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Archimedean Circles Induced by Skewed Arbeloi

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Abstract. Infinite Archimedean circles touching one of the inner circles of the arbelos are induced by skewed arbeloi.

Key Words: arbelos, skewed arbelos, infinite Archimedean circles
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1. Introduction

On the problem to find infinite Archimedean circles of the arbelos, the following four have priority to be considered:

(P1) to find infinite Archimedean circles touching the outer circle of the arbelos,
(P2) to find infinite Archimedean circles passing through the tangent point of the two inner circles,
(P3) to find infinite Archimedean circles touching the radical axis of the inner circles, and
(P4) to find infinite Archimedean circles touching one of the inner circles.

Solutions of (P1), (P2) and (P3) are given in [4], [6] and [2], respectively. But no solutions can be found for (P4). In this note we give a solution of this problem.

Let O be a point on a segment AB and α, β and γ be the circles with diameters OA, OB and AB, respectively. We show that infinite Archimedean circles of the arbelos formed by the circles α, β and γ can be obtained by considering arbitrary circles touching α and β at points different from O. Let a and b be the respective radii of the circles α and β. We set up a rectangular coordinate system with origin O so that A and B have coordinates $(2a, 0)$ and $(-2b, 0)$, respectively. We consider variable circles touching α and β at points different from O. Such a circle γ_z is expressed by the equation

$$\left(x - \frac{b-a}{z^2-1}\right)^2 + \left(y - \frac{2z\sqrt{ab}}{z^2-1}\right)^2 = \left(\frac{a+b}{z^2-1}\right)^2 \tag{1}$$

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for a real number $z \neq \pm 1$ [5]. The circle γ_z touches α and β internally if $|z| < 1$ and externally if $|z| > 1$. The external common tangents of α and β are expressed by the equation

$$(a-b)x \mp 2\sqrt{ab}y + 2ab = 0. \tag{2}$$

They are denoted by $\gamma_{\pm 1}$, where the double-signs correspond. The configuration $(\alpha, \beta, \gamma_z)$ is called a skewed arbelos. We denote the radii of Archimedean circles of the arbelos (α, β, γ) by r_{A}, i.e., $r_{\mathrm{A}} = ab/(a+b)$.

2. Archimedean circles

In this section we show that a skewed arbelos $(\alpha, \beta, \gamma_z)$ gives Archimedean circles of the arbelos (α, β, γ). For any two points V and W let (VW) denote the circle with diameter VW and $\mathcal{P}_W$ denote the perpendicular to the line AB passing through W in the case W lying on AB.

Lemma 1. *The following circles have radii $|AW||BV|/(2|AB|)$ for points V and W on the line AB.*

(i) *The circles touching (AV) externally, (AB) internally and $\mathcal{P}_W$ from the side opposite to B in the case V and W lying on the segment AB.*

(ii) *The circles touching (AB) and (AV) externally and $\mathcal{P}_W$ from the side same as A in the case A lying on the segments BW and BV.*

Proof. Let r be the radii of the touching circles and v and w be the respective x-coordinates of the points V and W (see Fig. 1). In the case (i), we get

$$((a+b)-r)^2 - ((a-b)-(w+r))^2 = ((2a-v)/2+r)^2 - ((2a+v)/2-(w+r))^2.$$

Solving the equation we get

$$r = \frac{(2a-w)(2b+v)}{4(a+b)} = \frac{|AW|BV|}{2|AB|}.$$

The other case is proved similarly. □

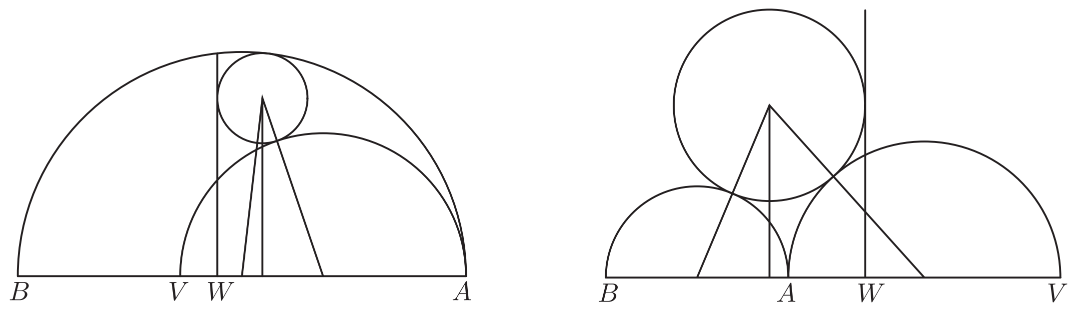

Figure 1: Circles of radii $|AW||BV|/(2|AB|)$

Let A_z be the tangent point of the circles α and γ_z. The point B_z is defined similarly (see Figs. 2, 3 and 4). The coordinates of A_z are

