

# Shape and Image Interrogation with Curvature Extremalities

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**Abstract.** Reliable detection of surface creases defined via loci of the principal curvatures along their corresponding curvature lines is important for many geometrical and graphical applications. Multivariate analogues of such creases have received a considerable attention in recent studies on multidimensional image visualization and analysis. In this paper, we propose a numerically efficient and reliable approach for estimating multidimensional curvature extremalities and detecting ridge-like structures in multidimensional images. The approach is based on local fitting of hypercubic polynomials and calculating their extremalities by using newly derived formulas. We also propose a new thresholding scheme for removing spurious and unessential extremalities. We test our approach by detecting crease structures on 2D and 3D real-world images and demonstrating their ability to capture salient geometric image features.

*Key Words:* Curvature extremalities, multidimensional ridges and creases, multivariate image analysis.

*MSC 2010:* 68U05, 53A30

## 1. Introduction

Accurate and robust detection of surface and creases and associated skeletal structures is important for many of computer graphics, geometric modeling, and medical imaging applications [9, 4, 7, 13]. Recent progress in multidimensional and multivariate data acquisition calls for developing advanced shape interrogation methods. In particular, extracting crease structures turns out to be useful for computational fluid dynamics and medical imaging purposes [18, 17, 12] (see also references therein).

In contrast to the so-called height ridges [8] and their multidimensional generalizations [4, 18, 17], the principal direction ridges are rarely used for multivariate data analysis. The principal direction ridges possess beautiful mathematical properties [10, 14], have numerous 2D surface-based analysis and modeling applications [9, 7, 13]. However their multivariate extensions may seem too difficult to deal with. In this paper <sup>1</sup>, we develop a computational theory of multivariate principal direction ridges.

## Our approach

In this work, we introduce a multivariate analogue of the principal direction ridges and propose an efficient numerical procedure for estimating multidimensional curvature extremalities. Our *ridge detection method* can be considered as an extension of our cubic-polynomial fitting scheme developed in [20] for tracing salient curvature extrema on 2D surfaces approximated by dense triangle meshes, see Fig. 1. We first derive a simple (and novel) formula for extremalities of hyper-polynomial surfaces, and then the extremalities of images are estimated by local fitting hyper-polynomial patches using our formula. We also introduce a differential invariant measure based on the curvature extremalities (we call it *cyclidity*), and use it for detecting salient ridge-like image structures.



Figure 1: Creases extracted on surface meshes by using the method of [20]. Red and blue lines correspond to the salient extrema of principal curvatures along their curvature lines (a subset of principal direction ridges called crest lines), respectively.

In our study, we deal with more difficult problems than those considered in [4, 18, 17] and study pure geometric surface ridges and their multidimensional analogues. Such ridges are invariant w.r.t. the Euclidean motions, scale changes, and generalized inversions (conformal invariant), and, therefore, being extracted from a multidimensional image, convey intrinsic information about image geometry.

## Paper organization

The rest of paper is organized as follows. Sections 2 and 3 define curvature extremalities and cyclidity measures for multidimensional surfaces, respectively. We derive a simple formula of the extremalities in Section 4. Our algorithm of calculating the extremalities and its numerical experiments are given in Sections 5 and 6, respectively. We conclude the paper in Section 7.

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<sup>1</sup>It is an extension of our previous work [19]. The main difference from [19] is our thresholding scheme consisting of the conformal invariant differential quantity (1) and its discrete implementation and numerical experiments.

## 2. Curvature extremalities

Consider a smooth hyper-surface ( $d$ -dimensional manifold)  $\mathcal{S}$  in  $\mathbb{R}^{d+1}$ . Let us assume that the hyper-surface is defined parametrically,  $\mathcal{S} = \mathcal{S}(\mathbf{x})$  where the vector of parameters

$$\mathbf{x} = (x_1, x_2, \dots, x_d)$$

lives in  $\mathbb{R}^d$ . The basic tangent vectors of  $\mathcal{S}$  at  $\mathbf{x}$  are given by  $\mathcal{S}_i = \frac{\partial \mathcal{S}}{\partial x_i}$ ,  $i = 1, \dots, d$ . The unit normal vector of  $\mathcal{S}$  is defined by the so-called wedge product of  $\mathcal{S}_1, \dots, \mathcal{S}_d$

$$\mathbf{n} = \frac{\bigwedge (\mathcal{S}_1, \dots, \mathcal{S}_d)}{|\bigwedge (\mathcal{S}_1, \dots, \mathcal{S}_d)|}, \quad \bigwedge (\mathcal{S}_1, \dots, \mathcal{S}_d) = \begin{vmatrix} \mathcal{S}_1^{(1)} & \dots & \mathcal{S}_1^{(d+1)} \\ \vdots & \ddots & \vdots \\ \mathcal{S}_d^{(1)} & \dots & \mathcal{S}_d^{(d+1)} \\ \mathbf{e}_1 & \dots & \mathbf{e}_{d+1} \end{vmatrix},$$

where  $\{\mathbf{e}_1, \dots, \mathbf{e}_{d+1}\}$  is the coordinate basis and  $[\mathcal{S}_i^{(1)}, \dots, \mathcal{S}_i^{(d+1)}]$  are components of  $\mathcal{S}_i$  w.r.t. the coordinate basis,  $i = 1, \dots, d$ .

Let us define the curvature tensor (generalization of the Waingarten map) of  $\mathcal{S}$  by the  $d \times d$  matrix  $\mathbf{W} = \mathbf{II}\mathbf{g}^{-1}$ , where the Riemannian covariant metric tensor  $\mathbf{g}$  is given by the  $i$ -th row and  $j$ -th column element  $\mathbf{g}_{ij} = \frac{\partial \mathcal{S}}{\partial x_i} \cdot \frac{\partial \mathcal{S}}{\partial x_j}$ , and the  $i$ -th row and  $j$ -th column element of the matrix  $\mathbf{II}$  is defined by  $\mathbf{II}_{ij} = \frac{\partial^2 \mathcal{S}}{\partial x_i \partial x_j} \cdot \mathbf{n}$ . Here  $\mathbf{a} \cdot \mathbf{b}$  is the inner product between  $\mathbf{a}$  and  $\mathbf{b}$ . The quadratic forms  $\mathbf{g}$  and  $\mathbf{II}$  are natural analogues of the two-dimensional first and second fundamental forms.

The  $d$  eigenvalues  $k_1 \leq k_2 \leq \dots \leq k_d$  and their corresponding eigenvectors  $\{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_d\}$  of  $\mathbf{W}$  give us the principal curvatures and directions of  $\mathcal{S}$ .

We introduce generalized curvature extremalities of  $\mathcal{S}$  as  $e_i = \partial k_i / \partial \mathbf{t}_i = \nabla k_i \cdot \mathbf{t}_i$ , where  $\nabla = \{\partial / \partial x_1, \partial / \partial x_2, \dots, \partial / \partial x_d\}$  is the standard gradient operator. Zero-crossings of  $e_i$ ,  $i = 1, 2, \dots, d$  describe principal direction ridges on  $\mathcal{S}$ .

