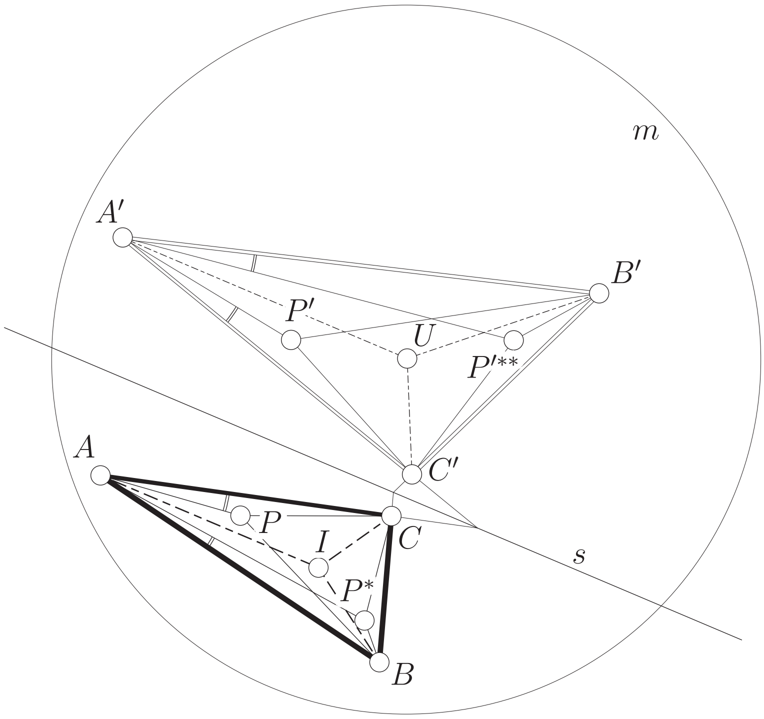

Figure 2: Applying the h-reflection in s provides: The h-isogonal transformation with its fundamental points A, B, C is conjugate to the Euclidean isogonal transformation with A', B', C' as fundamental points.

4. First Kiepert conic in hyperbolic geometry

We get the analogue to the Kiepert hyperbola of an arbitrary triangle ABC in the hyperbolic plane in much the same way as in the Euclidean plane by consistently interpreting each step in hyperbolic geometry and hyperbolic metrics. If ABC is an arbitrary hyperbolic triangle and if $A'BC$, $AB'C$, and ABC' are h-isosceles triangles with base angle ρ attached externally (or internally) to the edges AB, BC, CA, respectively, the points $A'B'C'$ form another triangle called *hyperbolic Kiepert triangle* to ABC. We prove the following

Theorem 1. *The triangle ABC and each hyperbolic Kiepert triangle $A'B'C'$ are perspective from some centre $K(\rho)$: AA', BB', and CC' have one point $K(\rho)$ in common. If the base angle ρ varies between $-\pi/2$ and $\pi/2$ the point $K(\rho)$ remains on a conic k called the first Kiepert conic in the hyperbolic plane (see Fig. 3). The conic k is distinguished as the conic through the vertices A, B, C, the hyperbolic centroid G and the hyperbolic orthocentre H of the triangle ABC.*

Proof. Let $A'BC$, $AB'C$ be h-isosceles triangles with base angle $\rho = \angle^h A'BC = \angle^h CAB'$. So, lines BA' and AB' of the pencils centred at B and A are linked projectively and the vertices A' and B' are projectively related elements of the point ranges on the h-perpendicular bisectors $s_{A'}$ of BC and $s_{B'}$ of CA. The two lines AA' and BB' are pairs of a projective mapping between the pencils of lines centred at A and B. These two pencils generate a conic $k_{AB} = k_{AB}(K^*)$ consisting of all intersection points $AA' \cap BB'$. This conic k_{AB} contains A and B. The h-centroid $G(\rho = 0)$, the h-orthocentre H $(\rho = \pi/2)$, and the third vertex C $(\rho = -\gamma)$ of the triangle ABC are points on the conic k_{AB}. Hence, the conic k_{AB} is determined by the five points A, B, C, G, and H.

