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Discovery of Dual Quaternions for Geodesy

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Abstract. The main aim of this paper is to show one application of dual quaternions in one of the challenging problem of geodesy. The Bursa-Wolf similarity transformation model is presented as a seven parameter model for transforming co-located 3D Cartesian coordinates between two datums. The transformation involves three translation parameters, three rotation elements and one scale factor. We will briefly introduce the theory of quaternions and dual quaternions. Consequently, it is shown that mathematical modelling based on dual quaternions is an elegant mathematical method which is used to represent rotation and translation parameters and a compact formula is derived for the Bursa-Wolf model.

Key Words: Dual quaternion, datum transformation, Bursa-Wolf model

MSC 2010: 51N20, 86A30

1. Introduction

A coordinate transformation allows us to take the coordinates of a point in one coordinate system and find the new coordinates of the same point in a second coordinate system. This mathematical operation is mainly used in geodesy but we can find its using in photogrammetry, Geographical Information Science (GIS), computer vision and other research areas.

Spatial data are connected to the geographical location which are expressed by coordinates based on a coordinate system. The basis of the coordinate system is called a geodetic datum which defines the size and shape of the Earth, and the origin and orientation of the coordinate systems used to map the Earth. There are many datums because different countries use their own datum. We can mention, for instance, that in geodesy datum transformations are used to convert coordinates related to the Czech national reference frame S-JTSK to the new reference frame WGS 84[1] (World Geodetic System).

Similarity transformation is a type of transformation, where the scale factor is the same in all directions. The seven parameter[2] similarity transformation is widely used for the datum

[1]Dating from 1984 and last revised in 2004.
[2]This transformation is also known as 3D similarity transformation or Helmert transformation.

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transformation since it, more or less, satisfies simplicity, efficiency, uniqueness and rigor. This transformation is composed of three translation parameters, one scale factor, and three rotation parameters. Therefore, coordinates from a three dimensional coordinate frame can be transformed into coordinates in another frame by translating the origin, applying rotation and modifying the scale. In practice, the seven transformation parameters are not always known. But if for some control points the coordinates with respect to two coordinate frames are known we can estimate the transformation parameters mentioned above. We can say that common coordinates of three points are sufficient for the solution of the seven parameters transformation. There are some popular seven parameter similarity transformation models such as the *Bursa-Wolf*, which we deal with, or the Molodensky model (see [9]). The similarity transformation model is often simplified to a linear one because its parameters can be easily computed (see [8]).

Existing solutions of seven parameter models solved by traditional algorithms based on rotation angles or recently quaternions are replaced by a new model based on dual quaternions. We will briefly present how to represent and improve the datum transformation by dual quaternions.

The remainder of the paper is organized as follows. Section 2 recalls some basic notions and facts from the quaternion geometry which are consequently used for introduction of dual quaternions. We introduce the definition of dual numbers and dual quaternions. Furthermore, dual quaternions are used for describing rigid transformations in the special Euclidean group. The following part is devoted to a practical application of dual quaternions. Subsequently, transformation model based on quaternion algorithm is reminded. Then the formula for the computation of rotation, translation and scale parameters in the Bursa-Wolf geodetic datum transformation model from two sets of co-located three-dimensional coordinates is derived. Furthermore, we will focus on comparing this algorithm with another algorithm based on quaternions.

2. Preliminaries

As we have mentioned, our results are based on fundamentals of quaternions. This paper explores the basics of the quaternion algebra, particularly its description of the three dimensional rotations. Let us therefore start our discussion with recalling some fundamental facts (see e.g., [4]).

2.1. Quaternions and rotations

A *quaternion* $\mathcal{Q}$ can be defined as follows

$$\mathcal{Q} = 1q_0 + \mathbf{i}\, q_1 + \mathbf{j}\, q_2 + \mathbf{k}\, q_3, \tag{1}$$

where $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are basis elements called *quaternion units* satisfying the relations $\mathbf{i}^2 = \mathbf{k}^2 = \mathbf{j}^2 = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = -1$, $\mathbf{i}\,\mathbf{j} = \mathbf{k}$, $\mathbf{j}\,\mathbf{i} = -\mathbf{k}$ and q_0, q_1, q_2, q_3 are real numbers. Otherwise, it can be written as comprising scalar and vector parts q_0 and $\mathbf{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. All units commute with real numbers. The set of quaternions is denoted $\mathbb{H}$. Hence, we write $\mathcal{Q} = (q_0, \mathbf{q})$. A *pure* quaternion is a quaternion with zero scalar part, i.e., $\mathcal{Q} = (0, \mathbf{q})$ (see Fig. 1). The set of pure quaternions[3] is denoted by $\mathbb{H}_p$.

[3]Sometimes called imaginary quaternions.

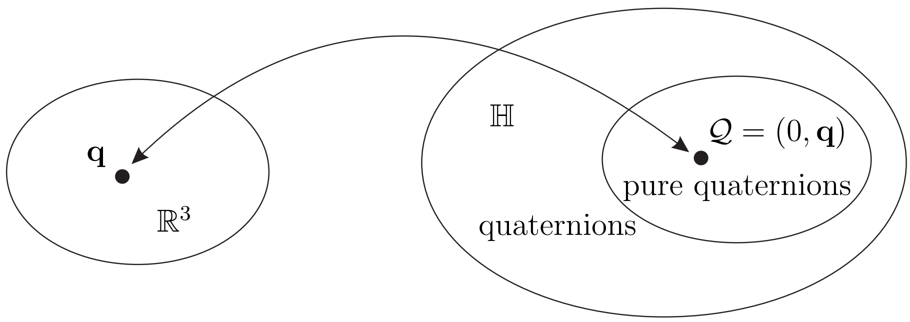

Figure 1: Representation of relation between quaternions and vectors in $\mathbb{R}^3$