## 3. Dupin's cyclides

The extremalities are closely related to a special family of surfaces called Dupin's cyclides which have been intensively studied in connection with various shape modeling tasks [3, 5]. The classical 2D Dupin's cyclides live in three-dimensional space and can be characterized by the conditions  $e_1 = 0 = e_2$ . A straightforward generalization leads us to the notion of Dupin hyper-surfaces defined by a system of  $d$  equations  $e_i = 0$ ,  $i = 1, 2, \dots, d$  and, therefore, constituting hyper-surfaces in  $\mathbb{R}^{d+1}$ .

Let us introduce a *cyclidity* by

$$C = \sum_i^d |e_i|.$$

The cyclidity measures how a point on  $\mathcal{S}$  is close to the Dupin hyper-surfaces. Recall the zero-crossings of extremalities correspond to the surface creases. On the other hand, the cyclides do not contain surface creases in general. Thus, the 2D analogue of cyclidity was

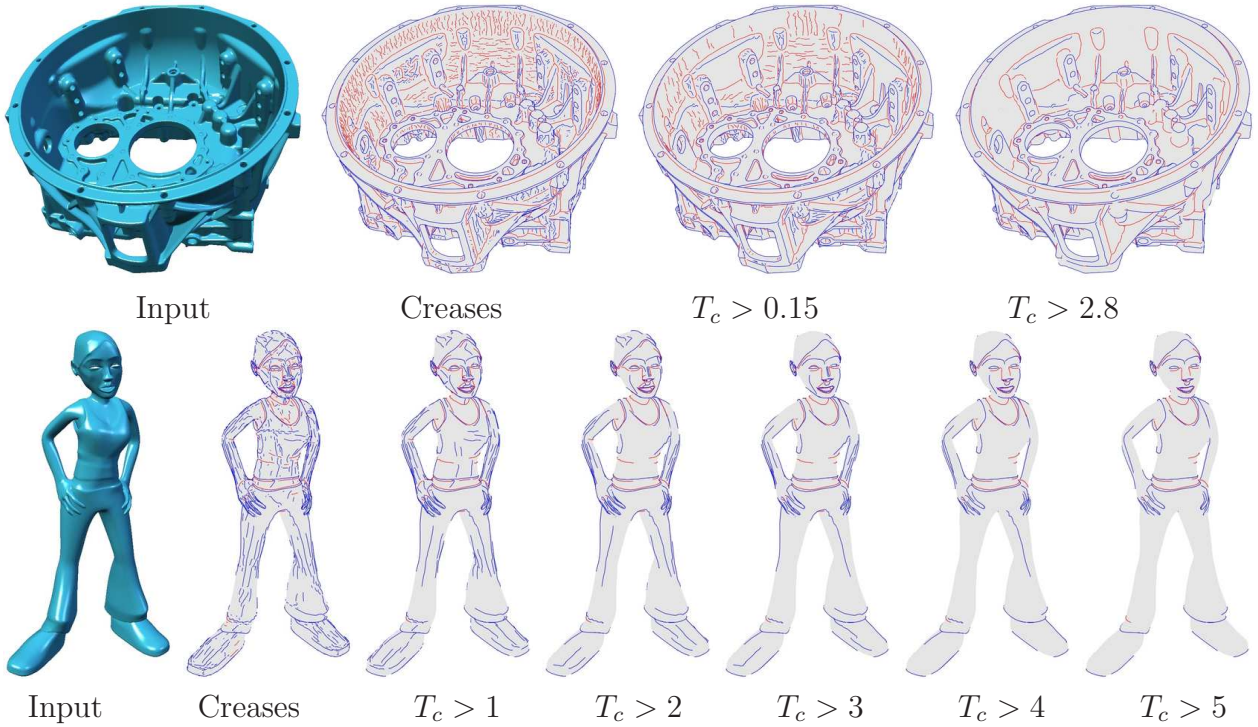


Figure 2: Filtering phantom surface creases (crest lines) via the conformal invariant differential quantity  $T_c$  [20]. The images show the filtered surface creases whose  $T_c$  is greater than the given thresholds.

employed in [20] to filter out the phantom surface creases around the surface parts close to the cyclides:

$$T_c = \int \sqrt{|e_1| + |e_2|} ds,$$

where the integral is calculated along each extracted creases,  $ds$  is the arclength element of the crease, and  $T_c$  is invariant w.r.t. the conformal transformations [20]. Figure 2 demonstrates how well our threshold  $T_c$  work to filter our phantom surface creases around the surface parts close to the Dupin's cyclides.

In this paper, we use the cyclidity together with  $\{e_i\}$  for extracting salient ridge-like image structures by employing the following integral (1).

$$T_{\mathbb{R}^d} = \int \left( \sum_{i=1}^d e_i^2 \right)^{\frac{d}{4}} dA, \quad (1)$$

where  $dA = \sqrt{1 + \sum_{i=1}^d \mathcal{S}_i^2} dx_1 dx_2 \dots dx_d$  is the area element of  $\mathcal{S}$ . We model that (1) is also the conformal invariant differential quantity, since the ridges are invariant under the conformal transformations.

#### 4. Formula of extremalities

Consider a multidimensional image  $x_{d+1} = I(\mathbf{x})$ ,  $\mathbf{x} = (x_1, x_2, \dots, x_d)$ . For the sake of simplicity, we deal with single-modality images, i.e., the image intensity  $I$  is a scalar function (our

results can be easily extended to multichannel images). Note that the equation  $x_{d+1} = I(\mathbf{x})$  defines a hyper-surface in the extended spatial-tonal image space  $\mathbb{R}^{d+1} = \{(x_1, \dots, x_d, x_{d+1})\}$ . Our main task now is to achieve fast and robust estimation of the extremalities  $\{e_i\}$ . Note that the curvature extremalities are shift and rotation invariant. The following formula is very useful because it significantly reduces computational time needed for the curvature extremalities.

### Main formula

Assume that the coordinates in  $\mathbb{R}^{d+1}$  are chosen such that the origin of coordinates  $\mathbf{0}$  is situated on  $\mathcal{S} = \{x_{d+1} = I(\mathbf{x})\}$  and the hyperplane formed by the coordinate axes  $x_1, \dots, x_d$  is orthogonal to  $\mathcal{S}$  at  $\mathbf{0}$ . Then the extremality  $e(\mathbf{x})$  of  $\mathcal{S}$  at  $\mathbf{x} = \mathbf{0}$  is given by

$$e = \sum_{i,j,l=1}^d I_{ijl} t_i t_j t_l, \quad (2)$$

where  $I_{ijl} = \frac{\partial^3 I(\mathbf{x})}{\partial x_i \partial x_j \partial x_l}$  are third-order partial derivatives of  $I(\mathbf{x})$  and  $\mathbf{t} = (t_1, t_2, \dots, t_d)$  is the principal direction corresponding to the curvature extremality  $e$ .