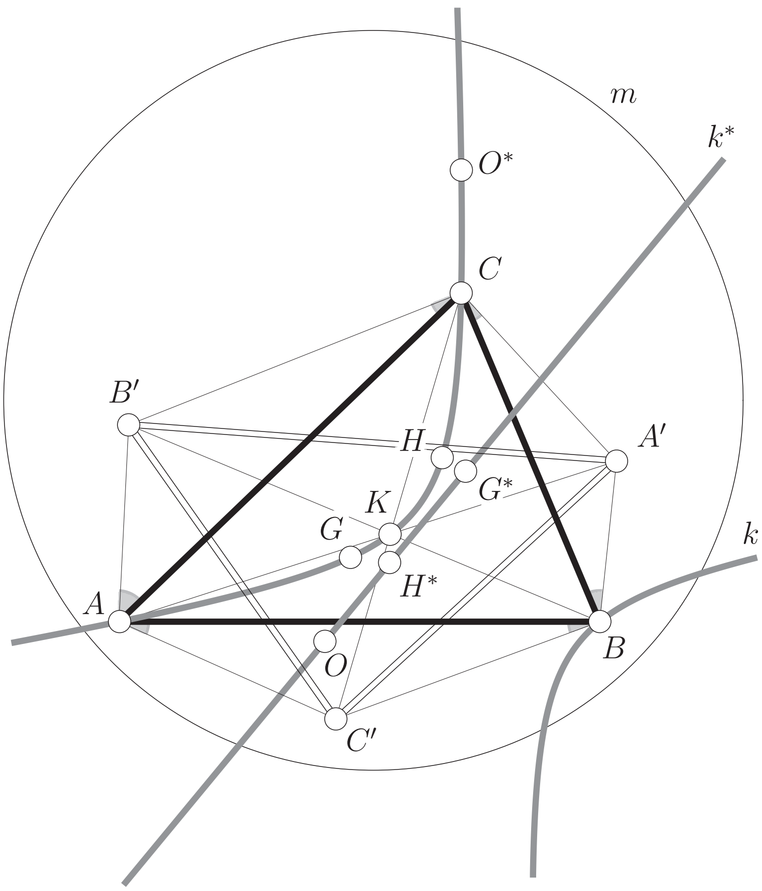

Figure 3: In the hyperbolic plane each Kiepert triangle $A'B'C'$ is perspective to the given triangle ABC from a centre K. All these centres K constitute the *first Kiepert conic* k. k contains the h-centre of gravity G, the h-orthocentre H, and the vertices A, B, C

By cyclic permutation of A, B, C we get another conic $k_{BC} = k_{BC}(K^{**})$ consisting of all intersection points $BB' \cap CC'$. Again, k_{BC} contains the five points B, C, A, G, and H which is why the conics $k_{AB}(K^*)$ and $k_{BC}(K^{**})$ coincide. We denote $k_{AB}(K^*) = k_{BC}(K^{**}) =: k(K)$ which we name the *first Kiepert conic*. As AA', BB', and CC' are concurrent, any pair of triangles ABC and $A'B'C'$ is perspective from a point K on the first Kiepert conic k. □

In Euclidean geometry the Kiepert triangle with base angle $\rho = \pi/2$ degenerates into the three points at infinity on the h-perpendicular bisectors of ABC. In the hyperbolic plane the Kiepert triangle $A'B'C'$ with base angle $\rho = \pi/2$ is the polar triangle of the triangle ABC with respect to the absolute conic m. But in both cases — Euclidean and hyperbolic — the intersection point of the lines AA', BB', and CC' is the orthocentre H of the triangle ABC and H is a point on the Kiepert hyperbola and on the first Kiepert conic. In Euclidean geometry the orthocentre H and the circumcentre O of a triangle are a pair of isogonal points. This is not true in the hyperbolic plane (see Fig. 3). Nonetheless, we can prove

