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General Rules of Fractals Construction from Polyhedra

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Abstract. The paper presents a method of construction of deterministic fractals based on uniform polyhedra using a contraction mapping procedure and an iterated function system algorithm. It was shown that the contraction mapping procedure, which implies the construction of fractals with non-overlapped and non-disjointed contractions, could produce only a limited number of fractals from uniform polyhedra, which is resulted by geometric specificity of some uniform polyhedra. The lists of uniform polyhedra from which fractals either can be constructed and not were presented and discussed. The contraction ratios and fractal dimensions of the constructed fractals were determined, some of uniform polyhedra-based fractals were presented graphically.

Key Words: fractals, uniform polyhedra, iterated function systems, contraction mapping

MSC 2010: 51M20, 28A80

1. Introduction

The fractal geometry became popular and found topical scientific interest when B. Mandelbrot introduced his works on complex fractals and fractal dimension and its application in statistical mathematics. Deterministic fractals found an application in some disciplines of science and engineering. They are used for testing ray-tracing algorithms during the rendering of spatial scenes [3], in description of dynamical systems [1], in light diffraction problems [10], hydrologic modeling [2], materials science [14], and many others.

The first strictly self-similar fractals were known much earlier before B. Mandelbrot. Two-dimensional deterministic fractals proposed by W. Sierpiński and his student [11,

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12], known as *Sierpiński triangle* and *Sierpiński carpet*, were developed in 1915 and 1916, respectively. Later, these fractals were generalized into the three-dimensional space and known as *Sierpiński gasket* and *Menger sponge*. Only the above-presented fractals are well-known and presented in many references [9]. However, there is much more deterministic geometric objects which are able to construct fractals from them. The authors of [7] generalized the fractal construction procedure from regular polygons, while the authors of [8] constructed fractals from Platonic solids. These works implied the interest of a generalization of 3D deterministic fractals. The results presented in [5] showed that among the 13 Archimedean solids only from 9 fractals can be constructed. This study stimulated the investigation of more general polyhedra with these properties.

For the generation of fractals two approaches can be used: deterministic or stochastic generation procedures. The *deterministic generation procedure* is based on the Iterated Function System (IFS) algorithm or the Multiple Reduction Copy Machine (MRCM), while the *stochastic generation procedure* is based on Barnsley's Chaos Game or its combination with IFS, e.g., Chaos Game for IFS connected in the net [9]. Both approaches represent complex geometry and have their own advantages: IFS-based algorithms use relatively simple mathematical operations on sets, while the Chaos Game-based algorithms are useful when high iterations of the fractal's attractor should be computed.

The aim of this paper is to determine new fractals based on uniform polyhedra, to determine from which uniform polyhedra fractals can be constructed and from which not, and to describe general rules of 3D fractals construction based on the investigated polyhedra. Based on a weak formulation of the fractal construction theorem presented in [5] it was generalized to all polyhedra with a description of cases when the polyhedron is not suitable for fractal construction.

2. Statements and algorithm

2.1. General considerations

Following the Sopov theorem [13] there are 75 possible uniform polyhedra (except prismatic polyhedra and Skilling's figure): 5 Platonic solids, 13 Archimedean solids, 4 Kepler-Poinsot solids and 53 other non-convex (star-)polyhedra. From most of them fractals can be constructed, which we prove further. In this study Platonic and Archimedean solids were excluded from consideration, because related results were already presented in [8] and [5]. Concerning the remaining 57 non-convex polyhedra, the possibility of fractal construction was studied in terms of the following definition of fractals.

Definition 1. *Let A_0^W be a uniform polyhedron with a set of vertices v_n of A_0^W with coordinates $v_{n,a}$ $(a = 1, 2, 3)$ in the Euclidean space $\mathbb{R}^3$, where W denotes an index of a uniform polyhedron following Wenninger's notation assumed in [15]; the subscript symbol of A denotes the number of contraction mapping iterations. Thus, the fractal based on a given polyhedron is defined as the attractor A_∞^W of IFS, which is the set of*

$$A_\infty^W = \bigcap_{i=0}^{\infty} h_i\left(A_0^W\right), \tag{1}$$

where $h_i()$ is an elementary similarity transformation.