### Derivation of main formula

Let

$$\mathbf{t} = (t_1, t_2, \dots, t_{d+1}) \text{ and } \mathbf{n} = (n_1, n_2, \dots, n_{d+1})$$

be a principal direction and the unit normal of  $\mathcal{S}$  at  $(\mathbf{x}, x_{d+1}) \in \mathbb{R}^{d+1}$ , respectively. Let us represent  $\mathcal{S}$  in implicit form

$$0 = I(\mathbf{x}) - x_{d+1} \equiv F(\mathbf{x}, x_{d+1}).$$

Similar to the 3D case (see, for example, [11, 1]) the extremality  $e$  of

$$F(\mathbf{x}, x_{d+1}) = 0$$

is given by

$$e = \frac{\partial k}{\partial \mathbf{t}} = \frac{\sum_{i,j,l=1}^{d+1} F_{ijl} t_i t_j t_l + 3k \sum_{i,j=1}^{d+1} F_{ij} t_i n_j}{|\nabla F|}, \quad (3)$$

where  $F_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j}$  and  $F_{ijl} = \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_l}$ .

Since  $F(\mathbf{x}, x_{d+1}) \equiv I(\mathbf{x}) - x_{d+1}$ , we have  $F_{i,d+1} = 0$ . Further,  $\nabla F \equiv (\nabla_{\mathbf{x}} I, -1)$  and  $\nabla_{\mathbf{x}} I(\mathbf{0}) = 0$ . Thus  $|\nabla F(\mathbf{0})| = 1$ . Note that the sum of  $F_{ij} t_i n_j$  in (3) the indices vary from 1 to  $d$  and, therefore, the sum vanishes because only the first  $d$  components of  $\mathbf{n}(\mathbf{0})$  are zeros. Formula (2) is proved.

For the sake of completeness, we present here a derivation of (3). Assume that the components of the surface orientation normal are given by  $n_i = -F_i/g$ , where  $g = |\nabla F|$  is the absolute value of the gradient. Let  $k$ ,  $\mathbf{t}$ , and  $s$  stand for a principal curvature, the associated principal direction, and the arclength parameter along the curvature line corresponding to  $k$  and  $\mathbf{t}$ . According to the Frenet formulas we have at a point on  $\mathcal{S}$

$$\frac{d\mathbf{t}}{ds} = k\mathbf{n}, \quad \frac{d\mathbf{n}}{ds} = -k\mathbf{t}.$$

Differentiating  $\sum_{i=1}^{d+1} F_i t_i = 0$  w.r.t.  $s$  yields

$$\frac{d}{ds} \left( \sum_{i=1}^{d+1} F_i t_i \right) = \sum_{i,j=1}^{d+1} F_{ij} t_i t_j + k \sum_{i=1}^{d+1} F_i n_i = \sum_{i,j=1}^{d+1} F_{ij} t_i t_j - k g.$$

Thus the principal curvatures  $k$  is given by

$$k = \frac{1}{g} \sum_{i,j=1}^{d+1} F_{ij} t_i t_j \quad (4)$$

Note that

$$\frac{dg}{ds} = \sum_{i=1}^{d+1} g_i t_i = \frac{1}{g} \sum_{i,j=1}^{d+1} F_{ij} t_i F_j = - \sum_{i,j=1}^{d+1} F_{ij} t_i n_j.$$

Finally differentiating (4) w.r.t.  $s$  yields

$$e = \frac{dk}{ds} = \frac{d}{ds} \left( \frac{1}{g} \sum_{i,j=1}^{d+1} F_{ij} t_i t_j \right) = \frac{1}{g} \left( \sum_{i,j,l=1}^{d+1} F_{ijl} t_i t_j t_l + 3k \sum_{i,j=1}^{d+1} F_{ij} t_i n_j \right)$$

which is desired formula (3).

### Relation to derivative of curvature tensor

Our formula (2) can be also obtained by extending the derivative-of-curvature tensor derived by RUSINKIEWICZ [16] (see formula (8) there) to  $d$ -manifolds. He used a  $2 \times 2 \times 2$  tensor composed by  $\{\partial \mathbf{W} / \partial x_1, \partial \mathbf{W} / \partial x_2\}$  for 2-manifolds. The curvature derivative w.r.t.  $\mathbf{t}$  was given by multiplying  $\mathbf{t}$  three times to the tensor. According to this observation, the extremality of  $\mathcal{S}$  is given by the gradient of the Weingarten map multiplying by  $\mathbf{t}$  three times:  $e = \nabla \mathbf{W}(\mathbf{t}, \mathbf{t}, \mathbf{t})$  where  $\nabla \mathbf{W}$  is a  $d \times d \times d$  tensor. Since the Weingarten map  $\mathbf{W}$  is equivalent to the Hessian of  $I(\mathbf{x})$  in our case because  $\mathbf{II}$  becomes an identity matrix at  $\mathbf{x} = \mathbf{0}$ , we have

$$e = \nabla \text{Hess}(I(\mathbf{x}))(\mathbf{t}, \mathbf{t}, \mathbf{t}) = \sum_{i,j,l=1}^d I_{ijl} t_i t_j t_l.$$

### Implementation to cubic polynomials

By using our formula (2), the following simple forms are derived for the 2D and 3D polynomials.

For a 2D cubic polynomial

$$f(x_1, x_2) = \frac{1}{2}(b_0 x_1^2 + 2b_1 x_1 x_2 + b_2 x_2^2) + \frac{1}{6}(c_0 x_1^3 + 3c_1 x_1^2 x_2 + 3c_2 x_1 x_2^2 + c_3 x_2^3),$$

we have

$$\mathbf{W} = \begin{pmatrix} b_0 & b_1 \\ b_1 & b_2 \end{pmatrix} \quad (5)$$

and

$$e = \begin{pmatrix} t_1^2 \\ t_2^2 \end{pmatrix}^T \begin{pmatrix} c_0 & 3c_1 \\ 3c_2 & c_3 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \quad (6)$$

at  $(x_1, x_2) = (0, 0)$ .

For a 3D cubic polynomial

$$\begin{aligned} f(x_1, x_2, x_3) = & \frac{1}{2}(b_0x_1^2 + 2b_1x_1x_2 + 2b_2x_1x_3 + b_3x_2^2 + 2b_4x_2x_3 + b_5x_3^2) + \\ & + \frac{1}{6}(c_0x_1^3 + 3c_1x_1^2x_2 + 3c_2x_1^2x_3 + 3c_3x_2^2x_1 + c_4x_2^3 + 3c_5x_2^2x_3 + \\ & + 3c_6x_1x_3^2 + 3c_7x_2x_3^2 + c_8x_3^3), \end{aligned}$$

we have

$$\mathbf{W} = \begin{pmatrix} b_0 & b_1 & b_2 \\ b_1 & b_3 & b_4 \\ b_2 & b_4 & b_5 \end{pmatrix} \quad (7)$$

and

$$e = \begin{pmatrix} t_1^2 \\ t_2^2 \\ t_3^2 \end{pmatrix}^T \begin{pmatrix} c_0 & 3c_1 & 3c_2 \\ 3c_3 & c_4 & 3c_5 \\ 3c_6 & 3c_7 & c_8 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \quad (8)$$

at  $(x_1, x_2, x_3) = (0, 0, 0)$ .