Theorem 2. *Let ABC be a triangle, F_1, D_1, E_1 the midpoints of the edges AB, BC, CA and $m = F_1D_1$, $n = D_1E_1$, $l = E_1F_1$ their connecting lines. The polar[1] triangle $A'B'C'$ to the trilateral lmn is a Kiepert triangle to ABC. These two triangles are perspective from the point O^* which is the h-isogonal point to the h-circumcentre O of ABC.*

Proof. The following considerations are set in *hyperbolic Cayley-Klein geometry*. As opposed to hyperbolic geometry all points of the projective plane — apart from those on the absolute conic m — are points of the model. In this geometry each edge of a triangle ABC determines two h-midpoints. The three pairs of h-midpoints of a triangle constitute a complete quadrilateral consisting of the three pairs of points (F_1, F_{-1}), (D_1, D_{-1}), (E_1, E_{-1}), and four lines m, n, l, and o with collinear triples (F_1, D_1, E_{-1}), (D_1, E_1, F_{-1}), (E_1, F_1, D_{-1}), (D_{-1}, E_{-1}, F_{-1}) ([8, pp. 434]). F_1, D_1, E_1 are interior points of m, i.e., the uniquely determined h-midpoints in the hyperbolic plane — see Fig. 4. Applying the polarity we obtain a new configuration. The six h-midpoints are mapped onto the six h-perpendicular bisectors; the four lines are mapped onto the four h-circumcentres A', B', C', and O of the triangle. A', B', C' are ultra-ideal points in the hyperbolic disk model, but O can either be an ordinary, an ideal or an ultra-ideal point.

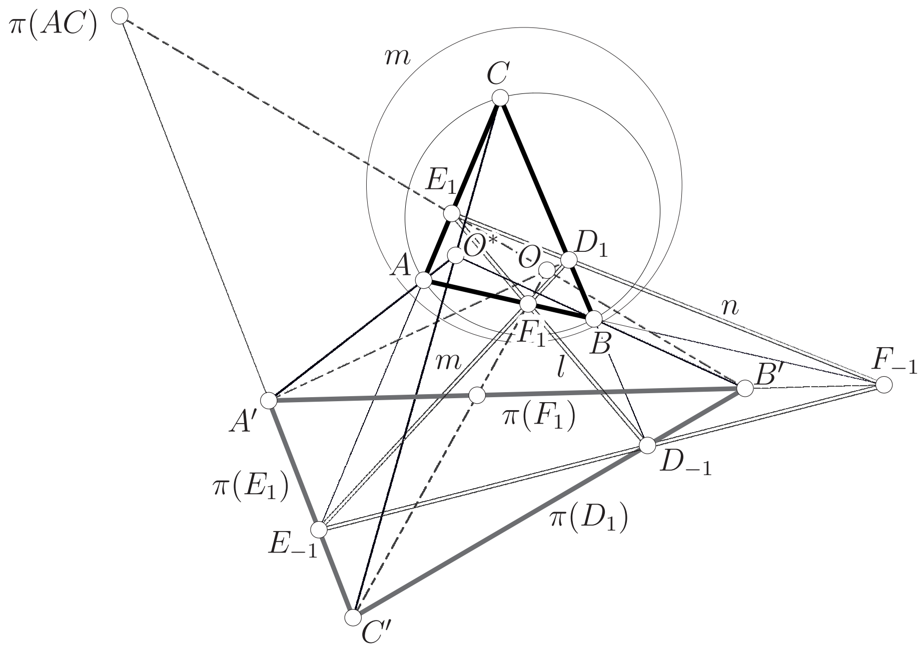

Figure 4: The triangles ABC and $A'B'C'$ are perspective from O^* which is h-isogonal to the h-circumcentre O of ABC.