The corresponding *conjugate* quaternion is

$$\mathcal{Q}^* = q_0 - \mathbf{i}\, q_1 - \mathbf{j}\, q_2 - \mathbf{k}\, q_3 = q_0 - \mathbf{q}. \tag{2}$$

From the definition we immediately have

$$(\mathcal{Q}^*)^* = q_0 - (-\mathbf{q}), \tag{3}$$
$$\mathcal{Q} + \mathcal{Q}^* = 2q_0. \tag{4}$$

The *product* of two quaternions $\mathcal{Q}$ and $\mathcal{P}$ is defined in a concise form as

$$\mathcal{Q}\mathcal{P} = [\, q_0 p_0 - \mathbf{q} \cdot \mathbf{p},\ \mathbf{q} \times \mathbf{p} + q_0 \mathbf{p} + p_0 \mathbf{q}\,], \tag{5}$$

where the symbols $\cdot$ and $\times$ are the standard dot product and the cross product in $\mathbb{R}^3$. Given two quaternions $\mathcal{Q}$ and $\mathcal{P}$, we can easily verify that

$$(\mathcal{Q}\mathcal{P})^* = \mathcal{P}^* \mathcal{Q}^*. \tag{6}$$

The *norm* of a quaternion is defined as

$$\| \mathcal{Q} \| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = \sqrt{\mathcal{Q}\mathcal{Q}^*}. \tag{7}$$

The norm of the product of two quaternions $\mathcal{Q}$ and $\mathcal{P}$ is the product of the individual norms, for we have

$$\| \mathcal{Q}\mathcal{P} \|^2 = \| \mathcal{Q} \|^2 \| \mathcal{P} \|^2. \tag{8}$$

The multiplicative *inverse* of each $\mathcal{Q} \neq 0$ is computed as

$$\mathcal{Q}^{-1} = \frac{\mathcal{Q}^*}{\|\mathcal{Q}\|^2}. \tag{9}$$

We can easily verify that $\mathcal{Q}^{-1}\mathcal{Q} = \mathcal{Q}\mathcal{Q}^{-1} = 1$. A quaternion is called the *unit* quaternion if $\|\mathcal{Q}\| = 1$. If $\mathcal{Q}$ is the unit quaternion then there exists a unit vector $\mathbf{n}$, an angle $\frac{\theta}{2} \in \langle -\pi, \pi \rangle$ such that

$$\mathcal{Q} = \left(\cos\frac{\theta}{2},\ \mathbf{n} \sin\frac{\theta}{2} \right). \tag{10}$$

The *special orthogonal group* is defined as

$$\mathbf{SO}(3) = \{ \mathbf{A} \in \mathbf{GL}(3, \mathbb{R}) \mid \mathbf{A}^T \mathbf{A} = \mathbf{I} \wedge \det \mathbf{A} = 1 \}. \tag{11}$$

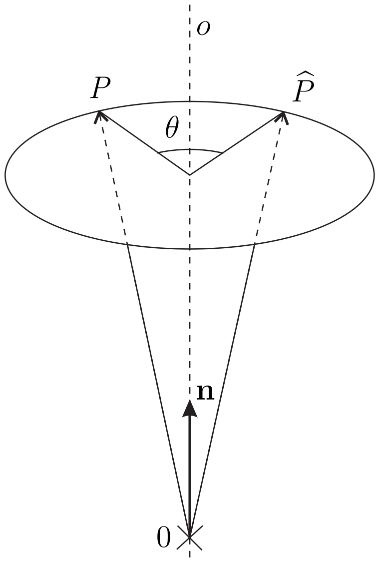

Figure 2: Rotation of a point P through the angle θ about the axis o given by the vector $\mathbf{n}$

The matrix $\mathbf{A}$ represents a rotation in $\mathbb{R}^3$ about the origin if and only if $\mathbf{A} \in \mathbf{SO}(3)$ (see [10] and the references therein). In the following statement we can easily see how the unit quaternions can represent rotations.

A unit quaternion $\mathcal{Q} = \left(\cos\frac{\theta}{2},\ \mathbf{n}\sin\frac{\theta}{2}\right)$ represents the rotation of a vector $\mathbf{p}$ through the angle θ about an axis through the origin in direction of the vector $\mathbf{n}$ (see Fig. 2). The vector $\mathbf{p}$ is represented by the pure quaternion $\mathcal{P} = (0, \mathbf{p})$. The rotated vector, represented as a pure quaternion, is

$$\widehat{\mathcal{P}} = \mathcal{Q}\mathcal{P}\mathcal{Q}^*. \tag{12}$$

We can find an elegant proof of this in [4]. First, it is shown how a vector $\mathbf{p}$ is rotated by θ along $\mathbf{n}$, using Sine, Cosine and the scalar and the vector products. Then it is shown that the same result is obtained by a rotation described by quaternions. Each element in $\mathbf{SO}(3)$ can be expressed using quaternions as (12) (see [5], for instance).

2.2. Dual numbers and dual quaternions

This section provides a brief introduction to the theory of dual numbers and dual quaternions. We focus only on the basics of this algebra. More details can be found in [2], [3] or [13].