The contraction process from A_k to A_{k+1}, $k \geq 0$, was realized using the Hutchinson operator:

$$H\left(A_{\infty}^{W}\right) = \bigcup_{i=1}^{N_k} h_i\left(A_k^W\right), \tag{2}$$

where N_k is a number of subsets for k-th iteration, thus

$$\forall_{v_n \in A_k^W}\ h_i\left(v_n\right) = \frac{v_n}{r(W)} - \frac{v_i\left(1 - r(W)\right)}{r(W)}, \tag{3}$$

where $r(W)$ is the contraction ratio of a polyhedron W, which ensures that the contractions of h_i are non-overlapped and non-disjointed: $h_i\left(A_k^W\right) \cap h_j\left(A_k^W\right) = \emptyset$, $i \neq j$. Such an object is the fractal of A_0^W in $\mathbb{R}^3$ with contraction ratio $r(W)$.

The dimension of fractals D is mostly fractional (with few exceptions) and for 3D fractals it should be $2 \geq D \geq 3$. For the definiton of fractal dimension a lot of formulations exist, but the most general one was proposed by F. Hausdorff in [4], which is defined as a power law:

$$D\left(W\right) = \frac{\ln\left(N\left(A_k^W\right)\right)}{r\left(A_k^W\right)}. \tag{4}$$

Basing on the above-presented Definition 1, the method of fractals construction was proposed.

2.2. Algorithm of fractals construction

The proposed algorithm of construction of fractals based on uniform polyhedra consists of the following steps:

- A given uniform polyhedron A_0^W with vertices $v_n \in \mathbb{R}^3$ was inscribed in a unit sphere $P \in \mathbb{R}^3$ with the central point placed in the origin c_0. Having A_0^W inscribed in P, the vertices v_n were determined.
- Then an basic edge of A_0^W was chosen and an orthogonal projection onto $\mathbb{R}^2$ was performed.
- In the orthogonal projection of A_0^W onto $\mathbb{R}^2$ the vector $\vec{a}$ between the chosen basic edge and the most distant left/right vertex was determined. The angle between $\vec{a}$ and a plane perpendicular to the base was determined.
- The maximal width of the orthogonal projection of A_0^W was determined. The ratio between the basic edge length and maximal width of A_0^W's orthogonal projection was computed, which is the contraction ratio $r\left(A_0^W\right)$.
- The central points $c_i(N)$ of contractions $h_i\left(A_0^W\right)$ were determined and A_0^W was replaced by A_1^W.

By repeating these operations i times the next iterations till A_i^W could be obtained. The algorithm implies that the number of contractions $h_i\left(A_0^W\right)$ is equal to the number of vertices $v_{n,i}$.

3. Construction of fractals from polyhedra and discussion

3.1. Uniform polyhedra

Applying the algorithm desribed in 2.2 we construct fractals from uniform polyhedra. It turns out that among in total 75 uniform polyhedra there are only 41 from which fractals can be derived. Furthermore, these 41 fractals can be grouped by common contraction ratios and common fractal dimension. Let us investigate this phenomenon.

Theorem 1. *There exist groups of fractals constructed from uniform polyhedra in terms of Definition 1, which have identical values of contraction ratio and fractal dimension.*

Proof: The fractals are grouped by the identical number of vertices of A_0^W with identical coordinates of these vertices. That is, the polyhedra of each group have the same convex hull, e.g., $\operatorname{conv}\left(v_{n,a}\left(A_0^{W002}\right)\right) \equiv \operatorname{conv}\left(v_{n,a}\left(A_0^{W067}\right)\right)$. $\square$

The complete list of uniform polyhedra from which we are able to construct fractals were presented in Table 1 and grouped by the contraction ratio $r(W)$ and fractal dimension $D(W)$. Some of aesthetically attractive initial iterations of fractals were shown in Fig. 1.