For the following computation in the projective plane we employ homogeneous coordinates $X = (x_1 : x_2 : x_3)^t$; we also use the vector representation $X = \lambda(x_1, x_2, x_3)^t, \lambda \in \mathbb{R}$. The points at infinity are given by $(x_1 : x_2 : 0)^t$. The absolute circle m with centre $(0 : 0 : 1)^t$ has the equation

$$-x_1^2 - x_2^2 + x_3^2 = X^t \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X = X^t A X = \Omega_X = 0. \tag{1}$$

Interior points of the disk m, i.e., points of the disk model of the hyperbolic plane are characterized by $\Omega_X = -x_1^2 - x_2^2 + x_3^2 > 0$. The h-midpoints of the edges AB, BC, CA are represented by

$$\begin{aligned} F_{\sigma_1} &= \sqrt{\Omega_B} A + \sigma_1 \sqrt{\Omega_A} B \\ D_{\sigma_2} &= \sqrt{\Omega_C} B + \sigma_2 \sqrt{\Omega_B} C \\ E_{\sigma_3} &= \sqrt{\Omega_A} C + \sigma_3 \sqrt{\Omega_C} A \end{aligned} \tag{2}$$

[1]Throughout this paper the concept of *polarity* relates to the absolute conic m of hyperbolic geometry.

with $\sigma_i \in \{1, -1\}$. The values $\sigma_i = 1$ deliver ordinary points in the hyperbolic disk model. Moreover, the points F_{σ_1}, D_{σ_2}, and E_{σ_3} are collinear iff $\sigma_1\sigma_2\sigma_3 = -1$. The equations of the connecting lines $m = F_1D_1, n = D_1E_1, l = E_1F_1$ and $o = D_{-1}E_{-1}$ are

$$\begin{array}{rl} m \ \ldots & \sqrt{\Omega_B}\,|A,C,X| + \sqrt{\Omega_C}\,|A,B,X| + \sqrt{\Omega_A}\,|B,C,X| = u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = 0 \\ n \ \ldots & \sqrt{\Omega_C}\,|B,A,X| + \sqrt{\Omega_A}\,|B,C,X| + \sqrt{\Omega_B}\,|C,A,X| = u_{21}x_1 + u_{22}x_2 + u_{23}x_3 = 0 \\ l \ \ldots & \sqrt{\Omega_A}\,|C,B,X| + \sqrt{\Omega_B}\,|C,A,X| + \sqrt{\Omega_C}\,|A,B,X| = u_{31}x_1 + u_{32}x_2 + u_{33}x_3 = 0 \\ o \ \ldots & -\sqrt{\Omega_B}\,|A,C,X| + \sqrt{\Omega_C}\,|A,B,X| + \sqrt{\Omega_A}\,|B,C,X| = u_{41}x_1 + u_{42}x_2 + u_{43}x_3 = 0. \end{array} \tag{3}$$

For the coordinates of the absolute poles to these lines we have

$$\begin{aligned} B' &= (-u_{11} : -u_{12} : u_{13})^t \\ C' &= (-u_{21} : -u_{22} : u_{23})^t \\ A' &= (-u_{31} : -u_{32} : u_{33})^t \\ O &= (-u_{41} : -u_{42} : u_{43})^t. \end{aligned} \tag{4}$$

