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Curvature Lines and Normal Congruences of Triangular Bézier Patches

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Abstract. This paper aims at giving a self-contained description of the focal surfaces of the normal congruence of a triangular Bézier patch in terms of the control points of the patch. The normal congruence of a surface is an Euclidean concept and it is algebraic, if the original surface is algebraic. To calculate the parameter representation of the normal congruence of a triangular Bézier patch we need derivatives and normal vectors of the patch. For calculating the pair of so called focal points on each generator of the congruence, one has to investigate the curvature lines of the patch in addition. Thus the results become already of high algebraic order even for quadratic or cubic patches. Therefore the treatment is restricted to quadratic and cubic triangular Bézier patches and focal points of its normal congruence are calculated only for the normals at very special points of the patch: the corner points and the "midpoint". Therewith one can deduce control point systems for 'low' order approximations of the two patches of focal surfaces of the normal congruence. Finally the calculations are applied to a numerical example. The paper is a first attempt to deal with the focal surfaces of a line congruence explicitly and might deliver fundamentals to treat refraction congruences, which have applications in geometric optics.

Key Words: triangular Bézier patch, normal congruence, curvature lines, focal points and surfaces.

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1. Introduction

Gaussian curvature and mean curvature of a rectangular or triangular Bézier patch is already well discussed, see e.g. [2, 12]. Even triangular Bézier patches (TBP) belong to common

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knowledge in CAGD, we here again recall basic facts for TBP as well as for differential geometric concepts to make the paper self contained. Most of Bézier-type approaches to describe an arbitrary surface S rely on piecewise defined surface patches over more or less regular triangulations, because it is always possible to triangulate an object. In fact, there are well-defined methods in computational geometry for constructing triangulations of a set of points (or more generally a simplex grid of hypersurfaces in n-spaces). Via smoothness conditions for the transitions from one patch to adjacent patches follow "inner control points" and finally, by subdivision one can create any desired refinement of the triangular grid on S, see [3, 9].

While TBP belongs to affine geometry, the normal congruence of a surface is an Euclidean concept and it is algebraic, if the original surface is algebraic. To calculate the parameter representation of the normal congruence of a triangular Bézier patch we need derivatives and normal vectors of the patch. Calculating the pair of so called *focal points* on each generator of the congruence affords another differentiation step. Thus the results become already of high algebraic order even for quadratic or cubic patches. So one might aim at a description of the focal surfaces by approximating patches of low order and rather improve the situation by subdividing the given TBP. Therefore the treatment here is restricted to quadratic and cubic triangular Bézier patches and the pair of focal points (and planes) of its normals are calculated only for the normals at very special points of the patch: the corner points and the "midpoint". By this one can deduce the control schemes for low order approximations of the pair of focal surfaces. There remain still many problems to be solved: e.g., detection of focal curves or how to handle ideal focal points and transition of the focal surfaces through infinity. Problems like these are reserved to another place.

Let us start with collecting well-known statements for TBP to introduce the symbols and the calculus we shall use: Univariate Bernstein polynomials are terms of the binomial expansion of $[t + (1 - t)]^n$. In the two-dimensional case, (generalized) Bernstein polynomials B^n_{ijk} are defined by

$$\begin{aligned} &B^n_{ijk}(u, v, w) = \frac{n!}{i!\, j!\, k!}\, u^i v^j w^k, \quad i + k + j = n, \\ &\text{whereby } 0 \le u, v, w \le 1 \text{ and } u + v + w = 1. \end{aligned} \tag{1}$$

We define $B^n_{ijk}(u, v, w) = 0$ if some of the indices in (i, j, k) are negative. Bernstein polynomials satisfy the following recursion:

$$B^n_{ijk}(U) = uB^{n-1}_{i-1,j,k}(U) + vB^{n-1}_{i,j-1,k}(U) + wB^{n-1}_{i,j,k-1}(U) \tag{2}$$

whereby U abbreviates the parameter triplet (u, v, w). The Bernstein polynomials fulfill the following conditions:

$$\begin{aligned} &\text{(A)} \quad \sum_{i,j,k\ge 0}^{i+j+k=n} B^n_{ijk}(U) \equiv 1, \quad u + v + w = 1. \\ &\text{(B)} \quad B^n_{ijk}(U) \ge 0 \quad \forall U \in \Delta; \quad \Delta \text{ is a triangle.} \end{aligned}$$