## 5. Interrogation algorithm

Let  $r$  and  $r_f$  be the user-specified fitting and thresholding radii, respectively. We perform the following local polynomial fitting procedure to each image element (pixels/voxels) in order to obtain  $\{e_i\}$  and  $C$ .

**Step 1:** For an image element at  $\mathbf{x}_i$ , consider its  $(2r + 1)^d$  neighboring elements centered at  $\mathbf{x}_i$ . Fit the polynomial  $f(\mathbf{x})$  to  $I(\mathbf{x}) - I(\mathbf{x}_i)$  in the least-squares sense by using the neighboring elements.

**Step 2:** Calculate the Weingarten map  $\mathbf{W}$  via the equations (5) or (7), and compute the eigenvalues (principal curvatures  $\{k_i\}$ ) and eigenvectors (principal directions  $\{\mathbf{t}_i\}$ ) of  $\mathbf{W}$ .

**Step 3:** Obtain the curvature extremalities  $\{e_i\}$  via the equations (6) or (8).

**Step 4:** Apply thresholding (if necessary) to the obtained extremalities. Consider the  $(2r_f + 1)^d$  neighboring elements centered at  $\mathbf{x}_i$ . In our implementation, we approximate the threshold (1) by

$$T_{\mathbb{R}^d} \approx \sum_{j=1}^{(2r_f+1)^d} a_j \left( \sum_{i=1}^d (\mathbf{t}_i \cdot \mathbf{t}_{i,j}) e_{i,j}^2 \right)^{\frac{d}{4}},$$

where  $a_j$ ,  $\mathbf{t}_{i,j}$ , and  $e_{i,j}$  are the area element, principal direction, and extremality of neighboring element centered at  $\mathbf{x}_i$ , respectively. We estimate how the neighboring element is close to be located on the ridge-like structure at  $\mathbf{x}_i$  by the inner-product of their principal directions  $(\mathbf{t}_i \cdot \mathbf{t}_{i,j})$ . The second order central finite difference scheme is employed

for approximating partial derivatives of  $\sqrt{1 + \sum_i^d \left| \frac{\partial I(\mathbf{x})}{\partial x_i} \right|^2} \approx a_j$ .

## Discussion

In our current implementation of polynomial fitting, we did not use the normals of  $\mathcal{S}$ , and we omitted a constant and linear terms. Although our fitting scheme provides smoother results than the case of incorporating these terms, adapting the normal-based [6] or/and osculating jets [2] fitting methods to our approach is promising to improve accuracy of polynomial fitting.

We employed same fitting radius for all image elements. Thus, the fitting process is time consuming when the radius is large (especially for  $d \geq 3$ ). Adaptive radius fitting with multiresolution or/and scale-space strategy may accelerate our approach, and also gain flexible control over extracting multi-scale features.

Future work includes developing a robust and efficient method to extract zero-crossings of the computed extremalities.

## 6. Numerical experiments

In our implementation, the singular value decomposition [15] for least-squares fitting and Jacobi method [15] for eigenanalysis (because  $\mathbf{W}$  is a real symmetric matrix in our case) were employed. All our numerical experiments reported in this paper were performed on a Core2 Extreme X9770 (3.2 GHz quad core, no parallelization was used) PC with 16GB RAM and 64 bit OS.



Figure 3: Input images: Lena ( $512^2$ ), Trui ( $256^2$ ), Mandrill ( $512 \times 506$ ), and Zebra ( $507 \times 379$ ).

We tested our approach on the real-world 2D and 3D images and find it useful for extracting meaningful image structures, see Figs. 3, 6, 7, 9, and 10. Since the extremalities vary with the High Dynamic Range (HDR), we applied the following normalization technique for visualization purposes. To map HDR values onto more intuitively visible domain,  $|e_i|$  and  $C$  are normalized into  $[0, 1]$  range and white (black) color corresponds to high (low) intensity value, while clipping 1% on the low (black) end and 2% on the on the high (white) end of the histogram (16 bit bins). We also used RGB color channels to visualize  $|e_i|$  and  $C$  simultaneously. For example, the 2D extremalities are visualized by  $\mathbf{e}_{rgb} = (R, G, B) = (1 - |e_1|, 1 - C, 1 - |e_2|)$  in Figs. 6 and 7. The convex and concave edge regions are well distinguished similar to true ridges in meshes (as in Fig. 1). Similar coloring technique was also employed in the 3D images, see caption of Figs. 9 and 10.

Our approach has a user-specified parameter for polynomial fitting: fitting radius ( $r$ ). The influence of changing the radius is demonstrated in Fig. 6. As we can see the multi-scale features as shadows in the left image of Fig. 5, increasing the radius provides extraction of large size ridge-like structures.

Figures 8 and 11 demonstrate our thresholding scheme applied to the extremalities of Lena and MRI-Head images (Figs. 7 and 10), respectively. We can see that the extremalities at the image regions close to the Dupin's cyclides (no creases) are adaptively removed according to increasing the threshold (1) while the salient ridge-like structures are preserved well.

The computational complexity of our approach is linear ( $O((2r + 1)^{2d})$ ) w.r.t. image size (fitting radius, since the singular value decomposition possess quadratic complexity).



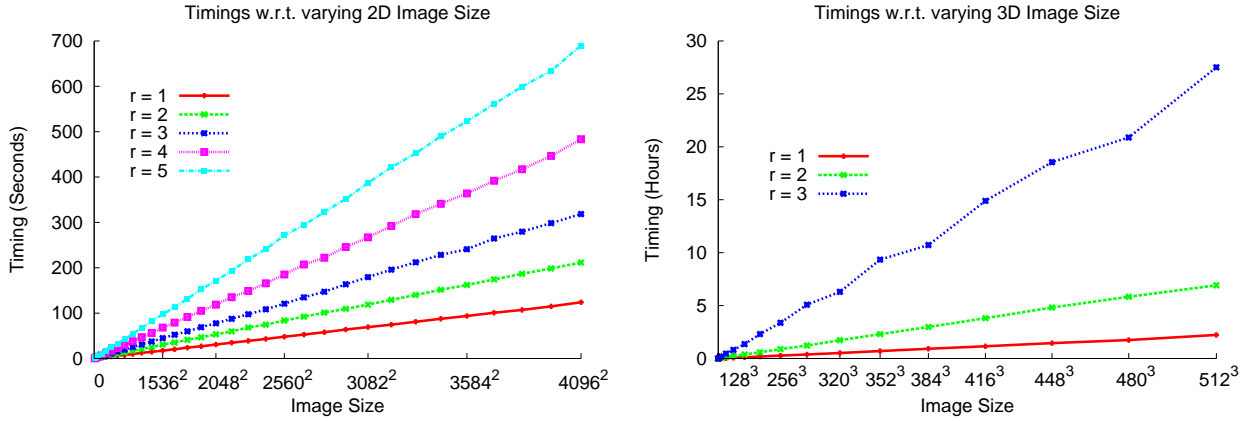


Figure 4: Computational time required for all necessary computations on 2D (left) and 3D (right) images.