To compute these points we can put — without loss of generality – $A = (0 : 0 : 1)^t, B = (\cos\beta : \sin\beta : b)^t$ and $C = (\cos\gamma : \sin\gamma : c)^t$ and we arrive at

$$B' = \lambda \begin{pmatrix} (c - \sqrt{-1 + c^2})\sin\beta - (b + \sqrt{-1 + b^2})\sin\gamma \\ -(c - \sqrt{-1 + c^2})\cos\beta + (b + \sqrt{-1 + b^2})\cos\gamma \\ -\sin(\beta - \gamma) \end{pmatrix} \quad \lambda \in \mathbb{R} \tag{5}$$

and

$$C' = \lambda \begin{pmatrix} (c + \sqrt{-1 + c^2})\sin\beta - (b - \sqrt{-1 + b^2})\sin\gamma \\ -(c + \sqrt{-1 + c^2})\cos\beta + (b - \sqrt{-1 + b^2})\cos\gamma \\ -\sin(\beta - \gamma) \end{pmatrix} \quad \lambda \in \mathbb{R}. \tag{6}$$

Properties of polarity w.r.t. a conic deliver that B' is a point on the h-perpendicular bisector of AC and C' is a point on the h-perpendicular bisector of AB.

To prove the equivalence $\rho = \angle^h CAB' = \angle^h BAC'$ it is sufficient to show that the lines AB' and AC' are symmetric w.r.t. the hyperbolic angle bisectors $w_A, \overline{w}_A$, i.e., the cross ratio of the lines $AB', AC', w_A, \overline{w}_A$ equals -1. Their equations are

$$\begin{array}{lll} AB' & \ldots & u_{12}x_1 - u_{11}x_2 = 0 \\ AC' & \ldots & u_{22}x_1 - u_{21}x_2 = 0 \\ w_A & \ldots & -(\sin\beta + b\sin\gamma)x_1 + (\cos\beta + \cos\gamma)x_2 = 0 \\ \bar{w}_A & \ldots & (\sin\beta - b\sin\gamma)x_1 - (\cos\beta - \cos\gamma)x_2 = 0 \end{array} \tag{7}$$

and, as anticipated, their cross ratio is

$$cr(AB', AC', w_A, \overline{w}_A) = -1. \tag{8}$$

Thus, the triangles $\Delta_1 = AB'C$ and $\Delta_2 = AC'B$ are isosceles with the same base angle ρ.

Cyclic permutation of the vertices in ABC conveys that C' and A' are vertices of triangles $\Delta_2 = BC'A$ and $\Delta_3 = BA'C$ which again are isosceles with base angle $\rho = \angle^h ABC' = \angle^h CBA'$. As Δ_2 appears in both cases, the first and the third triangle are isosceles with the same base angle ρ. $A'B'C'$ is a Kiepert triangle to ABC, perspective with respect to a centre $K = AA' \cap BB' \cap CC'$.

To verify that $K = O^*$ is the h-isogonal image point to the h-circumcentre O of ABC we have to replace B', C' by A', O and repeat the respective calculations. AA' and AO are h-symmetric w.r.t. $w_A, \bar{w}_A$, i.e., h-isogonal. Cyclic permutation delivers (BB', BO) and (CC', CO) are again two pairs of h-isononal lines. This delivers $K = O^*$ and finishes the proof. □

In Euclidean geometry the centre of gravity G of a triangle ABC is isogonal to the symmedian point G^*. The orthocentre H and the circumcentre O are a further pair of isogonal points. The *Brocard axis* r of a triangle ABC is defined as the straight line through the circumcentre O and the symmedian point G^*. The isogonal image of r is a conic through the fundamental points A, B, C of the isogonal quadratic transformation and through the images of O and G^*, i.e., the points $O^* = H$ and G. This conic is the Kiepert hyperbola.

In hyperbolic geometry the *h-Brocard axis* r to a triangle ABC is again defined as the straight line through the h-circumcentre O and the h-symmedian point G^*. As opposed to the Euclidean case, the h-isogonal image of the h-circumcentre O is different from the h-orthocentre H.