A *"triangular" Bézier patch* (TBP) is defined in terms of such trivariate Bernstein polynomials with an additional linear condition for the parameters (see Eq. (1)) as a set of convex combinations of control points P_{ijk}:

$$P(u, v, w) = \sum_{i+j+k=n} P_{ijk} B^n_{ijk}(u, v, w), \quad 1 \le u, v, w \ge 0, \quad u + v + w = 1. \tag{3}$$

Such a triangular patch is the result of a mapping of a "parameter triangle" with barycentric coordinates (u, v, w) into the d-space spanned by P_{ijk}. In case $d = 3$ the patch (4) belongs to an algebraic surface of order n.

Triangular patches can be joined smoothly along their common edge, see [2] and [4], whereby the order r of G^r-continuity depends on the r rows of control points of both patches along this common edge.

Remark 1. If we apply the local reparametrization:

$$s = u,\ t = \frac{v}{1-u} = \frac{v}{v+w}, \quad i+j+k = n,$$

to $P(u, v, w)$, the domain of the triangular patch $\{(u, v) \mid\ 0 \le u,\ v,\ u+v \le 1\}$ is transformed into the square $[0, 1] \times [0, 1]$, and we have

$$v = t(1-s),\ w = (1-t)(1-s).$$

Hence,

$$\begin{aligned}
\frac{n!}{i!\,j!\,k!}\,u^i v^j w^k &= \frac{n!}{i!\,j!\,k!}\,s^i t^j (1-s)^j (1-s)^k (1-t)^k \\
&= \frac{n!}{i!\,j!\,k!}\,s^i (1-s)^{j+k} t^j (1-t)^k \\
&= \frac{n!}{i!j!(n-i-j)!}\,s^i (1-s)^{n-i} t^j (1-t)^{n-i-j} \\
&= \frac{n!}{i!(n-i)!} \cdot \frac{(n-i)!}{j!\,(n-i)-j)!}\,s^i (1-s)^{n-i} t^j (1-t)^{(n-i)-j} \\
&= B_i^n(s) B_j^{n-i}(t).
\end{aligned}$$

It is clear that this describes the same patch as a ("rectangular") tensor product Bézier patch (TP-BP), but the re-parametrization changed the degree of the patch and it is singular in one corner.

Remark 2. Given three non-collinear points A, B, C in a plane, any other point P is determined by its barycentic coordinates (α, β, γ) as the linear combination

$$P = \alpha A + \beta B + \gamma C,$$

where $\alpha + \beta + \gamma = 1$. Recall, if $0 \le \alpha, \beta, \gamma \le 1$, then the point P lies within the triangle formed by vertices A, B, C. The construction of a triangular patch is based upon deriving a de Casteljau's type algorithm with recursion based on triangles instead of quadrangles. The structure of the control points is defined by a triply indexed set of points $P_{i,j,k}$ with $i + j + k = n$ forming a triangular grid of size n (see Fig. 1).