Dual numbers

Dual numbers were invented by Clifford in 1873 but their first applications to mechanics are due to Alexandr Petrovič Kotelnikov (1865–1944) (see [2] or [7] for more details). They are similar to complex numbers because any *dual number* z_d can be written as

$$z_d = a + \varepsilon a_\varepsilon, \tag{13}$$

where a is the non-dual part, a_ε the dual part and ε is a basis element called *dual unit.* The defining condition for the dual unit is $\varepsilon^2 = 0$. The set of dual numbers is denoted by $\mathbb{D}$. The *dual conjugate* is analogous to the complex conjugate, i.e.,

$$\overline{z_d} = a - \varepsilon a_\varepsilon. \tag{14}$$

The *multiplication* of two dual numbers is given as

$$z_d \hat{z}_d = a\hat{a} + \varepsilon(a\hat{a}_\varepsilon + a_\varepsilon \hat{a}). \tag{15}$$

Finally, note that *pure* dual numbers, i.e., dual numbers with the $a = 0$, do not have an inverse. This is a fundamental difference from complex numbers because every non-zero complex number has the inverse defined. An example of dual number is the *dual angle* between two skew lines in three dimensional space defined as

$$\alpha = \beta + \varepsilon s, \tag{16}$$

where β is the angle between their directions and s is the minimal distance between the lines along their common perpendicular line.

Dual quaternions

A *dual quaternion* $\mathcal{Q}_d$ can be written as the sum of two standard quaternions

$$\mathcal{Q}_d = \mathcal{Q} + \varepsilon \mathcal{Q}_\varepsilon, \tag{17}$$

where

$$\mathcal{Q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \quad \text{and} \quad \mathcal{Q}_\varepsilon = q_{0\varepsilon} + q_{1\varepsilon}\mathbf{i} + q_{2\varepsilon}\mathbf{j} + q_{3\varepsilon}\mathbf{k}, \tag{18}$$

are real quaternions and $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ are usual quaternion units. The dual unit ε commutes with quaternion units, for example $\mathbf{i}\,\varepsilon = \varepsilon\mathbf{i}$. The set of dual quaternions is denoted $\mathbb{H}_d$. A dual quaternion can also be considered as an 8-tuple of real numbers, or as

$$\begin{aligned} \mathcal{Q}_d &= q_{0d} + q_{1d}\mathbf{i} + q_{2d}\mathbf{j} + q_{3d}\mathbf{k} \\ &= (q_0 + \varepsilon q_{0\varepsilon}) + (q_1 + \varepsilon q_{1\varepsilon})\mathbf{i} + (q_2 + \varepsilon q_{2\varepsilon})\mathbf{j} + (q_3 + \varepsilon q_{3\varepsilon})\mathbf{k}\,, \end{aligned} \tag{19}$$

where q_{0d} is the scalar part (a dual number), (q_{1d}, q_{2d}, q_{3d}) is the vector part (a dual vector) (see [13]). The *product* of two dual quaternions $\mathcal{Q}_d$ and $\widehat{\mathcal{Q}}_d$is defined as

$$\mathcal{Q}_d \widehat{\mathcal{Q}}_d = \widehat{\mathcal{Q}} + \varepsilon(\mathcal{Q}\widehat{\mathcal{Q}}_\varepsilon + \mathcal{Q}_\varepsilon \widehat{\mathcal{Q}}).$$

The multiplication of dual quaternions is associative, distributive, but not commutative. The *conjugation* of a dual quaternion is defined using the classical quaternion conjugation

$$\mathcal{Q}_d{}^* = Q^* + \varepsilon \mathcal{Q}_\varepsilon^*. \tag{20}$$

However, the dual number conjugation (14) can be applied to dual quaternion conjugation and we get the *dual conjugate dual quaternion*

$$\overline{\mathcal{Q}_d^*} = \mathcal{Q}^* - \varepsilon \mathcal{Q}_\varepsilon^*. \tag{21}$$

The *norm* of the dual quaternion is the dual scalar and is defined as

$$\| \mathcal{Q}_d \| = \sqrt{(q_0 + \varepsilon q_{0\varepsilon})^2 + (q_1 + \varepsilon q_{1\varepsilon})^2 + (q_2 + \varepsilon q_{2\varepsilon})^2 + (q_3 + \varepsilon q_{3\varepsilon})^2} = \sqrt{\mathcal{Q}_d^* \mathcal{Q}_d}\,. \tag{22}$$

If the norm has a nonvanishing real part, then the dual quaternion $\mathcal{Q}_d$ has an *inverse*, which can be defined as

$$\mathcal{Q}_d^{-1} = \frac{\mathcal{Q}_d^*}{\| \mathcal{Q}_d \|^2}\,. \tag{23}$$

A dual quaternion is called dual *unit* quaternion if $\| \mathcal{Q}_d \| = 1$. Note that a dual quaternion $\mathcal{Q}_d$ is unit if and only if

$$\| \mathcal{Q} \| = 1 \quad \wedge \quad \mathcal{Q} \cdot \mathcal{Q}_\varepsilon = 0. \tag{24}$$

If we have a vector $\mathbf{p} = (p_1, p_2, p_3)$, we define the associated dual unit quaternion as

$$\mathcal{P}_d = 1 + \varepsilon(p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}), \tag{25}$$

which satisfy the previous statement. The *special Euclidean group* is defined as

$$\mathbf{SE}(3) = \left\{ \mathbf{A} \mid \mathbf{A} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix},\ \mathbf{R} \in \mathbf{SO}(3),\ \mathbf{t} \in \mathbb{R}^3 \right\}, \tag{26}$$

i.e., $\mathbf{SE}(3)$ is the set of all rigid transformations in three dimensions.

A new method to represent the rigid transformations is based on the use of dual quaternions (see [11]). Dual quaternions capture in their inner structure basic information about these transformations — namely the axis of rotation and the rotation angle about the axis and the translation about the axis. The composition of these transformations corresponds to the multiplication of the associated dual quaternions.