Thanks to Theorem 2 we can still prove

Theorem 3. *Let ABC be an arbitrary triangle in the hyperbolic plane. The h-isogonal image of the h-Brocard axis r is the first Kiepert conic (Fig. 3).*

Proof. The h-isogonal image of the h-Brocard axis r is a conic through the fundamental points A, B, C of the h-isogonal quadratic transformation and through the h-centre of gravity G as does the first Kiepert conic k. According to Theorem 2 the h-isogonal point O^* to the h-circumcentre O lies on k and, obviously, on the h-isogonal image of r. This image conic and the first Kiepert conic k coincide as they have five points in common. □

5. A theorem on perspective triangles

The following considerations are set in the real projective plane.

Theorem 4. *Let k be a conic and ABC an inscribed triangle. a', b', c' be three different straight lines through a point R, $a' \not\ni A$, $b' \not\ni B$, $c' \not\ni C$. Every point $K \in k$ defines a further triangle $A' = KA \cap a'$, $B' = KB \cap b'$, $C' = KC \cap c'$, i.e., $K \in k$ determines a triangle $A'B'C'$ which is perspective to ABC (see Fig. 5). Then we have*

a) The edges AB, BC, CA of ABC and $d_1 := [(AB \cap a'), (CB \cap c')]$, $d_2 := [(BC \cap b'), (AC \cap a')]$, $d_3 := [(CA \cap c'), (BA \cap b')]$ are tangents to a conic l.

b) To any pair of perspective triangles ABC and $A'B'C'$ the corresponding Desargues axis $d(K)$ is tangent to the conic l.

Proof. Ad a) It is possible to choose some appropriate projective coordinate system in the real projective plane such that the conic k is described by

$$k \ \ldots \ x_1x_2 - x_3^2 = 0. \tag{9}$$

The vertices of the triangle ABC can be chosen as $A = (0\colon 1\colon 0)^t$, $B = (0\colon 0\colon 1)^t$, $C = (1\colon 1\colon 1)^t$. Additionally, the equations of the straight lines a', b', c' can be written as

$$\begin{aligned} a' \ &\ldots \ a_1x_1 + a_2x_2 + a_3x_3 = 0 \\ b' \ &\ldots \ b_1x_1 + b_2x_2 + b_3x_3 = 0 \\ c' \ &\ldots \ c_1x_1 + c_2x_2 + c_3x_3 = 0. \end{aligned} \tag{10}$$

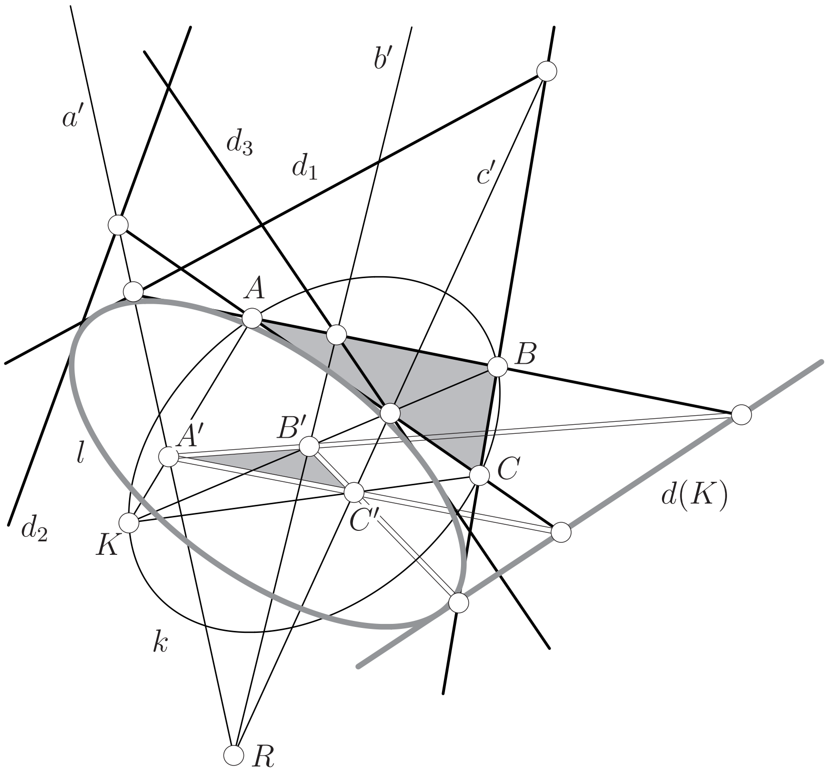

Figure 5: Each Desargues axis $d(K)$ is tangent to the conic l.