2. Directional derivatives and normal vectors

Even one knows the tangent plane at each of the corner points of the triangular patch, namely the plane spanned by the corner and the two adjacent control points, it might be useful to recall the concept of the so-called "directional derivative" of a patch in general. When

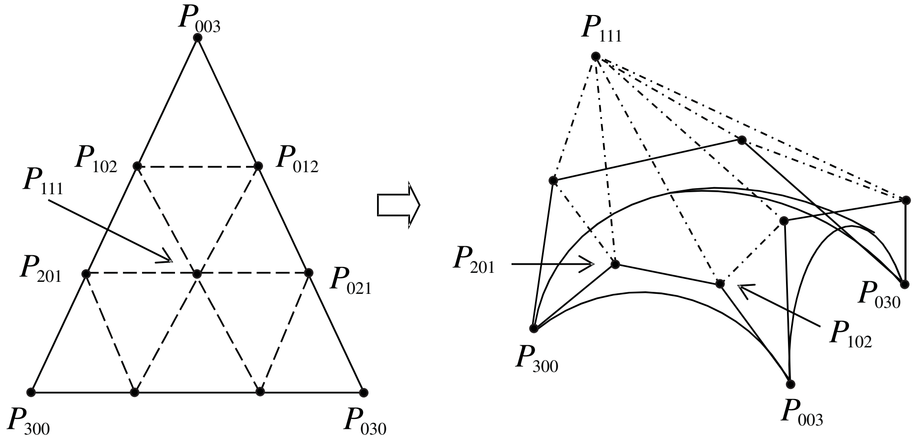

Figure 1: A cubic triangular patch grid.

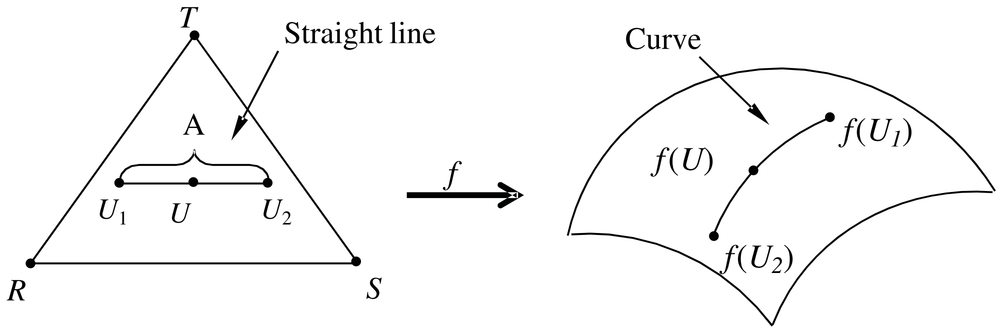

Figure 2: The segment A ("direction") is mapped onto a curve on the surface.

we discussed derivatives for tensor product (TP-) Bézier patches [10], we considered partial derivatives. The situation is different for triangular patches; the appropriate derivatives here are the "directional derivatives".

In Fig. 2, a geometric interpretation of the notion of the directional derivative shows that a straight line $U(t) = U + tA$ through a point U in the domain with direction vector $A := U_2 - U_1$ is mapped onto a curve $X(U(t))$ on the surface X. The tangent vector of this curve at $X(U)$ is the desired directional derivative. We follow [2] and use *"Farin abbreviations"* for index triplets denoting the point P_{ijk} by $P_{\mathbf{i}}$. We also use the abbreviations $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ and $|\mathbf{i}| = i + j + k, \;\; i, j, k \geq 0$.

Then we can give the r^{th} directional derivative of a Bézier triangle in a point $P(U)$ as follows:

$$D_A^r P^n(U) = \frac{n!}{(n-r)!} \sum_{|\mathbf{j}|=r} P_{\mathbf{j}}^{n-r}(U) B_{\mathbf{j}}^r(A). \tag{4}$$

A dual result is given by

$$D_A^r P^n(U) = \frac{n!}{(n-r)!} \sum_{|\mathbf{j}|=n-r} P_{\mathbf{j}}^r(A) B_{\mathbf{j}}^{n-r}(U). \tag{5}$$

For $r = 1$, the terms $P_{\mathbf{j}}^1(A)$ in Eq. (5) have a simple geometric interpretation, since $P_{\mathbf{j}}^1(A) = dP_{\mathbf{j}+e_1} + eP_{\mathbf{j}+e_2} + fP_{\mathbf{j}+e_3}$, whereby d, e, f are "barycentric coordinates" with respect to the triangles $P_{\mathbf{j}+e_1}, P_{\mathbf{j}+e_2}, P_{\mathbf{j}+e_3}$ and $|\mathbf{j}| = n-1$. The directional derivative of P^n is thus a triangular patch whose coefficients are the images of A on each sub-triangle in the control net (see Fig. 1).