Suppose that $\mathbf{p} = (p_1, p_2, p_3)$ is a position vector of a point P, $\mathbf{t} = (t_1, t_2, t_3)$ is a translation vector and $\mathcal{Q} = \left(\cos\frac{\theta}{2},\ \mathbf{n}\sin\frac{\theta}{2}\right)$ is a unit quaternion (see Fig. 3). Then we can express the image of the point P after this translation and this rotation as

$$\widehat{\mathcal{P}}_d = \mathcal{Q}_d\, \mathcal{P}_d\, \overline{\mathcal{Q}_d^*}, \tag{27}$$

where $\mathcal{P}_d$, $\mathcal{Q}_d$ are the dual unit quaternions and $\mathcal{T}$ is the pure quaternion fulfilling

$$\mathcal{Q}_d = \mathcal{Q} + \varepsilon\mathcal{Q}_\varepsilon = \mathcal{Q} + \varepsilon\frac{\mathcal{T}\mathcal{Q}}{2}, \quad \mathcal{T} = t_1\mathbf{i} + t_2\mathbf{j} + t_3\mathbf{k} \quad \text{and} \quad \mathcal{P}_d = 1 + \varepsilon(p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}). \tag{28}$$

To sum up, a dual unit quaternion naturally represents a rotation when the dual part $\mathcal{Q}_\varepsilon = 0$ (see (12)).

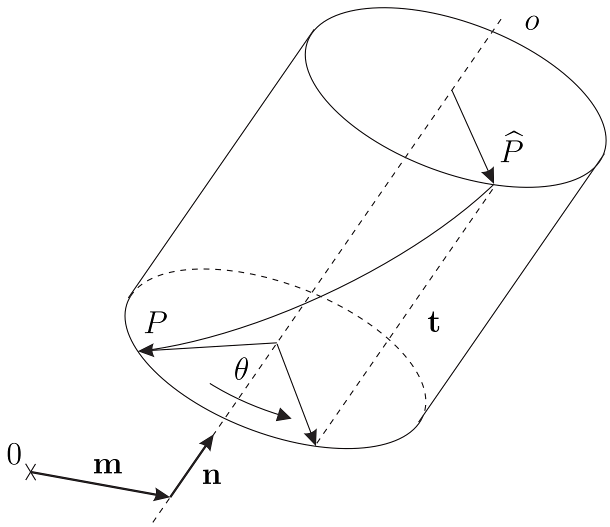

Figure 3: Transformation of a point P given by translation vector $\mathbf{t}$ and rotation through the angle θ about the axis o given by vector $\mathbf{n}$ and position vector $\mathbf{m}$

3. Dual quaternion datum transformation model

In this section, we will present a particular application of dual quaternions in the field of geodesy.

Bursa-Wolf similarity transformation model

One of the most commonly used transformation methods in the geodetic applications is the Bursa-Wolf similarity transformation model. The similarity transformation is popular due to the small number of parameters involved and the simplicity of the model. Our goal is to estimate all required parameters from co-located coordinates on two different datums. The Bursa-Wolf similarity transformation model can be written as

$$\mathbf{s}_i = \mathbf{t} + k\,\mathbf{R}\mathbf{p}_i, \tag{29}$$

where $\mathbf{s}_i = (s_{1i}, s_{2i}, s_{3i})^T \in \mathbb{R}^3$ and $\mathbf{p}_i = (p_{1i}, p_{2i}, p_{3i})^T \in \mathbb{R}^3$, $i = 1, \ldots, n$, are two sets of the co-located coordinates in the two different systems. $\mathbf{t} = (t_1, t_2, t_3)^T \in \mathbb{R}^3$ denotes the three translation parameters, k denotes the scale parameter and $\mathbf{R} \in \mathbf{SO}(3)$ is the rotation matrix containing three rotation parameters. In order to determine the mentioned parameters, the number n of the co-located coordinates $\mathbf{s}_i, \mathbf{p}_i$ must be greater than or equal to three.

3.1. Quaternion method

First, we remind one of the newest approaches which is used for solving this problem. In this case, the solution of Bursa-Wolf transformation model is based on quaternions. Since they are widely used to express the rotation which is in this model included. Let us therefore start with reminding this procedure. For a deeper discussion of this method we refer the reader to [12].

We define centrobaric coordinates $\triangle\mathbf{s}_i = (\triangle s_{1i}, \triangle s_{2i}, \triangle s_{3i})^T$, $\triangle\mathbf{p}_i = (\triangle p_{1i}, \triangle p_{2i}, \triangle p_{3i})^T$, $i = 1, \ldots, n$, for the sets of the co-located coordinates as

$$\triangle\mathbf{s}_i = \mathbf{s}_i - \frac{1}{n}\sum_{i=1}^{n}\mathbf{s}_i, \quad \text{and} \quad \triangle\mathbf{p}_i = \mathbf{p}_i - \frac{1}{n}\sum_{i=1}^{n}\mathbf{p}_i\,. \tag{30}$$

Notice, that the centrobaric coordinates satisfy the equality

$$\sum_{i=1}^{n}\triangle\mathbf{s}_i = \sum_{i=1}^{n}\triangle\mathbf{p}_i = \mathbf{0}. \tag{31}$$

If we substitute Eq. (30) into (29), we obtain

$$\triangle\mathbf{s}_i \;=\; \mathbf{t} + k\,\mathbf{R}\left(\triangle\mathbf{p}_i + \frac{1}{n}\sum_{i=1}^{n}\mathbf{p}_i\right) - \frac{1}{n}\sum_{i=1}^{n}\mathbf{s}_i \tag{32}$$

$$\;=\; \triangle\mathbf{t} + k\,\mathbf{R}\,\triangle\mathbf{p}_i, \quad \text{where} \quad \triangle\mathbf{t} = \mathbf{t} + k\,\mathbf{R}\frac{1}{n}\sum_{i=1}^{n}\mathbf{p}_i - \frac{1}{n}\sum_{i=1}^{n}\mathbf{s}_i. \tag{33}$$

Equation (32) is over-determined; therefore we denote the residual vectors $\mathbf{v}_i \in \mathbb{R}^3$ for $i = 1, \ldots, n$ as

$$\mathbf{v}_i = \triangle\mathbf{s}_i - \triangle\mathbf{t} - k\,\mathbf{R}\,\triangle\mathbf{p}_i\,. \tag{34}$$