Table 1: Uniform polyhedra able for the construction of fractals

<table>
<tr><th>Index</th><th>$r(W)$</th><th>$D(W)$</th><th>Index</th><th>$r(W)$</th><th>$D(W)$</th><th>Index</th><th>$r(W)$</th><th>$D(W)$</th></tr>
<tr><td>W001</td><td>2.0000</td><td>2.0000</td><td>W006</td><td rowspan="4">3.0000</td><td rowspan="4">2.2618</td><td>W010</td><td rowspan="4">6.8538</td><td rowspan="4">2.1271</td></tr>
<tr><td>W002</td><td rowspan="2">2.0000</td><td rowspan="2">2.5849</td><td>W011</td><td>W081</td></tr>
<tr><td>W067</td><td>W068</td><td>W088</td></tr>
<tr><td>W003</td><td>2.0000</td><td>3.0000</td><td>W078</td><td>W101</td></tr>
<tr><td>W016</td><td>8.4721</td><td>2.2404</td><td>W013</td><td rowspan="4">3.4142</td><td rowspan="4">2.5882</td><td>W012</td><td rowspan="9">4.2361</td><td rowspan="9">2.3561</td></tr>
<tr><td>W004</td><td rowspan="4">2.6180</td><td rowspan="4">2.5819</td><td>W069</td><td>W073</td></tr>
<tr><td>W020</td><td>W086</td><td>W089</td></tr>
<tr><td>W021</td><td>W092</td><td>W091</td></tr>
<tr><td>W041</td><td>W014</td><td rowspan="4">5.2361</td><td rowspan="4">2.4729</td><td>W094</td></tr>
<tr><td>W005</td><td rowspan="5">3.6181</td><td rowspan="5">2.3296</td><td>W072</td><td>W100</td></tr>
<tr><td>W022</td><td>W074</td><td>W102</td></tr>
<tr><td>W070</td><td>W097</td><td>W106</td></tr>
<tr><td>W087</td><td>W007</td><td>4.0000</td><td>2.2945</td><td>W107</td></tr>
<tr><td>W080</td><td>W009</td><td>5.8544</td><td>2.3168</td><td colspan="3"></td></tr>
</table>

There are 34 residual uniform polyhedra, from which no fractals can be constructed, because the contractions of A_0^W were overlapping or disjointed and there is more than one contraction ratio for these polyhedra.

Theorem 2. *For any uniform polyhedron A_0^W there exists a fractal construction based on Definition 1 iff the contraction ratio $r(W)$ is unique.*

Proof: Following the above-presented algorithm of fractal construction, the ratio between the basic edge length and the maximal width of the orthogonal projection of A_0^W must be the same regardless of a chosen basic edge. Since the points of $v_n\left(A_0^W\right)$ are the prisoner points of A_∞^W (see [9, p. 74] for the definition) there is a relation between points of contractions for arbitrary two iterations: $v_n\left(A_k^W\right) \equiv v_n\left(A_l^W\right)$, where $0 \leq k, l \leq \infty$. Thus, $h_i\left(A_k^W\right)$ has exactly one common point with A_k^W. $\square$