Again, let us set $r = 1$ in Eq. (4). Then,

$$D_A P^n(U) = n \sum_{|\mathbf{j}|=1} P_{\mathbf{j}}^{n-1}(U) B_{\mathbf{j}}^1(A) = n(dP_{e_1}^{n-1} + eP_{e_2}^{n-1} + fP_{e_3}^{n-1}). \tag{6}$$

Since this is true for all directions $A \in \mathrm{E}^2$, it follows that $P_{e_1}^{n-1}, P_{e_2}^{n-1}, P_{e_3}^{n-1}$ define the *tangent plane* at $P^n(U)$, in particular the three vertices $P_{0n0}, P_{0,n-1,1}, P_{1,n-1,0}$ span the tangent plane at P_{0n0} with analogous results for the remaining two corners. Also, we see that the de Casteljau algorithm produces derivative information as a byproduct of the iteration process.

Now, we can compute the first directional derivative at the corner control points of a triangular Bézier patch in the direction of the two boundary curves ending at each corner and then we can compute the normal vectors at these points: The directional derivative with respect to A_1 and A_2 along $u = 0$ is given by

$$\begin{aligned} D_{A_1} X^n(0, v, 1-v) &= n \sum_{j+k=n-1} P_{0jk}(A_1) B_{0jk}^{n-1}(0, v, 1-v), \\ D_{A_2} X^n(0, 1-w, w) &= n \sum_{j+k=n-1} P_{0jk}(A_2) B_{0jk}^{n-1}(0, 1-w, w), \end{aligned} \tag{7}$$

and for $v = 0$ and $w = 0$ we have similar expressions

$$\begin{aligned} D_{A_1} X^n(u, 0, 1-u) &= n \sum_{i+k=n-1} P_{i0k}(A_1) B_{i0k}^{n-1}(u, 0, 1-u), \\ D_{A_2} X^n(1-w, 0, w) &= n \sum_{i+k=n-1} P_{i0k}(A_2) B_{i0,}^{n-1}(1-w, 0, w), \end{aligned}$$

and

$$\begin{aligned} D_{A_1} X^n(u, 1-u, 0) &= n \sum_{i+j=n-r} P_{ij0}(A_1) B_{ij0}^{n-1}(u, 1-u, 0), \\ D_{A_2} X^n(1-v, v, 0) &= n \sum_{i+j=n-r} P_{ij0}(A_2) B_{ij0}^{n-1}(1-v, v, 0). \end{aligned}$$

Now, we will specialize the point (u, v, w) to a corner point and the vectors A_i passing through shall belong to the borders of the Bézier patch. (In the following we will restrict the explicit calculation to the cubic case only.)

The cubic case ($n = 3$):
Directional derivatives with respect to the vectors A_1 and A_2 evaluated along $u = 0$ are

$$\begin{aligned} D_{A_1} X^3(0, v, 1-v) &= 3\,\{P_{002}^1(A_1) B_{002}^2(0, v, 1-v) + P_{020}^1(A_1) B_{020}^2(0, v, 1-v) \\ &\quad + P_{011}^1(A_1) B_{011}^2(0, v, 1-v)\} \\ &= 3\,\{(d_1 P_{102} + e_1 P_{012} + f_1 P_{003})\,(1-v)^2 \\ &\quad + (d_1 P_{120} + e_1 P_{030} + f_1 P_{021})\,v^2 \\ &\quad + 2\,(d_1 P_{111} + e_1 P_{021} + f_1 P_{012})\,v(1-v)\} \end{aligned} \tag{8}$$