Now we get the following optimization problem to solve for the required parameters

$$\min_{k,\triangle\mathbf{t},\mathbf{R}} \sum_{i=1}^{n} \mathbf{v}_i^T \mathbf{v}_i = \min_{k,\triangle\mathbf{t},\mathbf{R}} \sum_{i=1}^{n} (\triangle\mathbf{s}_i - \triangle\mathbf{t} - k\,\mathbf{R}\,\triangle\mathbf{p}_i)^T (\triangle\mathbf{s}_i - \triangle\mathbf{t} - k\,\mathbf{R}\,\triangle\mathbf{p}_i) \tag{35}$$

$$= \min_{k,\triangle\mathbf{t},\mathbf{R}} \left[n\triangle\mathbf{t}^T\triangle\mathbf{t} + \sum_{i=1}^{n} (\triangle\mathbf{s}_i - k\,\mathbf{R}\,\triangle\mathbf{p}_i)^T (\triangle\mathbf{s}_i - k\,\mathbf{R}\,\triangle\mathbf{p}_i) \right]. \tag{36}$$

Subsequently, it must be satisfied

$$\triangle\mathbf{t} = \mathbf{0}, \quad \text{i.e.,} \quad \mathbf{t} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{s}_i - \frac{k}{n}\mathbf{R}\sum_{i=1}^{n} \mathbf{p}_i\,. \tag{37}$$

Equation (35) is modified and re-arranged

$$\min_{k,\mathbf{R}} \left[\sum_{i=1}^{n} \triangle\mathbf{s}_i^T \triangle\mathbf{s}_i - 2k \sum_{i=1}^{n} \triangle\mathbf{s}_i^T\,\mathbf{R}\,\triangle\mathbf{p}_i + k^2 \sum_{i=1}^{n} \triangle\mathbf{p}_i^T \triangle\mathbf{p}_i \right]. \tag{38}$$

It should be noted that a necessary condition for an extremum of the function

$$F(k, x_1, \ldots, x_n) = k^2 A - kB(x_1, \ldots, x_n)$$

with a constant A is an extremum of the function B and $2kA - B = 0$, i.e., $k = B/2A$. Therefore, we get the new optimization problem

$$\max_{\mathbf{R}} \sum_{i=1}^{n} \triangle\mathbf{s}_i^T\,\mathbf{R}\,\triangle\mathbf{p}_i \tag{39}$$

and a new equation which allows us to determine the scale parameter k as

$$k = \sum_{i=1}^{n} \triangle\mathbf{s}_i^T\,\mathbf{R}\,\triangle\mathbf{p}_i \Big/ \sum_{i=1}^{n} \triangle\mathbf{p}_i^T \triangle\mathbf{p}_i. \tag{40}$$

As we have mentioned before, quaternions can represent rotation. Therefore, we substitute the quaternion $\mathcal{Q} = q_0 + \mathbf{q}$ to represent the rotation matrix $\mathbf{R}$. Then the maximization problem can be solved as

$$\max_{\mathbf{R}} \sum_{i=1}^{n} \triangle\mathbf{s}_i^T\,\mathbf{R}\,\triangle\mathbf{p}_i = \max_{\mathcal{Q}} \sum_{i=1}^{n} (q_0, \mathbf{q}^T)\, N \begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix}, \tag{41}$$

where

$$N = \sum_{i=1}^{n} \begin{bmatrix} \triangle\mathbf{s}_i^T \triangle\mathbf{p}_i & -\triangle\mathbf{s}_i^T \mathbf{C}(\triangle\mathbf{p}_i) \\ -\mathbf{C}(\triangle\mathbf{s}_i)\triangle\mathbf{p}_i & \triangle\mathbf{s}_i\,\triangle\mathbf{p}_i^T + \mathbf{C}(\triangle\mathbf{s}_i)\mathbf{C}(\triangle\mathbf{p}_i) \end{bmatrix}, \tag{42}$$

and $\mathbf{C}(\mathbf{q})$ is the skew-symmetric matrix

$$\mathbf{C}(\mathbf{q}) = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}. \tag{43}$$

The matrix N is real-symmetric and it contains four real-valued eigenvalues and their corresponding eigenvectors. Then the solution of the maximization problem of Eq. (41) is equal to the eigenvector corresponding to the maximal eigenvalue of N. Thus we get the solution of the unit quaternion $\mathcal{Q}$, i.e., the unit quaternion representing the best rotation is the eigenvector associated with the eigenvalue of a symmetric matrix. This quaternion is determined uniquely up to its sign. Then we compute the rotation matrix $\mathbf{R}$ as

$$\mathbf{R} = \left[q_0^2 - \mathbf{q}^T\mathbf{q}\right]\mathbf{I} + 2[\mathbf{q}\,\mathbf{q}^T + q_0\,\mathbf{C}(\mathbf{q})], \tag{44}$$

where $\mathbf{I}$ denotes the 3×3 identity matrix. The rotation angles can be computed by using

$$\theta_x = \arctan\frac{r_{23}}{r_{33}}, \quad \theta_y = \arcsin(-r_{13}), \quad \theta_z = \arctan\frac{r_{12}}{r_{11}}, \tag{45}$$

where r_{ij} is the element of the rotation matrix $\mathbf{R}$ in the i-th row and j-th column and $\theta_x, \theta_y, \theta_z$ are the rotation angles around corresponding coordinate axes. Then the scale parameter k is computed by Eq. (40). Finally, by (37) we get the translation parameters.

3.2. Improved method using dual quaternions

Now we use dual quaternions for a description of the datum transformation, where the matrix representation of dual quaternions helps us to simplify manipulations of equations (see[14] for the definition of the matrix form).