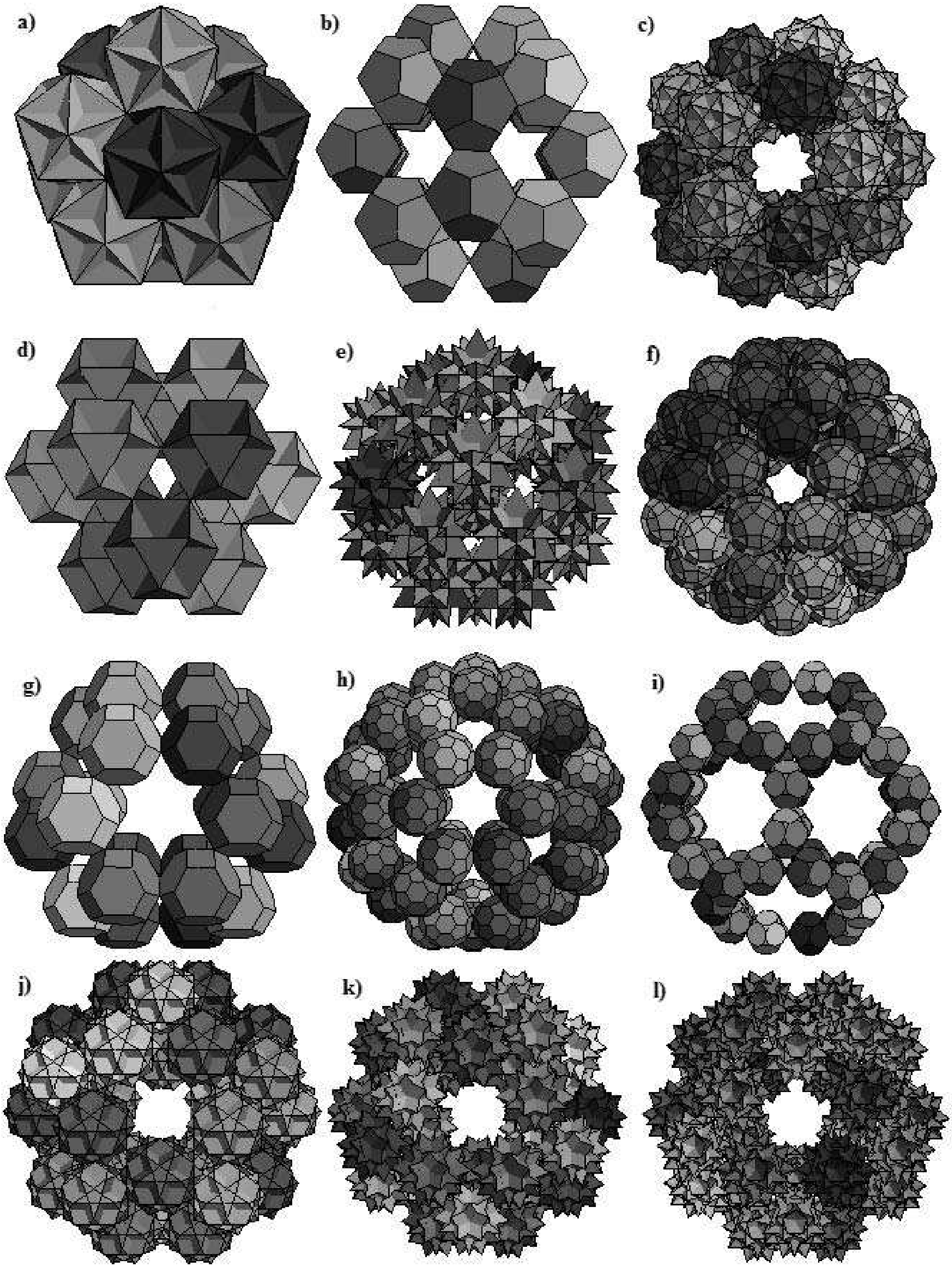

Figure 1: First interations of fractals of a) W021, b) W005, c) W087, d) W068, e) W092, f) W074, g) W007, h) W009, i) W010, j) W073, k) W094, l) W107.