$$\begin{aligned} D_{A_2}X^3(0,1-w,w) &= 3\left\{P^1_{0,0,2}(A_2)B^2_{0,0,2}(0,1-w,w)+P^1_{0,2,0}(A_2)B^2_{0,2,0}(0,1-w,w)\right.\\ &\quad \left.+P^1_{0,1,1}(A_2)B^2_{0,1,1}(0,1-w,w\ )\right\}\\ &= 3\left\{(d_2P_{102}+e_2P_{012}+f_2P_{003})\,w^2\right.\\ &\quad +(d_2P_{120}+e_2P_{030}+f_2P_{021})\,(1-w)^2\\ &\quad \left.+2\,(d_2P_{111}+e_2P_{021}+f_2P_{012})\,w(1-w)\right\} \end{aligned} \tag{9}$$

The directional derivatives along $v=0$ and along $w=$ are similar expressions.

Specializing the coefficient triplets d_i, e_j, f_k in equations (8) and (9) according to the coordinates of the corner points as well as of the tangent directions and using the forward difference operator D we receive the directional derivatives at these special Bézier control points as expected (see Fig. 3):

$$\text{At } P_{300}: \quad \begin{aligned} D_{(1,-1,0)}X^3(1,0,0) &= 3\Delta\left(P_{300},P_{210}\right),\\ D_{(1,0,-1)}X^3(1,0,0) &= 3\Delta\left(P_{300},P_{201}\right). \end{aligned} \tag{10}$$

$$\text{At } P_{030}: \quad \begin{aligned} D_{(-1,1,0)}X^3(0,1,0) &= 3\Delta\left(P_{030},P_{120}\right),\\ D_{(0,1,-1)}X^3(0,1,0) &= 3\Delta\left(P_{030},P_{021}\right). \end{aligned} \tag{11}$$

$$\text{At } P_{003}: \quad \begin{aligned} D_{(0,-1,1)}X^3(0,0,1) &= 3\Delta\left(P_{003},P_{012}\right),\\ D_{(-1,0,1)}X^3(0,0,1) &= 3\Delta\left(P_{003},P_{102}\right). \end{aligned} \tag{12}$$

The normal vectors $\overrightarrow{N}$ at these points are simply the cross products of each tangent vector pair.

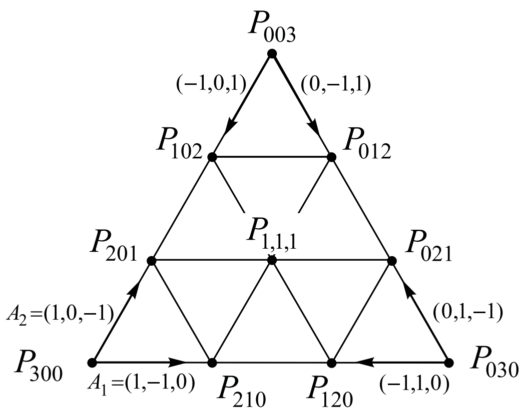

Figure 3: Directional derivatives: the coefficients of the directional derivative of a triangular cubic Bézier patch.

3. First and second fundamental form, main curvature directions

Even the calculation of main curvatures of a triangular Bézier patch (TBP) remains algebraic, it seems to be of no practical use to calculate them in general, as they would be of extremely high order even when starting with a low order TBP. Therefore we finally will restrict the calculations to the corner points of a cubic triangular Bézier patch only. Because of the special use of indices i, j and the restriction to dimension 3 we use the old-fashioned symbols E, F, G instead of the more modern g_{ij}. Analogously, L, M, N will denote the coefficients of the 2^{nd} fundamental form.