The previous model based on quaternions is now adjusted to the use of dual quaternions. The dual quaternion transformation algorithm can be summarized in the following steps:

Algorithm 1 Dual quaternion transformation algorithm

Input: Cartesian coordinates of n stations given in a local and a global reference system.

1: Compute centrobaric coordinates $\triangle\mathbf{s}_i = (\triangle s_{1i}, \triangle s_{2i}, \triangle s_{3i})^T$, $\triangle\mathbf{p}_i = (\triangle p_{1i}, \triangle p_{2i}, \triangle p_{3i})^T$ using (30).

2: Express the dual unit quaternion $\mathcal{V}_{d_i}$ with the parameters $q_0, \cdots, q_3, \triangle t_1, \cdots, \triangle t_3, k$ using (47) and determine the corresponding vector $\mathbf{v}_i = (v_{1i}, v_{2i}, v_{3i})^T$.

3: Compute the required parameters $q_0, \cdots, q_3, \triangle t_1, \cdots, \triangle t_3, k$ by (52) determined by conditions (53).

4: Compute the rotation matrix $\mathbf{R}$ by (44) and then the rotation angles $\theta_x, \theta_y, \theta_z$ using (45).

5: Compute the translation vector $\mathbf{t}$ using the modified Eq. (33).

Output: Three rotation parameters $\theta_x, \theta_y, \theta_z$, three translation parameters t_1, t_2, t_3 and the scale parameter k.

It is possible to express the residual vector $\mathbf{v}_i$ in the form of dual quaternions. First, we modify (34). The scale parameter $k \in \mathbb{R}$ is a constant, therefore

$$\mathbf{v}_i = \triangle\mathbf{s}_i - \triangle\mathbf{t} - k\,\mathbf{R}\triangle\mathbf{p}_i = \triangle\mathbf{s}_i - \triangle\mathbf{t} - \mathbf{R}k\triangle\mathbf{p}_i = -(\triangle\mathbf{l}_i + \mathbf{R}\,\triangle\mathbf{r}_i), \tag{46}$$
$$\text{where} \quad \triangle\mathbf{l}_i = \triangle\mathbf{t} - \triangle\mathbf{s}_i \quad \text{and} \quad \triangle\mathbf{r}_i = k\triangle\mathbf{p}_i\,.$$

Equation (46) expresses a rotation of the vector $\triangle\mathbf{r}_i$ and then a translation given by the translation vector $\triangle\mathbf{l}_i$. We can express this equation according to (27) using dual quaternions in the form

$$\mathcal{V}_{d_i} = -\mathcal{Q}_{d_i}\mathcal{R}_{d_i}\overline{\mathcal{Q}^*_{d_i}}, \tag{47}$$

where $\mathcal{R}_{d_i}$ is a dual unit quaternion

$$\mathcal{R}_{d_i} = 1 + \varepsilon(\triangle r_{1i}\mathbf{i} + \triangle r_{2i}\mathbf{j} + \triangle r_{3i}\mathbf{k}) = 1 + k\varepsilon(\triangle p_{1i}\mathbf{i} + \triangle p_{2i}\mathbf{j} + \triangle p_{3i}\mathbf{k}), \tag{48}$$

and $\mathcal{Q}_{d_i}$ is a dual unit quaternion

$$\mathcal{Q}_{d_i} = \mathcal{Q} + \varepsilon\mathcal{Q}_\varepsilon = \mathcal{Q} + \varepsilon\frac{\mathcal{L}_i\mathcal{Q}}{2}, \quad \text{where} \quad \mathcal{Q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \tag{49}$$

$$\text{and} \quad \mathcal{L}_i = (\triangle t_1 - \triangle s_{1i})\mathbf{i} + (\triangle t_2 - \triangle s_{2i})\mathbf{j} + (\triangle t_3 - \triangle s_{3i})\mathbf{k}. \tag{50}$$

The quaternion $\mathcal{Q}$ is a unit quaternion and $\mathcal{L}$ is a pure quaternion. Since $\mathcal{Q}_{d_i}$ is a dual unit quaternion, we must apply the conditions (24), i.e.,

$$\| \mathcal{Q} \| = 1 \quad \wedge \quad \mathcal{Q} \cdot \mathcal{Q}_\varepsilon = 0. \tag{51}$$

From Eq. (47) we get the dual unit quaternion of the form $\mathcal{V}_{d_i} = 1 + \varepsilon(v_{1i}\mathbf{i} + v_{2i}\mathbf{j} + v_{3i}\mathbf{k})$ corresponding to the vector $\mathbf{v}_i = (v_{1i}, v_{2i}, v_{3i})^T$, where the terms v_{1i}, v_{2i}, v_{3i} contain seven parameters to be solved, i.e., $q_0, \cdots, q_3, \triangle t_1, \cdots, \triangle t_3, k$. Further, the transformation parameters can be determined by solving this optimization problem

$$\min_{q_0,\cdots,q_3,\triangle t_1,\cdots,\triangle t_3,k} \sum_{i=1}^{n} \mathbf{v}_i^T\mathbf{v}_i \;=\; (v_{1i}, v_{2i}, v_{3i})^T(v_{1i}, v_{2i}, v_{3i}), \tag{52}$$

$$\| \mathcal{Q} \| \;=\; 1 \quad \wedge \quad \mathcal{Q} \cdot \mathcal{Q}_\varepsilon = 0. \tag{53}$$