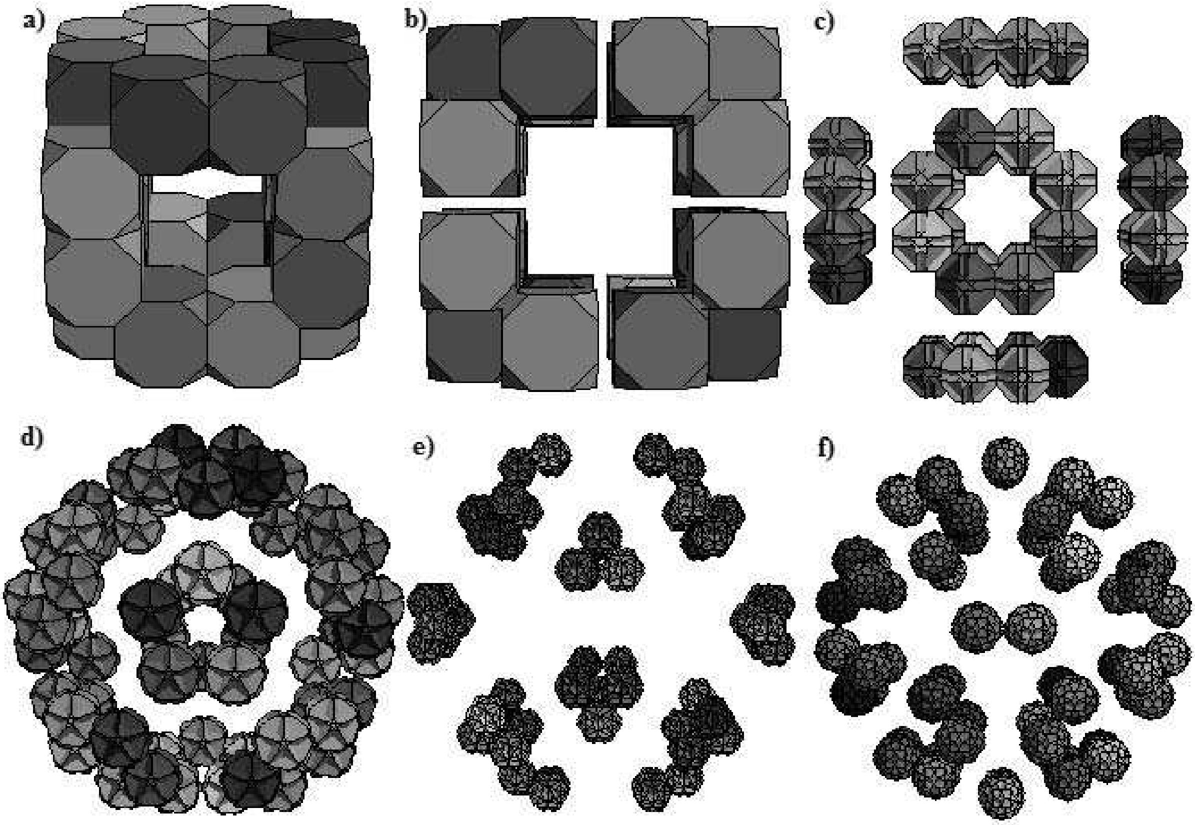

Figure 2: First iterations of non-fractals of W008 with a) $r_1 = 2 + \sqrt{2}$, b) $r_2 \approx 3.8476$ and other examples of non-fractals with genus of c) W079, d) W075, e) W095, f) W076.

Lemma 1. *If the affine transformation T of A_0^W does not exist, then $A_0^W\,(T, r(W))$ also does not exist.*

Proof: Let us consider regular and quasi-regular polyhedra, which are genera of non-convex uniform polyhedra through tessellation, truncation, stellation or other operations. As it is proved, they create groups of uniform polyhedra, where from every polyhedron with a single group a fractal can be constructed or can't. The ability of fractal construction for a given group of polyhedra depends on the existence of affine transformations and the possibility of isomorphic transformations on edges and vertices, which is closely connected with symmetry properties. If these statements do not hold (e.g., lack of affine transformation), A_0^W has more than one $r(W)$. □

For instance, let us consider W008, which has two types of possible basic edges: on the triangular face and the octagonal face. Depending on the choice, two contraction ratios could be obtained: $r_1(W008) = 2 + \sqrt{2}$ and $r_2(W008) \approx 3.8476$. The attempts of fractal constructions with these contraction ratios give overlapped contractions for r_1 and disjointed contractions for r_2 (see Fig. 2a,b). Additional cases of polyhedra, which do not fulfill Theorem 2, were presented in Fig. 2 c–f.