Starting from the scheme of Bernstein polynomials $B(u, v, w)$

$$B(u,v,w)=\left\{\begin{array}{ccccccc} & & & w^3 & & & \\ & & 3uw^2 & & 3vw^2 & & \\ & 3u^2w & & 6uvw & & 3v^2w & \\ u^3 & & 3u^2v & & 3uv^2 & & v^3 \end{array}\right.$$

Re-parametrising the patch in an (u, v)-representation $(w = 1-u-v)$ we receive the schemes of the first and second partial derivatives as

$$\frac{\partial B(u,v,w)}{\partial u}=\left\{\begin{array}{ccccccc} & & & -3w^2 & & & \\ & & 3w^2-6uw & & -6vw & & \\ & 6uw-3u^2 & & 6vw-6uw & & -3v^2 & \\ 3u^2 & & 6uv & & 3v^2 & & 0 \end{array}\right.$$

$$\frac{\partial B(u,v,w)}{\partial v}=\left\{\begin{array}{ccccccc} & & & -3w^2 & & & \\ & & -6uw & & 3w^2-6vw & & \\ & -3u^2 & & 6uw-6uv & & 6vw-3v^2 & \\ 0 & & 3u^2 & & 6uv & & 3v^2 \end{array}\right.$$

$$\frac{\partial^2 B(u,v,w)}{\partial u^2}=\left\{\begin{array}{ccccccc} & & & 6w & & & \\ & & 6u-12w & & 6v & & \\ & 6w-12u & & -12v & & 0 & \\ 6u & & 6v & & 0 & & 0 \end{array}\right.$$

$$\frac{\partial^2 B(u,v,w)}{\partial v^2}=\left\{\begin{array}{ccccccc} & & & 6w & & & \\ & & 6u & & 6v-12w & & \\ & 0 & & -12u & & 6w-12v & \\ 0 & & 0 & & 6u & & 6v \end{array}\right.$$

$$\frac{\partial^2 B(u,v,w)}{\partial u\partial v}=\left\{\begin{array}{ccccccc} & & & 6w & & & \\ & & 6u-6w & & 6v-6w & & \\ & -6u & & 6w-6u-6v & & -6v & \\ 0 & & 6u & & 6v & & 0 \end{array}\right.$$

By the (u, v)-representation $(w = 1-u-v)$ the triangular patch is treated as a tensor product Bézier patch and we have to exclude one corner, where the new parameter net is singular. By additionally using the (v, w)-representation $(u = 1-w-v)$ and the (w, u)-representation $(v = 1-u-w)$, we cover the whole patch and keep some cyclic symmetry in our formulas. We are now in a position to calculate the coefficients of the 1^{st} and 2^{nd} fundamental forms, at first for the (u, v)-representation:

At P_{300}:

$$\begin{pmatrix} E \\ F \\ G \end{pmatrix} = \begin{pmatrix} |3(P_{300} - P_{201})|^2 \\ 9(P_{300} - P_{201}) \cdot (P_{210} - P_{201}) \\ |3(P_{210} - P_{201})|^2 \end{pmatrix}, \quad \vec{N} = 9(P_{300} - P_{201}) \times (P_{210} - P_{201}),$$

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} \frac{6\vec{N}}{\|\vec{N}\|}\left((P_{300} - P_{201}) + (P_{102} - P_{201})\right) \\ \frac{6\vec{N}}{\|\vec{N}\|}\left((P_{102} - P_{111}) + (P_{210} - P_{201})\right) \\ \frac{6\vec{N}}{\|\vec{N}\|}\left((P_{102} - P_{111}) + (P_{120} - P_{111})\right) \end{pmatrix}.$$

At P_{030}:

$$\begin{pmatrix} E \\ F \\ G \end{pmatrix} = \begin{pmatrix} |3(P_{120} - P_{021})|^2 \\ 9(P_{120} - P_{021}) \cdot (P_{030} - P_{021}) \\ |3(P_{030} - P_{021})|^2 \end{pmatrix}, \quad \vec{N} = 9(P_{120} - P_{021}) \times (P_{030} - P_{021}),$$

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} \frac{6\vec{N}}{\|\vec{N}\|}\left((P_{012} - P_{111}) + (P_{210} - P_{111})\right) \\ \frac{6\vec{N}}{\|\vec{N}\|}\left((P_{012} - P_{021}) + (P_{030} - P_{021})\right) \\ \frac{6\vec{N}}{\|\vec{N}\|}\left((P_{012} - P_{111}) + (P_{120} - P_{021})\right) \end{pmatrix}.$$