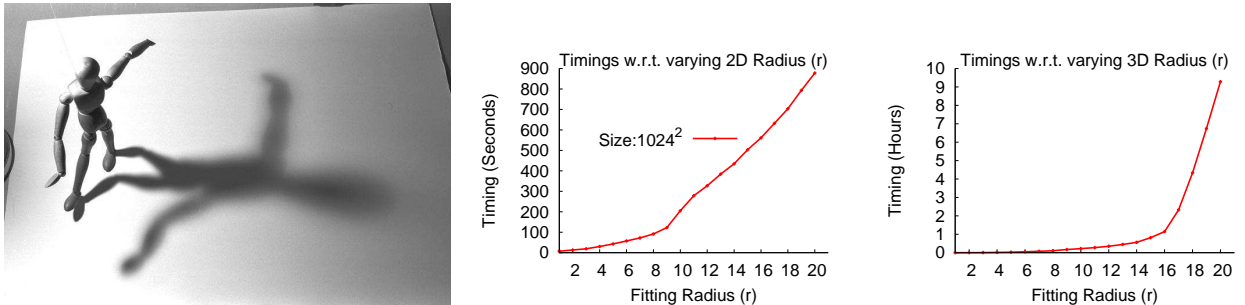


Figure 5: Left: image consists of multi-scale features. Center and right: computational time on 2D (center) and 3D (right) images w.r.t. varying fitting radius ( $r$ ).

We examined computational time of our approach on varying image size (Fig. 4) and radius (center and right images of Fig. 5). The slope of resulting timing lines are reasonably close to the complexity. Our approach approximately processes 66K pixels and 8K voxels per second in average for small fitting radius ( $r \leq 5$  and  $r \leq 3$ ), respectively.

## 7. Conclusion

In this paper, we have developed a numerically efficient approach to extracting the curvature extremalities from multidimensional images. We have also introduced the formula and differential invariant measure for extremalities. Our numerical experiments demonstrate that the proposed approach is capable of detecting salient ridge-like image structures on real-world images. We believe that our study in a multidimensional analogue of the surface creases as geometric shape descriptors will be beneficial to obtain new aspects of feature analysis in early vision tasks.

## Acknowledgements

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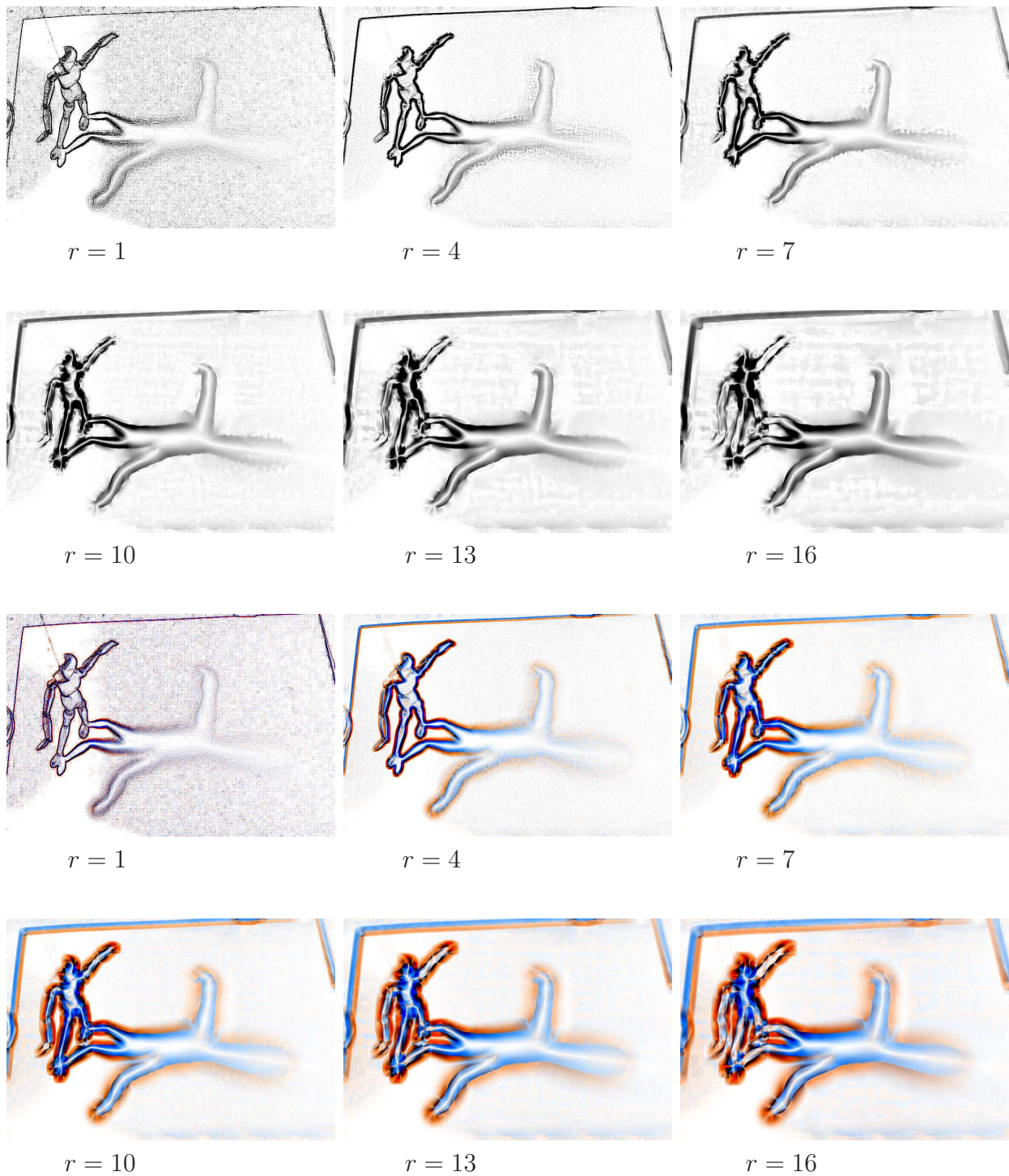


Figure 6: Cyclidity ( $C$ , top two rows) and extremalities ( $e_{rgb}$ , bottom two rows) with varying the fitting radius  $r$ . The image consisting of 768x512 pixels shown in the left image of Fig. 5 took 2.9 ( $r = 1$ ), 11 ( $r = 4$ ), 27.2 ( $r = 7$ ), 88.7 ( $r = 10$ ), 172 ( $r = 13$ ), and 251 ( $r = 16$ ) seconds, respectively for all necessary computations. Increasing the radius for our approach extracts the large size ridge-like structures. See the shadow regions in the left image of Fig. 5 and corresponding parts in the above images.

