Journal for Geometry and Graphics
Volume 16 (2012), No. 1, 41–46.

Characterizations of Euclidean Hyperspheres Under Relatively Normalized Convex Hypersurfaces

Georg Stamou

Department of Mathematics, Aristotle University of Thessaloniki
54124 Thessaloniki, Greece
email: stamoug@math.auth.gr

Abstract. In this paper we treat convex hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ which are relatively normalized. The relative normalizations are either independent of geometric magnitudes of the considered convex hypersurface Φ or characterized by the fact that the corresponding support functions depend on elementary symmetric functions of the (Euclidean) principal curvatures of Φ. In the first case two characterizations of Euclidean hyperspheres are given via inequalities. In the second case it is proved that if the Pick-invariant vanishes identically, then Euclidean hyperspheres are obtained too.

Key Words: Convex hypersurfaces, relative normalizations, Pick-invariant, Euclidean hyperspheres

MSC 2010: 53A07

1. Preliminaries

In the Euclidean space $\mathbb{R}^{n+1}$ let $\Phi = (M, \bar{x})$ be a C^r-hypersurface defined by an n-dimensional, oriented, connected C^r-manifold M $(r \geq 3)$ and by a C^r-immersion $\bar{x}\colon\ M \to \mathbb{R}^{n+1}$. In the sequel, we suppose that the principal curvatures of Φ are positive (convex hypersurface). A C^s-mapping $\bar{y}\colon\ M \to \mathbb{R}^{n+1}$ $(r > s \geq 1)$ is called a C^s-*relative normalization*[1] if

$$\bar{y}(a) \notin T_P\Phi, \quad \frac{\partial \bar{y}}{\partial u^i}(a) \in T_P\Phi \ \ (i = 1, 2, \ldots, n), \quad P = \bar{x}(a) \tag{1}$$

at every point $P \in \Phi$, where $T_P\Phi$ is the tangent vector space of Φ at P and $u^1, u^2, \ldots, u^n$ are local coordinates. The *covector* $\bar{X}$ of the tangent vector space is defined by

$$\left\langle \bar{X}, \frac{\partial \bar{x}}{\partial u^i} \right\rangle = 0 \quad (i = 1, 2, \ldots, n) \quad \text{and} \quad \langle \bar{X}, \bar{y} \rangle = 1, \tag{2}$$

[1] For the basic concepts of relative Differential Geometry see [3].

ISSN 1433-8157/\$ 2.50

where $\langle\ ,\ \rangle$ denotes the standard scalar product in $\mathbb{R}^{n+1}$. Using $\bar{X}$, the (definite) *relative metric* G is introduced by

$$G_{ij} = \Big\langle \bar{X}, \frac{\partial^2 \bar{x}}{\partial u^i \partial u^j} \Big\rangle \tag{3}$$

and without loss of generality let it be *positive* definite. From now on we shall use G_{ij} as the fundamental tensor for "raising and lowering the indices" in the sense of classical tensor notation. Let $\bar{\xi}\colon M \to \mathbb{R}^{n+1}$ be the *Euclidean normalization* of Φ. The *support function of the relative normalization* $\bar{y}$ is defined by

$$q := \langle \bar{\xi}, \bar{y} \rangle\colon M \to \mathbb{R}, \quad q \in C^s(M). \tag{4}$$

By virtue of (1) we have $q \neq 0$ (without loss of generality let $q > 0$); moreover, because of (2) holds

$$\bar{X} = q^{-1} \bar{\xi}. \tag{5}$$

From (3) and (5) we obtain

$$G_{ij} = q^{-1} h_{ij}, \tag{6}$$

where h_{ij} are the components of the second fundamental form of Φ. We mention that, when the support function q is given, the relative normalization $\bar{y}$ is uniquely determined and possesses the following parametrization (see [2, p. 197])

$$\bar{y} = -h^{(ij)} \frac{\partial q}{\partial u^i} \frac{\partial \bar{x}}{\partial u^j} + q \bar{\xi}, \tag{7}$$

where $h^{(ij)}$ are the components of the inverse tensor of h_{ij}.

Let ${}^G\nabla_i f$ denote the covariant derivative of a differentiable function $f\colon M \to \mathbb{R}$ with respect to G. The (symmetric) *Darboux-tensor* is defined by

$$A_{jkl} := \Big\langle \bar{X}, {}^G\nabla_l\, {}^G\nabla_k \frac{\partial \bar{x}}{\partial u^j} \Big\rangle, \tag{8}$$

and the *Pick-invariant* by

$$J := \frac{1}{n\,(n-1)} A_{jkl} A^{jkl}, \tag{9}$$

where J is non-negative because G is positive definite.