At P_{003}:

$$\begin{pmatrix} E \\ F \\ G \end{pmatrix} = \begin{pmatrix} |3(P_{102} - P_{003})|^2 \\ 9(P_{102} - P_{003}) \cdot (P_{012} - P_{003}) \\ |3(P_{012} - P_{003})|^2 \end{pmatrix}, \quad \vec{N} = 9(P_{102} - P_{003}) \times (P_{012} - P_{003}),$$

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} \frac{6\vec{N}}{\|\vec{N}\|}\left((P_{003} - P_{102}) + (P_{201} - P_{102})\right) \\ \frac{6\vec{N}}{\|\vec{N}\|}\left((P_{003} - P_{012}) + (P_{111} - P_{102})\right) \\ \frac{6\vec{N}}{\|\vec{N}\|}\left((P_{003} - P_{012}) + (P_{021} - P_{012})\right) \end{pmatrix}.$$

Analogue results would follow for the other two representations, the (u, w)-resp. the (v, w)-representation.

Now we calculate the main curvatures κ_1, κ_2 and the orthogonal pairs of curvature directions λ_1, λ_2 at these corner points in the same way as in [10]:

$$\lambda_1, \lambda_2 = \frac{-\begin{vmatrix} E & G \\ L & N \end{vmatrix} \pm \sqrt{\begin{vmatrix} E & G \\ L & N \end{vmatrix}^2 - 4\begin{vmatrix} F & G \\ M & N \end{vmatrix} \cdot \begin{vmatrix} E & F \\ L & M \end{vmatrix}}}{2(FN - MG)}$$

$$\kappa_i(\lambda_i) = \frac{L + 2M\lambda_i + N\lambda_i^2}{E + 2F\lambda_i + G\lambda_i^2}, \quad i = 1, 2.$$

Finally we calculate the focal points F_1, F_2 of each corner normal and the normal of at P_{111} of the normal congruence by

$$\vec{F_i} = P(u, v, w) + \frac{1}{\kappa_i} \cdot \frac{\vec{N}}{\|\vec{N}\|}, \quad i = 1, 2.$$

Therewith, for each of the patches of the two focal surfaces Φ_1, Φ_2 belonging to the TBP S we again get a corner triangle and an additional point and in each of these four points the tangent plane of Φ_i is given, too. Even so the interpolation of these data by a TBP of low degree might deviate extremely from the true focal surface Φ_i! To avoid large distance between the focal points of adjacent congruence rays one had to subdivide the given TBP S and use the presented calculation for the partial triangles of S. For parabolic points of S one of the focal points F_i is a point at infinity and the presented apparatus cannot directly be used for providing correct data to interpolate both focal surfaces of S. For regions of S with points "close to parabolic points", i.e. one of the κ_i-values is almost zero the calculation should restrict to one focal patch only; subdivision of S should consider such a region for its own.

4. Computational example

The cubic triangular Bézier patch in Fig. 4 is created by considering all values of the parameters u, v, w with $0 \le u, v, w \le 1$ and $u + v + w = 1$. Let the family of points $P_{i,j,k}$ where $i+j+k \le 3$ which are called a (triangular) control net, or Bézier net denoted by the following grid of 10 points and we will consider the standard frame $((1,0,0), (0,1,0), (0,0,1))$:

			$\boldsymbol{P}_{003}$ $(3,6,0)$			
		$\boldsymbol{P}_{102}$ $(2,4,2)$		$\boldsymbol{P}_{012}$ $(4,4,2)$		
	$\boldsymbol{P}_{201}$ $(1,2,2)$		$\boldsymbol{P}_{111}$ $(3,2,3)$		$\boldsymbol{P}_{021}$ $(5,2,2)$	
$\boldsymbol{P}_{300}$ $(0,0,0)$		$\boldsymbol{P}_{210}$ $(2,0,2)$		$\boldsymbol{P}_{120}$ $(4,0,2)$		$\boldsymbol{P}_{030}$ $(6,0,0)$