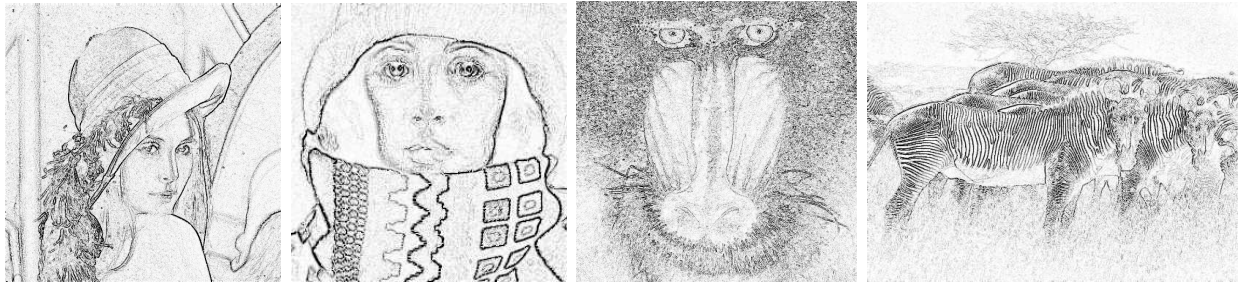
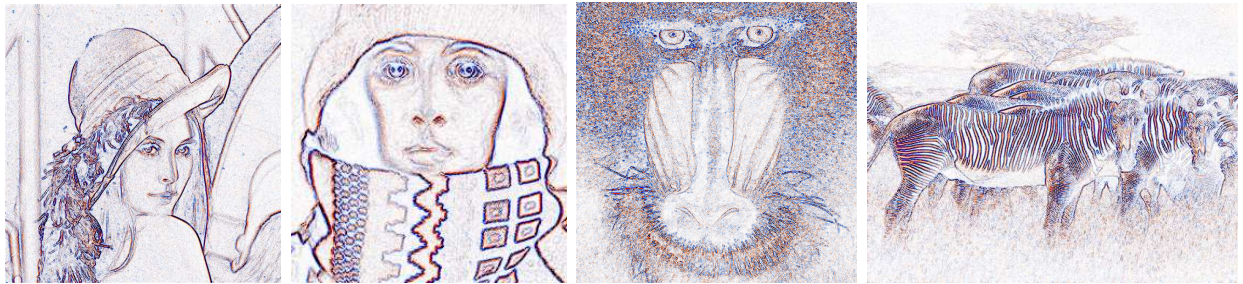
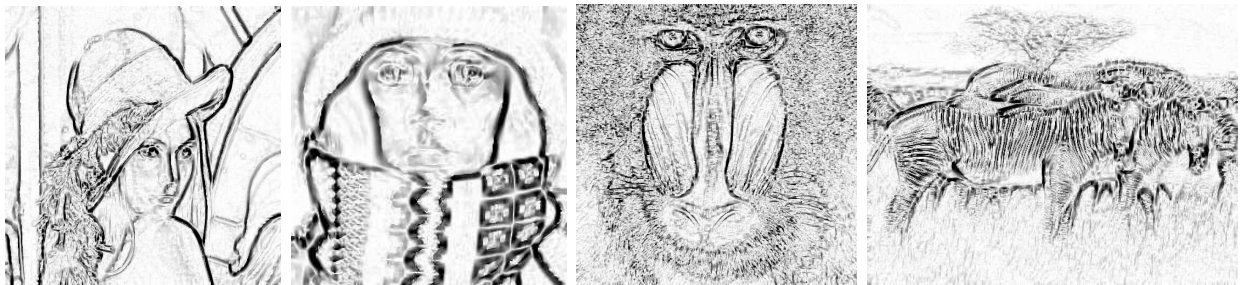
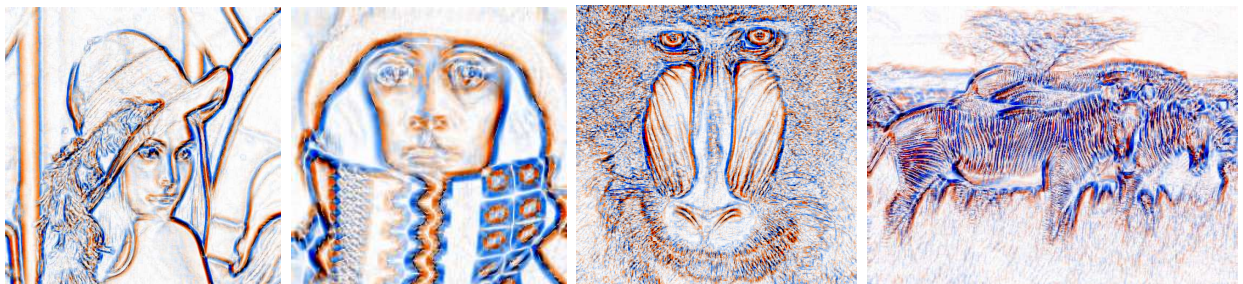
Cyclidity ( $r = 1$ ):  $C$ Extremalities ( $r = 1$ ):  $\mathbf{e}_{rgb}$ , timing: 1.96, 0.48, 1.93, and 1.44 seconds.Cyclidity ( $r = 5$ ):  $C$ Extremalities ( $r = 5$ ):  $\mathbf{e}_{rgb}$ , timing: 10.66, 2.71, 10.46, and 8.05 seconds.

Figure 7: The images show both cyclidity  $C$  and extremalities  $\mathbf{e}_{rgb}$  by using  $3^2$  and  $11^2$  pixel neighborhoods for polynomial fitting, where the input images are given in Fig. 3. Timings listed above present computational times required for all necessary computations. Here red and blue regions correspond to low value (close to zero) of  $|e_1|$  and  $|e_2|$ , respectively.