The relative curvature theory is based on the symmetric tensor

$$B_{ij} = \Big\langle \frac{\partial \bar{X}}{\partial u^i}, \frac{\partial \bar{y}}{\partial u^j} \Big\rangle. \tag{10}$$

Especially, the *mean relative curvature* H is defined as follows

$$H = \frac{1}{n} \operatorname{tr}\left(B_i^j\right) \quad (\text{with } B_i^j = B_{ik} G^{kj}). \tag{11}$$

We denote by K_I the Gaussian curvature, H_I the mean curvature of Φ, S_{II} the scalar curvature of the second fundamental form II of Φ and S the scalar curvature of relative metric. Furthermore, let ∇_{II} resp. $\triangle_{II}$ denote the first resp. second Beltrami differential operator with respect to the fundamental form II. According to [4] the following formula holds true

$$S_{II} = H_I - \frac{\nabla_{II}\,(\ln K_I)}{4n\,(n-1)} + Q, \tag{12}$$

where Q is some non-negative function. For the relative magnitudes S, H and J the following metric formulae are valid (see [2]):

$$S = qS_{II} + \frac{1}{n} q \triangle_{II} (\ln q) + \frac{2-n}{4n} q \nabla_{II} (\ln q), \tag{13}$$

$$H = qH_I + \frac{1}{n} \left[\triangle_{II} q - \nabla_{II} \left(q, \ \ln K_I^{1/2} \right) \right], \tag{14}$$

$$J = \frac{3(n+2)}{4n(n-1)} q \nabla_{II} \left(\ln q, \ \ln q - \ln K_I^{2/(n+2)} \right) + qQ. \tag{15}$$

2. Arbitrary relative normalizations

Let the considered hypersurface in the previous section be closed. In [6] the following integral formulae were proved:

$$\text{(A)} \quad \int_M q^{\nu-1} (S - qS_{II})\, do_{II} = \frac{2-n-4\nu}{4n} \int_M q^{\nu} \nabla_{II} (\ln q)\, do_{II},$$

$$\text{(B)} \quad \int_M q^{\nu-1} (H - qH_I)\, do_I = \frac{1-\nu}{n} \int_M q^{\nu} \nabla_{II} (\ln q)\, do_I,$$

where do_I resp. do_{II} is the element of area with respect to the first resp. second fundamental form of Φ and $\nu \in \mathbb{R}$. From these formulae one obtains immediately the result [6]:
If one of the functions $S - qS_{II}$, $H - qH_I$ does not change sign on M, then we have $q =$ const., i.e., the relative normalization $\bar{y}$ is constantly proportional to the Euclidean normalization (see also [2]).

Besides the formulae (A) and (B), one can prove that the following formulae also hold true:

$$\text{(C)} \quad \int_M q^{\nu-1} \left(S - qS_{II} + \frac{n-1}{3} J \right) do_I = \frac{1-\nu}{n} \int_M q^{\nu} \nabla_{II} (\ln q)\, do_I + \frac{n-1}{3} \int_M q^{\nu} Q\, do_I,$$

$$\text{(D)} \quad \int_M q^{\nu-1} \left(H - qH_I - \frac{n-1}{3} J \right) do_{II} = \frac{2-n-4\nu}{4n} \int_M q^{\nu} \nabla_{II} (\ln q)\, do_{II} - \frac{n-1}{3} \int_M q^{\nu} Q\, do_{II}.$$

To *prove* this, we firstly verify that the above mentioned curvatures of Φ and the Pick-invariant are related as follows:

$$S - qS_{II} = H - qH_I - \frac{n-1}{3} J + \frac{n-1}{3} qQ. \tag{16}$$

After multiplication of (16) by the function $q^{\nu-1}$, we integrate over all M and taking into account (A) and (B), we obtain the integral formulae (C) and (D). Applying these formulae we can prove the following

Proposition 1. *Let $\Phi = (M, \bar{x})$ be a relatively normalized closed convex C^3-hypersurface in the space $\mathbb{R}^{n+1}$ with the support function of the relative normalization $q \in C^2$, for which one of the following conditions on M is valid:*

$$(a) \qquad S - qS_{II} \leq -\frac{n-1}{3}J,$$
$$(b) \qquad H - qH_I \geq \frac{n-1}{3}J.$$

Then $q =$ const. and Φ is an Euclidean hypersphere.