We can compute the normal vectors and focal points at three corner control points and the "midpoint" on the patch:

At P_{300}:

$$\begin{pmatrix} E \\ F \\ G \end{pmatrix} = \begin{pmatrix} 81 \\ 27 \\ 45 \end{pmatrix}, \qquad \vec{N} = [-36,\ 18,\ 36],$$

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \\ -8 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -7.6056 \\ -0.3944 \end{pmatrix}.$$

At P_{030}:

$$\begin{pmatrix} E \\ F \\ G \end{pmatrix} = \begin{pmatrix} 45 \\ 27 \\ 81 \end{pmatrix}, \qquad \vec{N} = [-36,\ -18,\ 36],$$

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} -85 \\ 4 \\ -8 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -0.13141 \\ -2.53525 \end{pmatrix}.$$

At P_{003}:

$$\begin{pmatrix} E \\ F \\ G \end{pmatrix} = \begin{pmatrix} 81 \\ 63 \\ 81 \end{pmatrix}, \qquad \vec{N} = [0,\ 36,\ 36]\,,$$
$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} -8.4852 \\ -4.2426 \\ -8.4852 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

At P_{111}:

$$P_{111} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix},$$
$$\begin{pmatrix} E \\ F \\ G \end{pmatrix} = \begin{pmatrix} 45 \\ 27 \\ 45 \end{pmatrix}, \qquad \vec{N} = [0,\ 0,\ 36]\,,$$
$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} -126 \\ -62 \\ -12 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence we have the quantities λ_1, λ_2 which define directions in the uv-plane and the principal curvature directions, through the points P_{300}, P_{030}, P_{003} and P_{111}, which are defined as those directions and the normal curvature $\kappa(\lambda)$ obtains extreme values for directions λ_1 and λ_2. Finally we get the two focal points at each point as follow (see Fig. 4):

$$F_i(u,v,w) = P(u,v,w) + \frac{1}{\kappa_i(\lambda_i)}, \quad \frac{\vec{N}}{\|\vec{N}\|}, \quad i = 1,2.$$

At P_{300}:

$$\begin{pmatrix} \kappa_1(\lambda_1) \\ \kappa_2(\lambda_2) \end{pmatrix} = \begin{pmatrix} -0.18032 \\ -0.09129 \end{pmatrix}, \ F_1 = [3.6972,\ 1.8485,\ -3.6972]\,,\ F_2 = [7.3020,\ 3.6510,\ -7.3020]\,.$$

At P_{030}:

$$\begin{pmatrix} \kappa_1(\lambda_1) \\ \kappa_2(\lambda_2) \end{pmatrix} = \begin{pmatrix} -0.18032 \\ -0.09129 \end{pmatrix}, \ F_1 = [2.3029,\ 1.8485, -3.6971]\,,\ F_2 = [-1.3027,\ 3.6513, -7.3027]\,.$$

At P_{003}:

$$\begin{pmatrix} \kappa_1(\lambda_1) \\ \kappa_2(\lambda_2) \end{pmatrix} = \begin{pmatrix} -0.08839 \\ -0.23570 \end{pmatrix}, \ F_1 = [3,\ -2,\ -8]\,,\ F_2 = [3,\ 3,\ -3]\,.$$

At P_{111}:

$$\begin{pmatrix} \kappa_1(\lambda_1) \\ \kappa_2(\lambda_2) \end{pmatrix} = \begin{pmatrix} -0.3325005 \\ -1.8194333 \end{pmatrix}, \ F_1 = [3,\ 2,\ -1]\,,\ F_2 = [3,\ 2,\ -2]\,.$$

Figure 4: A cubic triangular Bézier patch with normals and their focal points at the corner control points and the "midpoint" P_{111} of the patch.

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