The lines a', b', c' are concurrent but different. Hence, the rank of the system of linear equations is 2 and its coefficient determinant Δ (10) vanishes, i.e.

$$\Delta = -a_3b_2c_1 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 + a_1b_2c_3 \quad = \quad 0. \tag{11}$$

The conic l is defined by the tangents AB, BC, CA, and the lines $d_1 = [AB \cap a', BC \cap c']$ and $d_2 = [AC \cap a', BC \cap b']$. We compute its equation in terms of dual coordinates u_1, u_2, u_3

$$q(u_1, u_2, u_3) := B_{12}(u_2^2 + u_1u_2) + B_{13}(u_3^2 + u_1u_3) + B_{23}u_2u_3 = 0 \tag{12}$$

with

$$\begin{aligned} B_{12} &= b_3[-a_3(c_1 + c_2) + (a_1 + a_2)c_3] \\ B_{13} &= a_2[b_3(c_1 + c_2) - (b_1 + b_2)c_3] \\ B_{23} &= a_3(b_1 + b_2)(c_1 + c_2) - (a_1 + a_3)b_3(c_1 + c_2) - (a_2 - a_3)(b_1 + b_2)c_3. \end{aligned} \tag{13}$$

The dual coordinates of d_3 are

$$d_3 \;\ldots\; (b_2c_1 - b_3c_2 + b_2c_3 : b_2c_2 : b_3c_2). \tag{14}$$

Due to (11) the coordinates (14) fulfill the equation (12). This implies: d_3 is tangent to l.

Ad b) The conic k can be parameterized as $K = K(t_0, t_1) = (t_0t_1 : t_0^2 : t_1^2)$. The Desargues axis $d(t_0, t_1)$ to the perspective triangles ABC and $A'B'C'$ coincides with the points $I = BC \cap B'C'$, $II = CA \cap C'A'$ and $III = AB \cap A'B'$. In order to determine the the dual coordinates of $d(t_0 : t_1)$ we compute

$$II(t_0 : t_1) = \begin{pmatrix} a_2(c_2t_0^2 + c_1t_0t_1 + c_3t_1^2) \\ (*) \\ a_2(c_2t_0^2 + c_1t_0t_1 + c_3t_1^2) \end{pmatrix} \tag{15}$$

with

$$(*) = -a_2(c_1+c_3)t_0^2 - [a_1(c_1+c_2+c_3) - a_2c_1]t_0t_1 - [a_3(c_1+c_2+c_3) - a_3c_3]t_1^2 \tag{16}$$

and

$$III(t_0 \colon t_1) = \begin{pmatrix} 0 \\ -b_3(a_2t_0^2 + a_1t_0t_1 + a_3t_1^2) \\ a_2(b_2t_0^2 + b_1t_0t_1 + b_3t_1^2) \end{pmatrix}. \tag{17}$$

The dual coordinates $u_i = u_i(t_0 : t_1),\ i = 1, 2, 3$, are homogeneous polynomials of degree 4 in the variables $t_0 : t_1$.