Proof. Let the condition (a) and $\nu < 1$ hold. From the integral formula (C) we deduce $q =$ const. and $Q = 0$. The latter relation means that the covariant derivative of the second fundamental form with respect to the first fundamental form of Φ vanishes (see [4, p. 232]). This fact characterizes, according to [5, p. 142], the Euclidean hyperspheres. We prove analogously the assertion of the Proposition, if the condition (b) holds. In this case we apply the integral formula (D). ☐

3. Special relative normalizations

In this section we consider special relative normalizations of the hypersurface $\Phi = (M, \bar{x})$ and we suppose that the Pick-invariant vanishes. A result in this direction has been proved by F. Manhart [2, p. 200] which asserts:

A (not necessarily closed) convex C^3-hypersurface $\Phi \subset \mathbb{R}^{n+1}$, which is relatively normalized by ${}^{(\alpha)}\bar{y}$ with the corresponding support function ${}^{(\alpha)}q = K_I^{\alpha}$ $(\alpha \neq 1/(n+2))$ and whose Pick-invariant vanishes identically, lies on an Euclidean hypersphere.

This result is generalized by the following

Proposition 2. *Let $\Phi = (M, \bar{x})$ be a (not necessarily closed) convex C^3-hypersurface in the space $\mathbb{R}^{n+1}$ and $f\colon M \to \mathbb{R}^+$ be a C^2-function. Furthermore, let Φ be relatively normalized by $\bar{y}$ with the corresponding support function $q = f(K_I)$.*

If $\bar{y}$ is not constantly proportional to the equiaffine normalization and if the Pick-invariant vanishes identically on M, then Φ lies on an Euclidean hypersphere.

Proof. It can be proved by the same arguments as those used in the above mentioned Proposition of F. Manhart. Setting $q = K_I^{1/(n+2)}$ in (15) we find the affine Pick-invariant

$$J_{\text{AFF}} = \frac{1}{4n(n+2)} K_I^{1/(n+2)} \nabla_{II}(\ln K_I) + K_I^{1/(n+2)} \left(S_{II} - H_I\right). \tag{17}$$

On account of (12) and (17) for $q = f(K_I)$ we obtain from (15)

$$\begin{aligned} J =& \frac{3f(K_I)}{4n(n-1)(n+2)} \left[(n+2)K_I \frac{f'(K_I)}{f(K_I)} - 1\right]^2 \nabla_{II}(\ln K_I) \\ &+ f(K_I) K_I^{-1/(n+2)} J_{\text{AFF}}. \end{aligned} \tag{18}$$

Let $J = 0$ on M. Due to the fact that each term on the right-hand side of (18) is nonnegative, it follows that both terms vanish. However, the vanishing of J_{AFF} characterizes hyperquadrics in case of locally strongly convex hypersurfaces. By assumption $q \neq cK_I^{1/(n+2)}$ ($c =$ const. > 0), thus we obtain $K_I =$ const. But the local constancy of the Gaussian curvature of a hyperquadric means that it lies on an Euclidean hypersphere. ☐

From (15) one obtains

$$J = 0 \iff 3q(n+2)\nabla_{II}(\ln q) - 6\nabla_{II}(q, \ln K_I) + 4n(n-1)qQ = 0. \tag{19}$$

Let now $q = g(H_I)$, where $g\colon M \to \mathbb{R}^+$ is a C^2-function. Then (19) leads to

$$3(n+2)g(H_I)\nabla_{II}(\ln g(H_I)) + 4n(n-1)g(H_I)Q = 6g'(H_I)\frac{\nabla_{II}(H_I, K_I)}{K_I}. \tag{20}$$

Let $g'(H_I) \leq 0$. If $g'(H_I)_{|a_0} < 0$ at a point $a_0 \in M$, then it follows from (20)

$$\frac{\nabla_{II}(H_I, K_I)}{K_I}\Big|_{a_0} \leq 0. \tag{21}$$

Let $g'(H_I)_{|a_1} = 0$ at a point $a_1 \in M$. We prove that the above inequality is also valid at this point. Let, contrary to (21),

$$\frac{\nabla_{II}(H_I, K_I)}{K_I}\Big|_{a_1} > 0. \tag{22}$$

Due to continuity reasons, there exists a neighbourhood $M_1 \subset M$ of the point a_1, so that

$$\frac{\nabla_{II}(H_I, K_I)}{K_I}\Big|_{a} > 0 \quad \forall\, a \in M_1. \tag{23}$$