We can use a nonlinear method to solve this minimization problem, i.e., *Lagrange multipliers*. This method can also be accommodated to multiple constraints. Thus the problem (52) under the condition (24) can be expressed as minimizing the following Lagrange function

$$L(q_0, \cdots, q_3, \triangle t_1, \cdots, \triangle t_3, k, \alpha, \beta) = \sum_{i=1}^{n} \mathbf{v}_i^T\mathbf{v}_i + \alpha(\sqrt{\mathcal{Q}\mathcal{Q}^*} - 1) + \beta(\mathcal{Q} \cdot \mathcal{Q}_\varepsilon), \tag{54}$$

where α and β are the Lagrange multipliers to be determined. The solution by minimizing the Lagrange function (54) is equivalent to solving the following non-linear system of equations

$$\begin{array}{ccccc} \dfrac{\partial L}{\partial q_0} = 0, & \dfrac{\partial L}{\partial q_2} = 0, & \dfrac{\partial L}{\partial \triangle t_1} = 0, & \dfrac{\partial L}{\partial \triangle t_3} = 0, & \dfrac{\partial L}{\partial k} = 0, \\ \dfrac{\partial L}{\partial q_1} = 0, & \dfrac{\partial L}{\partial q_3} = 0, & \dfrac{\partial L}{\partial \triangle t_2} = 0, & \dfrac{\partial}{\partial \alpha} = 0, & \dfrac{\partial L}{\partial \beta} = 0, \end{array} \tag{55}$$

where $q_0, \cdots, q_3, \triangle t_1, \cdots, \triangle t_3, k, \alpha, \beta$ denotes the unknowns parameters to be solved. Since Eq. (55) is non-linear, we can find the solution numerically, e.g., by using the CA-system Mathematica. Finally, the translation vector $\mathbf{t}$ can be determined using (33).

4. Example

We consider Cartesian coordinates of seven stations given in the local and global reference systems (WGS 84) as in Table 1. Values of this stations are taken from [6]. We are looking for the seven parameters of the datum transformation. This numerical example is presented to demonstrate the functionality of the designed method. The values of these stations are frequently used as in [12] or [15].

Table 1: Coordinates for local system (system A), coordinates for WGS 84 (system B)

	System A			*System B*		
Station name	X(m)	Y(m)	Z(m)	X(m)	Y(m)	Z(m)
Solitude	4157870.237	664818.678	4775416.524	4157222.543	664789.307	4774952.099
Buoch Zeil	4149691.049	688865.785	4779096.588	4149043.336	688836.443	4778632.188
Hohenneuffen	4173451.354	690369.375	4758594.075	4172803.511	690340.078	4758129.701
Kuehlenberg	4177796.064	643026.700	4761228.899	4177148.376	642997.635	4760764.800
Ex Mergelaec	4137659.549	671837.337	4791592.531	4137012.190	671808.029	4791128.215
Ex Hof Asperg	4146940.228	666982.151	4784324.099	4146292.729	666952.887	4783859.856
Ex Kaisersbach	4139407.506	702700.227	4786016.645	4138759.902	702670.738	4785552.196

Now we compute the transformation parameters $\theta_x, \theta_y, \theta_z, t_1, t_2, t_3, k$ from the local geodetic system to WGS 84. We use the CA-system Mathematica to find the transformation parameters, where it is convenient to express dual quaternions in the 8×8 matrix form. The optimization problem was solved using *Lagrange multipliers*.

The quaternion $\mathcal{Q}$ and the translation $\triangle\mathbf{t}$ are shown in Table 2. The final list of results given from Eqs. (45), (33) and *Lagrange multipliers* are listed in Table 3.

In addition to this transformation, we compute the residual value to each point, i.e., the difference between coordinates of the system A and the new coordinates of the system A'. The coordinates of the system A' are determined using the computed transformation parameters $\mathbf{R}, \mathbf{t}, k$ and the substitution (29). The residuals and the new values of the stations are given in Table 4. Transformation parameters and transformed coordinates are equal to the parameters described in [6] and [12].

Table 2: Quaternion and translation parameter

Quaternion $\mathcal{Q}$		*Translation* $\triangle\boldsymbol{t}$	
q_0	-0.9999999987474	$\triangle t_1$	-6.649×10^{-10}
q_1	-0.0000024204319	$\triangle t_2$	-3×10^{-13}
q_2	0.0000021663738	$\triangle t_3$	2.658×10^{-10}
q_3	0.0000024073178		

Table 3: Final results of the dual quaternion transformation algorithm

Rotation angles		*Translation* $\boldsymbol{t}$		*Scale* k	
θ_x	$-0.99850''$	t_1	$641.8908\,m$		
θ_y	$0.89370''$	t_2	$68.6570\,m$	k	1.0000055825199
θ_z	$0.99309''$	t_3	$416.4101\,m$		

Table 4: Transformed Cartesian coordinates of System A into System B (Table 1) using the seven datum transformation parameters of Table 3 computed with the dual quaternion algorithm

Station name	X (m)	Y (m)	Z (m)
System A: Solitude	4157870.237	664818.678	4775416.524
System B	4157222.543	664789.307	4774952.099
Transformed value: A'	4157870.143	664818.543	4775416.384
Residual	0.0940	0.1351	0.1402
System A: Buoch Zeil	4149691.049	688865.785	4779096.588
System B	4149043.336	688836.443	4778632.188
Transformed value: A'	4149690.990	688865.835	4779096.574
Residual	0.0588	−0.0497	0.0137
System A: Hohenneuffen	4173451.354	690369.375	4758594.075
System B	4172803.511	690340.078	4758129.701
Transformed value: A'	4173451.394	690369.463	4758594.083
Residual	−0.0399	−0.0879	−0.0081
System A: Kuehlenberg	4177796.064	643026.700	4761228.899
System B	4177148.376	642997.635	4760764.800
Transformed value: A'	4177796.044	643026.722	4761228.986
Residual	0.0203	−0.0221	−0.0875
System A: Ex Mergelaec	4137659.549	671837.337	4791592.531
System B	4137012.190	671808.029	4791128.215
Transformed value: A'	4137659.641	671837.323	4791592.536
Residual	0.0919	0.0139	−0.0055
System A: Ex Hof Asperg	4146940.228	666982.151	4784324.099
System B	4146292.729	666952.887	4783859.856
Transformed value: A'	4146940.240	666982.145	4784324.154
Residual	−0.0118	0.0065	−0.0546
System A: Ex Kaisersbach	4139407.506	702700.227	4786016.645
System B	4138759.902	702670.738	4785552.196
Transformed value: A'	4139407.535	702700.223	4786016.643
Residual	−0.0294	0.0041	0.0017

5. Algorithm test

In this part of the article we will focus on testing two different approaches to solve the datum transformation model. We found a new formula for the Bursa-Wolf transformation model above and our purpose is to compare this algorithm based on dual quaternions with the algorithm based on quaternions.