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} (**) \\ -a_2(b_2t_0^2 + b_1t_0t_1 + b_3t_1^2)(c_2t_0^2 + c_1t_0t_1 + c_3t_1^2) \\ -b_3(a_2t_0^2 + a_1t_0t_1 + a_3t_1^2)(c_2t_0^2 + c_1t_0t_1 + c_3t_1^2) \end{pmatrix} \tag{18}$$

with

$$\begin{aligned}(**) = {} & -t_1(a_1t_0 + a_3t_1)\left\{b_2(c_1+c_2+c_3)t_0^2 + b_1(c_1+c_2+c_3)t_0t_1 - b_3(t_0-t_1)\cdot\right. \\ & \cdot [c_1t_1 + c_2(t_0+t_1)]\} - a_2\left\{b_2t_0^2(t_0-t_1)[c_1t_0 + c_3(t_0+t_1)] + b_1t_0(t_0-t_1)t_1\cdot\right. \\ & \left.\cdot [c_1t_0 + c_3(t_0+t_1)] - b_3[c_2t_0^4 + t_1(c_3t_1^3 + c_1t_0(t_0^2 - t_0t_1 + t_1^2))]\right\}.\end{aligned} \tag{19}$$

Eq. (18) describes the complete set of Desargues axes $\{d(K)|K \in k\}$. Substituting the dual coordinates (18) of all Desargues axes $d(K)$ into the polynomial function $q(u_1, u_2, u_3)$ defined by (12) we obtain

$$\begin{aligned} q(u_1(t_0,t_1), u_2(t_0,t_1), u_3(t_0,t_1)) = q^*(t_0,t_1) = a_2b_3(c_1+c_2+c_3)^3\Delta\cdot \\ \cdot t_0^3\,(t_0-t_1)^5\,t_1^2\,(a_2t_0^2 + a_1t_0t_1 + a_3t_1^2)(b_2t_0^2 + b_1t_0t_1 + b_3t_1^2)(c_2t_0^2 + c_1t_0t_1 + c_3t_1^2). \end{aligned} \tag{20}$$

As one of the factors of (20) is the determinant Δ from (11) the polynomial function (20) vanishes identically. As a consequence, each Desargues axis $d(K)$ belongs to l which finishes the proof. □

As an application of Theorem 4 we can now prove the existence of the second Kiepert conic in the hyperbolic plane.

6. Second Kiepert conic in hyperbolic geometry

According to Theorem 1 a triangle ABC and each Kiepert triangle $A'B'C'$ are perspective from a centre $K(\rho)$ on the first Kiepert conic k which is circumscribed to the triangle ABC. The h-perpendicular bisectors s_c, s_a, s_b of the triangle edges AB, BC, and CA are three straight lines through the h-circumcentre O.

This is why the preconditions of Theorem 4 are fulfilled and the following theorem can be viewed as a direct consequence of Theorem 4.

Theorem 5. *Let ABC be a triangle in the hyperbolic plane and $A'B'C'$ any of its Kiepert triangles. Then the envelope of the axis $d(\rho)$ is a conic l inscribed to the triangle ABC. l is called the second Kiepert conic in the hyperbolic plane.*

7. Summary and outlook

In this paper we have shown that any triangle in hyperbolic geometry determines a first and a second Kiepert conic. These two conics are the hyperbolic counterpart to the Kiepert hyperbola and Kiepert parabola in Euclidean geometry.

It takes a combination of synthetic considerations and analytical methods to find out propositions in real projective geometry which permit an interesting interpretation in a non-Euclidean context.

Having gathered this experience we are determined to investigate triangles and their Kiepert conics in the more general context of regular Cayley-Klein geometries. The more general viewpoint leads to a quadruple of first Kiepert conics and, consequently, a quadruple of second Kiepert conics.

In affine Cayley-Klein geometries, however, the results are a good deal different. There are some investigation for the Euclidean geometry and the isotropic plane revealing phenomenons which do not have an analogon in regular Cayley-Klein geometries. E.g., Napoleon's theorem holds in the Euclidean as well as in the isotropic plane (cp. [7], [9]), pretty much in contrast to the hyperbolic and elliptic case.

8. Acknowledgment

We gratefully thank one of the reviewers for valuable remarks which finally led to a major improvement of this paper.

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Received August 9, 2010; final form April 25, 2012