$$\left(\frac{2ab}{az^2+b}, \frac{2az\sqrt{ab}}{az^2+b}\right)$$

(see [3]). Let $\mathcal{H}_\alpha$ be the line perpendicular to the line joining the centers of the circles α and γ_z and passing through O. The line $\mathcal{H}_\beta$ is defined similarly. They are expressed by the equations

$$(az^2 - b)x - 2z\sqrt{ab}y = 0 \quad \text{and} \quad (a - bz^2)x - 2z\sqrt{ab}y = 0 \tag{3}$$

respectively. It coincides with the radical axis of α and β if $z = 0$. If γ_z is one of the external common tangents of α and β, we regard that $\mathcal{H}_\alpha$ is the line parallel to the tangent and passing through O. The line $\mathcal{H}_\beta$ is defined similarly. The foot of the perpendicular to $\mathcal{H}_\alpha$ from the point A_z is denoted by F_α. The distance between A_z and $\mathcal{H}_\alpha$ is

$$|A_zF_\alpha| = \frac{2ab}{az^2 + b}. \tag{4}$$

Theorem 2. *The circles touching γ_z, α and $\mathcal{H}_\alpha$ from the same side as A_z and different from the circle (A_zF_α) are Archimedean circles of the arbelos (α, β, γ).*

Proof. We assume that γ_z touches α and β internally. Let S and T be the remaining intersections of α and γ_z with the line A_zF_α respectively. Then by (1) we get

$$|ST| = 2\frac{a+b}{1-z^2} - 2a = \frac{2(az^2+b)}{1-z^2}.$$

Therefore by (i) of Lemma 1 with (4), the radii of the touching circles are

$$\frac{|A_zF_\alpha| \cdot |ST|}{2|A_zT|} = \frac{2ab}{az^2+b}\frac{2(az^2+b)}{1-z^2}\frac{1-z^2}{4(a+b)} = r_\mathrm{A}.$$

The case in which γ_z touches α and β externally can be proved in a similar way. If γ_z is one of the external common tangents, the distance between γ_z and O is $2r_\mathrm{A}$ by (2). □

For an Archimedean circle δ touching α externally at a point different from O, there is a circle touching α, β and δ at points different from O. Let γ_z be this circle. Then the skewed arbelos $(\alpha, \beta, \gamma_z)$ gives the Archimedean circle δ by Theorem 2. Hence any Archimedean circle touching α externally at points different from O is obtained by Theorem 2.