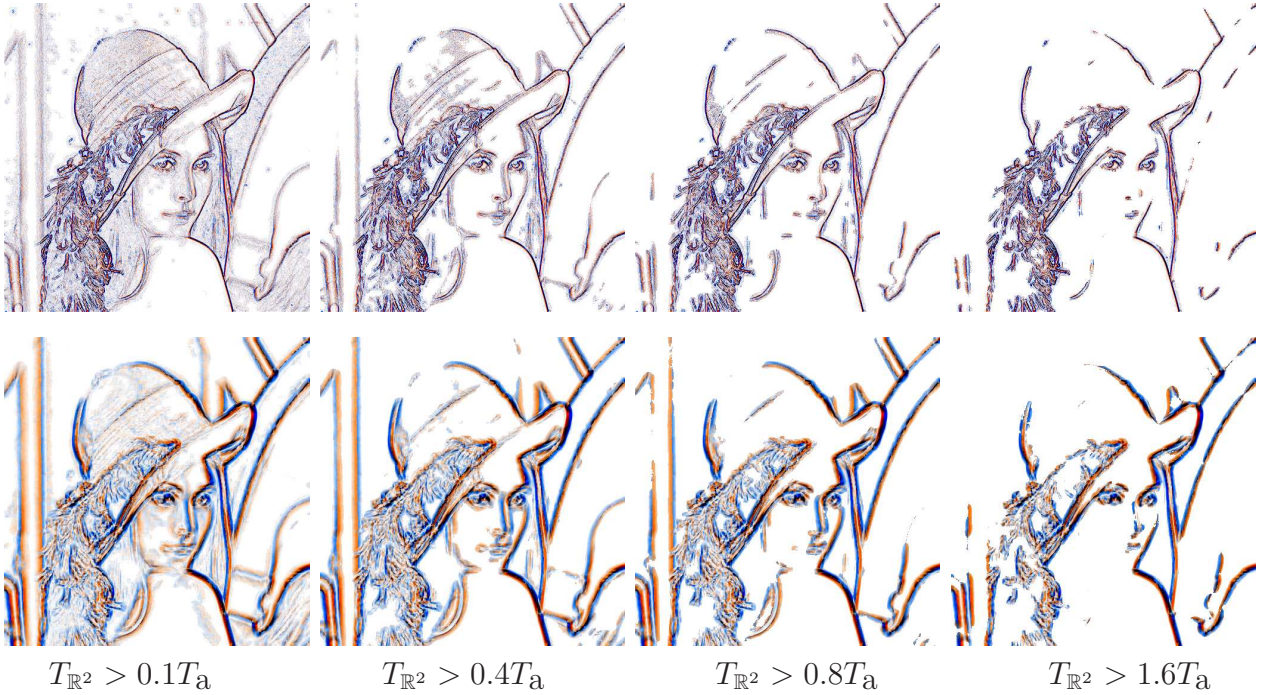


Figure 8: Filtering phantom ridge-like structures via the conformal invariant differential quantity (1), where the input image is shown in Fig. 3, the extremalities ( $\mathbf{e}_{rgb}$ ) corresponding to  $T_{\mathbb{R}^2} \geq 0$  are illustrated in Fig. 7 (upper:  $r = 1$  and bottom:  $r = 5$ ), we use  $r_f = 5$ , and  $T_a$  is an average of  $T_{\mathbb{R}^2}$  within the input image.

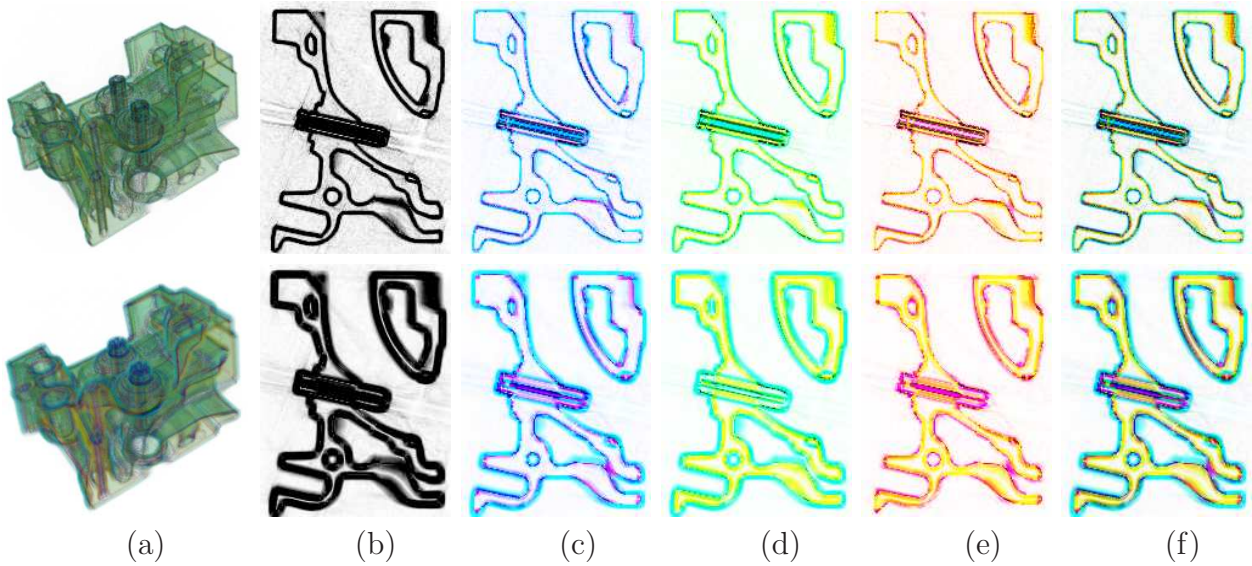


Figure 9: 3D extremalities of CT-Engine ( $256^2 \times 110$ ) took 6.4 and 85 minutes respectively for  $r = 1$  and  $r = 3$ . (a): volume rendering of  $\{|e_1|, |e_2|, |e_3|\}$  where RGB color channels are given by  $(R, G, B) = (1 - |e_1|, 1 - |e_2|, 1 - |e_3|)$ . (f): sectional images of (a). (b, c, d, e): sectional images where their locations correspond to (f), and their RGB color channels consist of  $(1 - C, 1 - C, 1 - C)$ ,  $(1 - |e_1|, 1 - |e_2|, 1 - C)$ ,  $(1 - |e_1|, 1 - C, 1 - |e_3|)$ , and  $(1 - C, 1 - |e_2|, 1 - |e_3|)$ , respectively. The upper and bottom images are obtained by using  $3^3$  ( $r = 1$ ) and  $7^3$  ( $r = 3$ ) voxel neighborhoods to the fitting process, respectively.