5.1. Descriptive statistics

This section provides a simple view at the descriptive statistics obtained from 3070 tested points at algorithm mentioned above, i.e., dual quaternion algorithm (*DQ algorithm*) and quaternion algorithm (*Q algorithm*). The tested points were given in a local and a global system. To find the difference or distance between two sets of coordinates we use *Euclidean metric*.

Comparing the algorithms can be summarized in the following steps, which are the same for both of them:

Algorithm 2 Differences between systems

Input: Cartesian coordinates of 3070 stations given in a local (system A) and a global (WGS 84–system B) reference system.

1: Compute rotations, translations and scale parameters using DQ/Q algorithm.
2: Compute new coordinates A' of the system A. The coordinates of the system A' are determined using computed transformation parameters $\mathbf{R}, \mathbf{t}, k$ and substitution (29).
3: Compute difference between system A and system A' using Euclidean metric.

Output: Set of values which indicates the difference between the system A and the system A' for both algorithms.

Furthermore, we get the set of values for the DQ algorithm and for the Q algorithm. To get some information about these data, we use descriptive statistics. An overview of these values can be found in Table 5. We can see from the descriptive statistics that we are not dealing with normally distributed data but rather the log-normally distributed data and this is the reason why we apply nonparametric test, a *sign test* (see [1] for more details).

Table 5: Descriptive statistics

	DQ algorithm	*Q algorithm*
minimum	0.2708	0.3333
maximum	9.2231	8.7695
average	3.4163	3.2243
sample standard deviation	1.7049	1.6847
sample variance	2.9075	2.8390
sample quantile$_{05}$	1.0087	0.9058
sample quantile$_{25}$	2.1176	1.8654
sample quantile$_{50}$	3.2470	3.0371
sample quantile$_{75}$	4.3874	4.3317
sample quantile$_{95}$	6.7545	6.3291

5.2. Sign test — DQ and Q algorithm

The sign test is used to test the null hypothesis about a median $\tilde{Z}$ of a continuous distribution.

The observations in a sample of size n are $Z_1, Z_2, \ldots, Z_n$. Observations are obtained as the difference $d(A'_Q, A) - d(A'_{DQ}, A)$[4], where A'_Q and A'_{DQ} are new sets of points in WGS 84 and A is the original set of points in WGS 84.

The null hypothesis is that the median $\tilde{Z}$ is equal to the given value 0. Suppose that Z^+ is a sum of values, where $Z_i > 0$ and Z^- is a sum of values, where $Z_i < 0$. Values of Z which are exactly equal to 0 are ignored. The sum $Z^+ + Z^-$ may therefore be less than n. The null hypothesis $H_0\colon\ \tilde{Z} = 0$ is tested that two set of values are of equal size, i.e., there is no significant difference between the methods. We use only values outside of the interval $[-1, 1]$, which we consider as a significant difference. We get $\sum Z^+ = 580$ and $\sum Z^- = 487$. In fact these values count how often one method gives better results.

One-sided hypotheses:

- For one-sided hypothesis $H_1 : \tilde{Z} > 0$, we obtained $F_{Z^+}(x) = 0,9978$. While $0,9978 > 0,05 \Rightarrow$ we do not reject H_0 on the significance level $0,05$. It means that there is no significant difference between the methods.
- For one-sided hypotheses $H_1 : \tilde{Z} < 0$, we obtained $F_{Z^-}(x) = 0,0022$. While $0,0022 < 0,05 \Rightarrow$ we reject H_0 on the significance level $0,05$ and we accept H_1 that DQ algorithm is better if we take into account only significant differences.

6. Conclusion

This paper is focused on one particular example of practical applications of dual quaternions. Since the algebra of this structure is very popular and frequently used in various mathematical fields nowadays we try to show some of their modern applications. The next part of the paper is devoted to dual quaternions and their application. There is a strong motivation for dealing with the problem of finding parameters of transformation of two coordinate systems, the local and the world one. Datum transformation is widely used in geodesy. This paper describes one of the methods for the determination of the datum transformation parameters. We use a nonlinear transformation model. In this model we can easily use a description by dual quaternions. Dual quaternions allow us to describe any rigid transformation, i.e., a composition of rotations and translations. The main advantage of this approach is the simplification of the original solution of the datum transformation. The maximum error of this method can be estimated by the error matrix and it is similar to other methods, e.g., based on quaternions.

This paper presents one numerical example to demonstrate the introduced formula describing the datum transformation. There exist various modifications of this model, therefore we try to compare the latest approaches, i.e., the model based on dual quaternions with the model based on quaternions. Descriptive statistics shows that both models have probably a log-normal distribution. In order to deal with comparing both algorithms, we have applied a non parametric test, a sign test. By comparing known algorithms it was found that the accuracy of our algorithm based on dual quaternions is higher if we take into account only significant differences. Consequently, advantages of the novel approach lie in the fact that there is no need for a linearization of the nonlinear transformation model and the accuracy of this model is better considering given conditions.

[4] The Euclidean distance between two points A, B is denoted by $d(A, B)$.

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