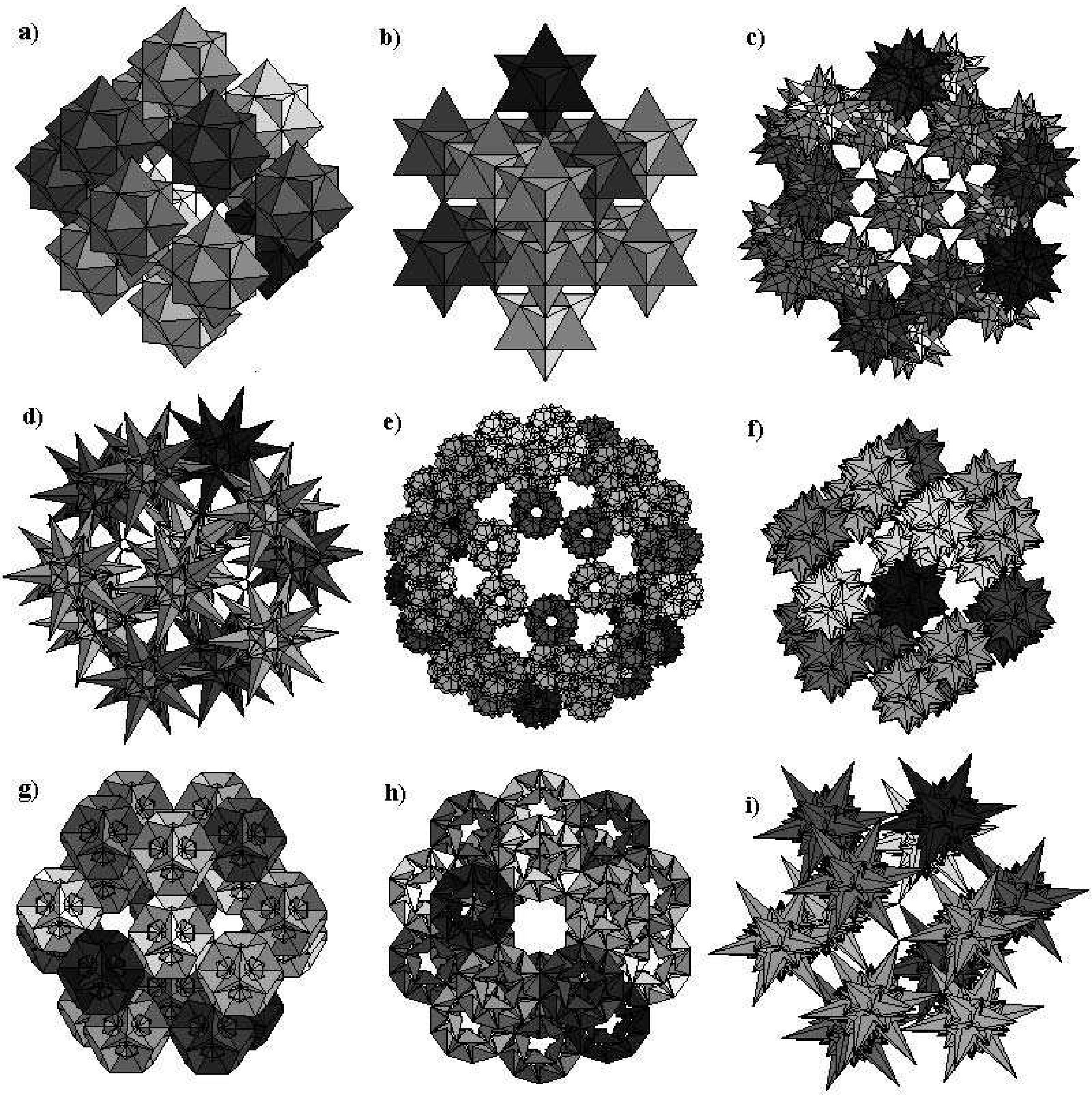

Figure 3: First iterations of fractals a) on rhombidodecahedral convex hull with genus of W043, b) with genus of W019, c) on dodecahedral convex hull with genus of W032, d) on icosahedral convex hull with genus of W027, e) on truncated icosahedral convex hull with genus of 13th stellation of W005, f) on dodecahedral convex hull with genus of 20th stellation of W005, g) on dodecahedral convex hull with genus of 23rd stellation of W005, h) on dodecahedral convex hull with genus of 34th stellation of W005, i) on icosahedral convex hull with genus of 50th stellation of W005.

3.2. Wenninger's stellations and non-uniform polyhedra

Apart from uniform polyhedra Wenninger [15] proposed non-uniform ones obtained by stellation of regular and semi-regular polyhedra — total 44 stellations. With some exceptions (i.e., W019÷W022) these polyhedra cannot be genera for fractals in terms of Definition 1. However, fractals could be constructed using these polyhedra following the assumption, that only $v_n \in P$ are considered for the contraction process from A_k^W to A_{k+1}^W.

Theorem 3. *Fractals can be constructed from any non-uniform polyhedron A_0^Q, where Q is an arbitrary non-uniform polyhedron, iff* conv $\left(A_0^Q\right)$ *is suitable for the construction of a fractal.*

Proof: When the convex hull conv $\left(A_0^Q\right)$ of a polyhedron satisfies Theorem 2 and Lemma 1, then $v_n\left(A_0^Q\right) \equiv v_n\left(\mathrm{conv}\left(A_0^Q\right)\right)$ for $v_n \in P$, which guarantees that $h_i\left(A_k^Q\right) \cap h_j\left(A_k^Q\right) = \emptyset$ for $i, j \geq 0$. □

The convex hulls of non-uniform Wenninger's polyhedra are regular and semi-regular polyhedra in most cases. However, some of them are Catalan polyhedra or Archimedean duals, which contain only three polyhedra suitable for fractal construction: the rhombic dodecahedron, the tetrakis hexahedron and the pentakis dodecahedron [6]. Exemplary fractals constructed from non-uniform Wenninger's polyhedra were presented in Fig. 3. There is a very big number of non-uniform polyhedra (including Johnson solids, Stewart toroids, compounds of polyhedra, geodesic domes etc.), so it is not possible to classify all of them with regard to the ability of fractals construction on their convex hulls. However, if a given polyhedron satisfies Theorem 3 the fractal can be constructed as well.

4. Conclusions

The polyhedra, both uniform and non-uniform ones, were analyzed in terms of their possibility of fractal construction. It was determined, that fractals could be constructed from 41 uniform solids, the list of them was presented. For uniform polyhedra, both genera and non-genera, the comments for factors which have an influence on the ability of fractals construction were presented. It was proved, that the uniqueness of a contraction ratio regardless of the choice of a basic edge is an absolutely necessary condition for the construction of fractals with non-adjacent and non-overlapped contractions. The genera of fractals were grouped by values of contraction ratio and fractal dimension. It was also shown, that fractals could be constructed from non-uniform polyhedra by using their convex hulls for placing contractions. Further research will be focused on the ability of fractals construction from polytopes of higher dimensions.

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