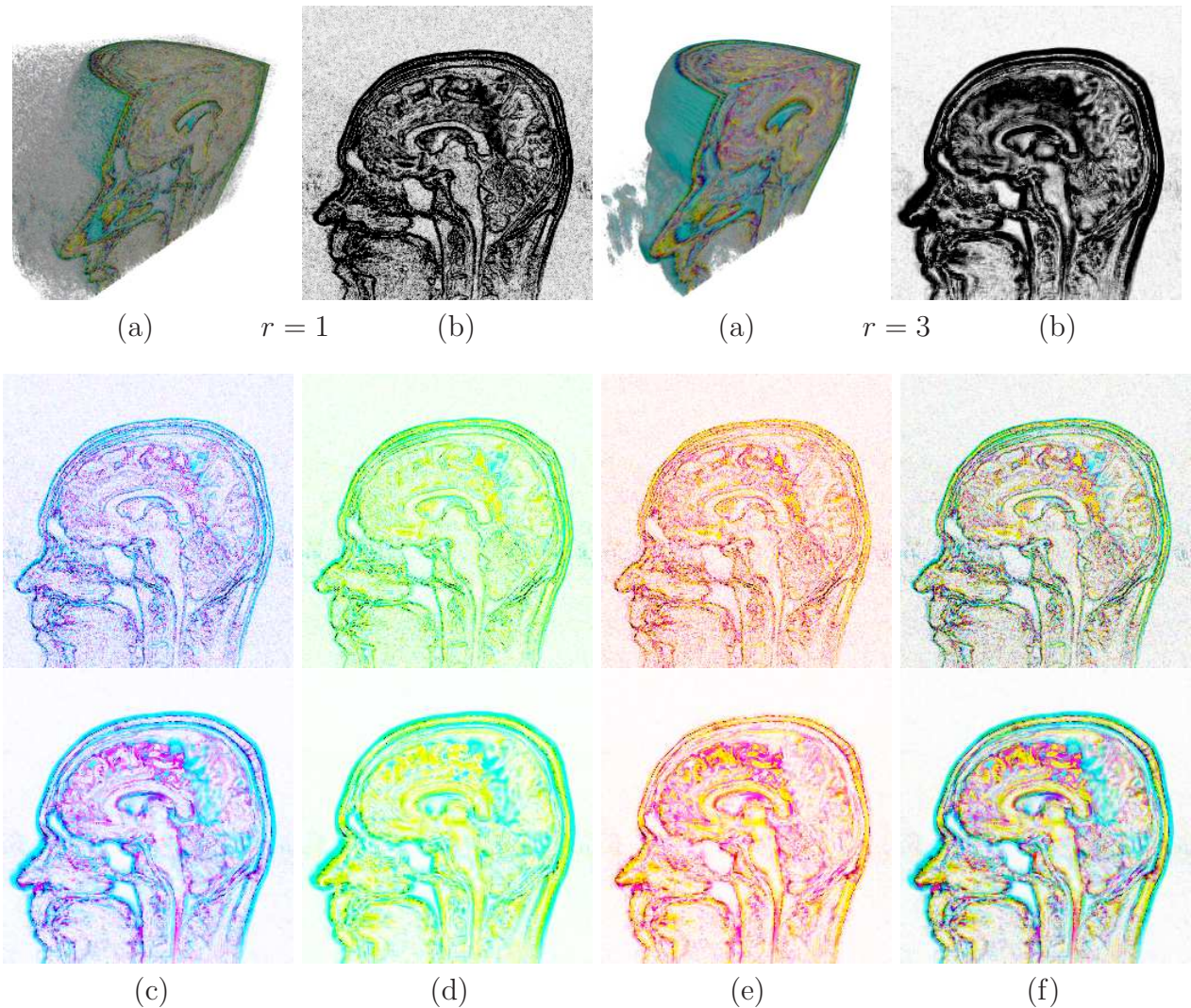


Figure 10: 3D extremalities of MRI-Head ( $256^3$ ) took 17 and 195 minutes respectively for  $r = 1$  and  $r = 3$ . (a): volume rendering of  $\{|e_1|, |e_2|, |e_3|\}$  where RGB color channels are given by  $(R,G,B) = (1 - |e_1|, 1 - |e_2|, 1 - |e_3|)$ . (f): sectional images of (a). (b,c,d,e): sectional images where their locations correspond to (f), and their RGB color channels consist of  $(1-C, 1-C, 1-C)$ ,  $(1-|e_1|, 1-|e_2|, 1-C)$ ,  $(1-|e_1|, 1-C, 1-|e_3|)$ , and  $(1-C, 1-|e_2|, 1-|e_3|)$ , respectively. The upper and bottom images are obtained by using  $3^3$  ( $r = 1$ ) and  $7^3$  ( $r = 3$ ) voxel neighborhoods to the fitting process, respectively.

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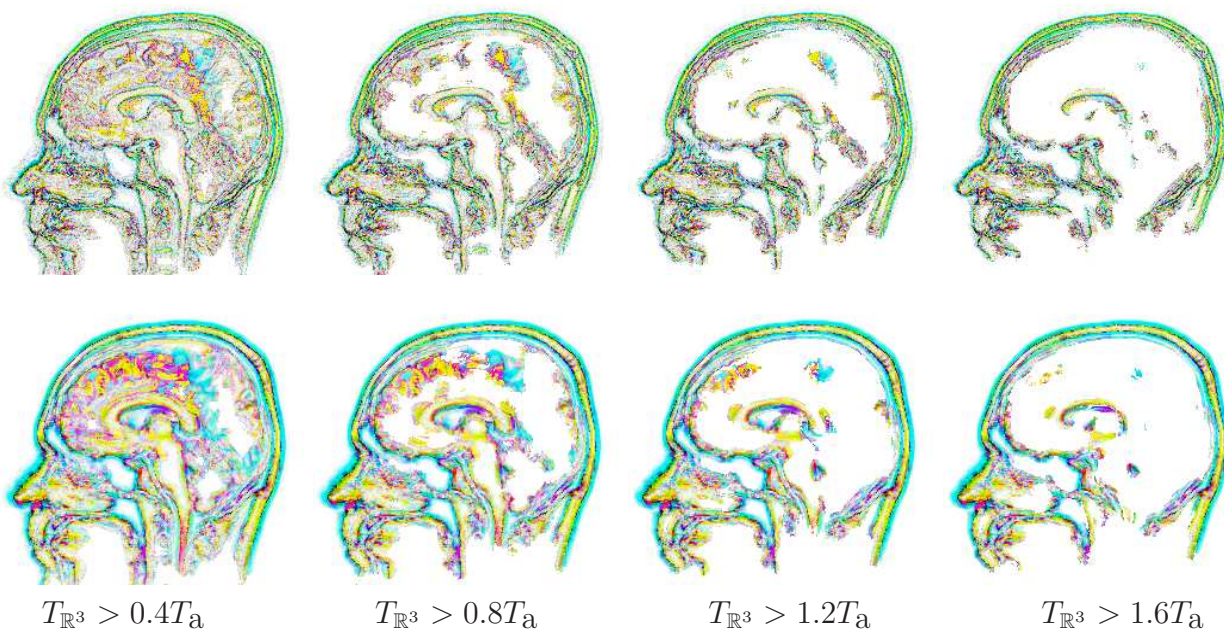


Figure 11: Filtering phantom ridge-like structures via the conformal invariant differential quantity (1), where the input image and its extremalities  $(R,G,B) = (1 - |e_1|, 1 - |e_2|, 1 - |e_3|)$  corresponding to  $T_{\mathbb{R}^3} \geq 0$  are illustrated in Fig. 10 (upper:  $r = 1$  and bottom:  $r = 3$ ), we use  $r_f = 5$ , and  $T_a$  is an average of  $T_{\mathbb{R}^3}$  within the input image.

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