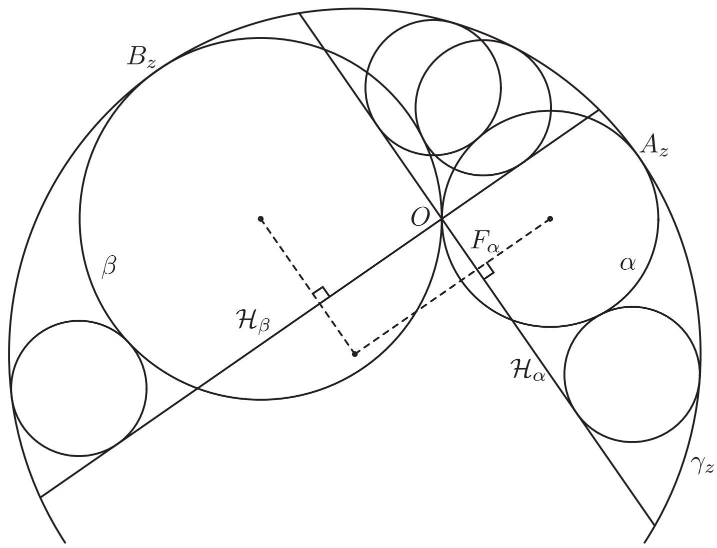

Figure 2: $|z| < 1$

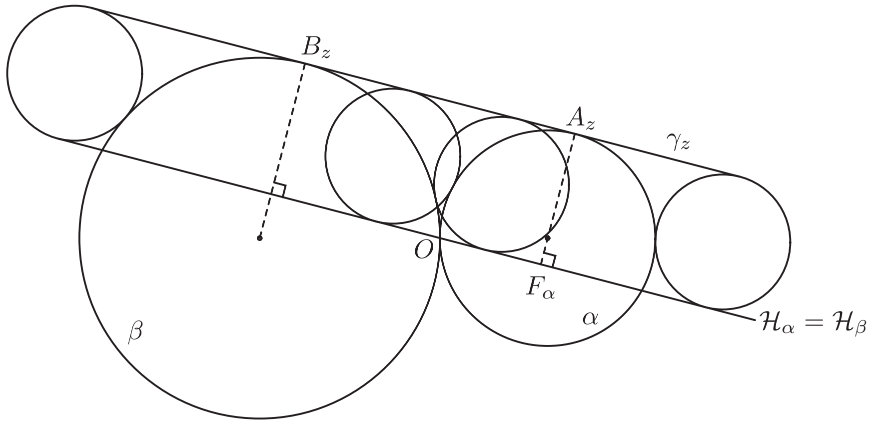

Figure 3: $|z| = 1$

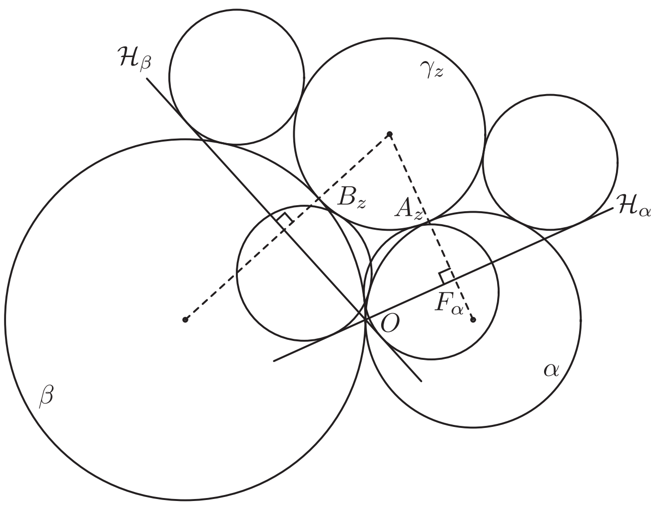

Figure 4: $|z| > 1$

3. A special case

For a skewed arbelos $(\alpha, \beta, \gamma_z)$, we get four Archimedean circles two of which touch α and the other two touch β. We now consider the special case in which two of them coincide. There are four such cases. But two of them are obtained by reflecting the remaining two in the line AB. Therefore we consider the two cases $z > 0$.

In the remaining of this section, the double-signs correspond. If γ_s coincides with the coincident Archimedean circles[1], then $(a+b)/(s^2-1) = r_{\mathrm{A}}$ and $s > 1$. Solving the equation, we get $s = \sqrt{(a^2+3ab+b^2)/(ab)}$. While two circles γ_z and γ_w touch if and only if $z - w = \pm t$, where $t = (a+b)/\sqrt{ab}$ [5]. Therefore if the skewed arbelos $(\alpha, \beta, \gamma_z)$ gives the Archimedean circle γ_s, we get $z = s \pm t$. In this case, the y-coordinate of the center of γ_z is $2z\sqrt{ab}/(z^2-1) = \pm r_{\mathrm{A}}$. This implies that the circle concentric to γ_z and touching AB is an Archimedean circle of the arbelos (α, β, γ) (see Fig. 5). Let θ_α be the angle that the line $\mathcal{H}_\alpha$ makes with the x-axis. The angle θ_β is defined similarly. Then we get $\tan(\theta_\alpha - \theta_\beta) = \mp 4/3$ in the case $z = s \pm t$ by (3). Hence the angle between the lines $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ is constant for valuable a and b in this case.

[1]The circle is denoted by U_1 in [1].

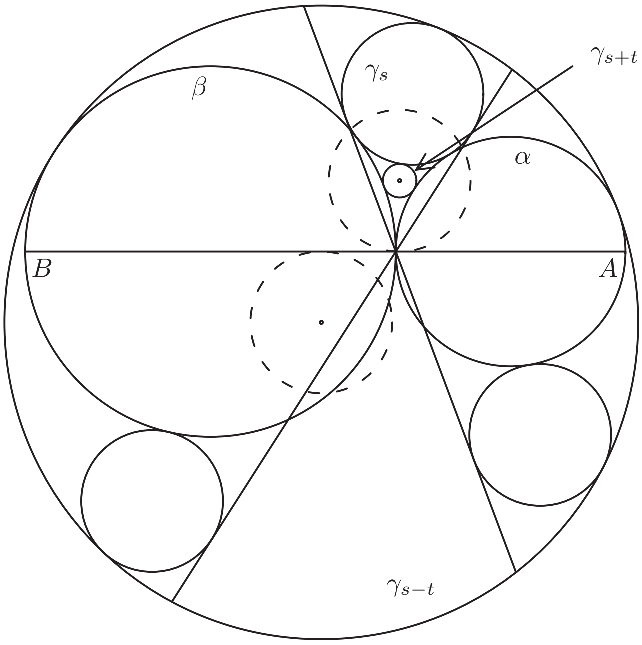

Figure 5: Two more Archimedean circles in the coincident case

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