Then, the right-hand side of (20) is non-positive for each $a \in M_1$ and both terms on the left-hand side of (20) must vanish. Consequently, we have $g(H_I)|_a = \text{const.}$ and $Q(a) = 0$ $\forall\, a \in M_1$. This means that the hypersurface $\Phi_1 := (M_1, \bar{x})$ lies on an Euclidean hypersphere. But in this case we would have

$$\frac{\nabla_{II}(H_I, K_I)}{K_I} = 0 \quad \forall\, a \in M_1, \tag{24}$$

i.e., a contradiction to assumption (22). From the above process it follows that the inequality $\frac{\nabla_{II}(H_I, K_I)}{K_I} \leq 0$ is satisfied for all points of M and hence

$$\int_M \frac{\nabla_{II}(H_I, K_I)}{K_I} do_I \leq 0. \tag{25}$$

Let now the convex hypersurface Φ be closed. Besides (25) and according to [1, p. 52], the following inequality holds true

$$\int_M \frac{\nabla_{II}(H_I, K_I)}{K_I} do_I \geq 0. \tag{26}$$

The equality holds iff Φ is an Euclidean hypersphere. From (25) and (26) we deduce:

Proposition 3. *Let $\Phi = (M, \bar{x})$ be a closed convex C^3-hypersurface in the space $\mathbb{R}^{n+1}$ and $g\colon M \to \mathbb{R}^+$ be a C^2-function with $g'(H_I) \leq 0$. Furthermore, let Φ be relatively normalized with the support function of the relative normalization $q = g(H_I)$.*
If the Pick-invariant vanishes identically on M, then Φ is an Euclidean hypersphere.

Let H_{n-1} denote the $(n-1)$-th mean curvature of the convex hypersurface Φ, defined by

$$H_{n-1} = \binom{n}{n-1}^{-1} \sum_{1\le i_1\le i_2\le\cdots\le i_{n-1}\le n} \prod_{j=1}^{n-1} k_{i_j}, \tag{27}$$

where $k_1, k_2, \ldots, k_n$ are the (Euclidean) principal curvatures of Φ. Let $q = h\big(\frac{H_{n-1}}{K_I}\big)$, where $h: M \to \mathbb{R}^+$ is a C^2-function with $h'\big(\frac{H_{n-1}}{K_I}\big) \ge 0$. We suppose that the corresponding Pick-invariant vanishes on M. Following similar arguments, we obtain the inequality

$$\int_M \nabla_{II}\left(\frac{H_{n-1}}{K_I}, K_I\right) do_I \ge 0. \tag{28}$$

However, according to [1, p. 52], the following inequality also holds

$$\int_M \nabla_{II}\left(\frac{H_{n-1}}{K_I}, K_I\right) do_I \le 0 \tag{29}$$

and the equality is valid iff Φ is an Euclidean hypersphere. From these inequalities we conclude:

Proposition 4. *Let $\Phi = (M, \bar{x})$ be a closed convex C^3-hypersurface in the space $\mathbb{R}^{n+1}$ and $h\colon M \to \mathbb{R}^+$ be a C^2-function with $h'\big(\frac{H_{n-1}}{K_I}\big) \ge 0$. Furthermore, let Φ be relatively normalized with the support function $q = h\big(\frac{H_{n-1}}{K_I}\big)$ of the relative normalization.*
If the Pick-invariant vanishes identically on M, then Φ is an Euclidean hypersphere.

References

[1] Th. Hasanis: *On convex hypersurfaces in E^{n+1}*. Colloq. Math. **48**, 49–55 (1984).

[2] F. Manhart: *Relativgeometrische Kennzeichnungen euklidischer Hypersphären*. Geom. Dedicata **29**, 193–207 (1989).

[3] R. Schneider: *Zur affinen Differentialgeometrie im Großen I*. Math. Z. **101**, 375–406 (1967).

[4] R. Schneider: *Closed convex hypersurfaces with second fundamental form of constant curvature*. Proc. Amer. Soc. **35**, 230–233 (1972).

[5] U. Simon, A. Weinstein: *Anwendungen der de Rahmschen Zerlegung auf Probleme der lokalen Flächentheorie*. Manuscripta Math. **1**, 139–146 (1969).

[6] G. Stamou: *Kennzeichnungen euklidischer Hypersphären durch relativgeometrische Größen*. Studia Sci. Math. Hungar. **40**, 349–358 (2003).

Received March 20, 2012