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Generalization of a Geometry Problem Posed by Fermat

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Abstract. We consider a configuration generated by a point on a sphere and a rectangular hexahedron whose top face is inscribed to the equator of the sphere. Our results could be regarded as the space version of a geometric problem in the plane posed by FERMAT in 1658 that was first proved by EULER almost one century later in 1750. In fact, both the ratio of the semicircle's diameter and the rectangle's height in FERMAT's theorem and of the sphere's diameter and the hexahedron's height in our generalization of it is $\sqrt{2}$. We also give four additional invariant and equivalent properties for this configuration. Moreover, when the above ratio is $\frac{\sqrt{2}}{3}$ another sum of squares does not depend on the position of a point on the sphere.

Key Words: 3-dimensional Euclidean space, sphere, hexahedron, sum of squares of distances, Fermat geometric problem

MSC 2010: 51M04

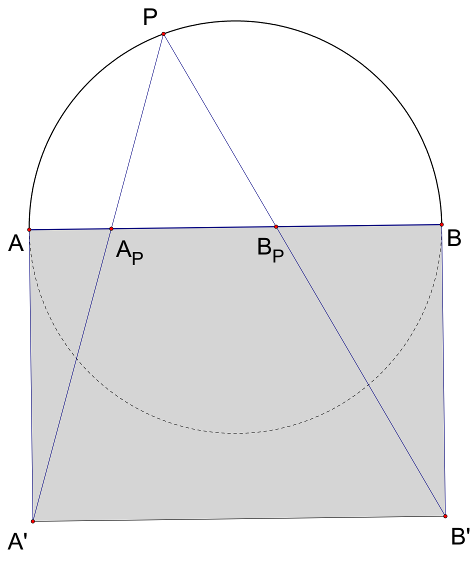

Figure 1: The configuration of Fermat's problem

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1. Space version of Fermat's problem

In this paper we consider the space version of the following geometric problem in the plane that was first formulated in a letter by Pierre DE FERMAT (see Fig. 1 and [8, pp. 402–408]).

Fermat's Problem. *Let P be a point on the (semi)circle that has the top side AB of the rectangle $ABB'A'$ as a diameter. Let $\frac{|AB|}{|AA'|} = \sqrt{2}$. Let the segments PA' and PB' intersect the side AB in the points A_P and B_P. Then $|AB_P|^2 + |BA_P|^2 = |AB|^2$.*

The great Leonhard EULER in [7] has provided the first rather long proof, which is old fashioned (for his time), and avoids the analytic geometry (which offers rather simple proofs as we shall see later). Several more concise synthetic proofs are now known (see [11], [9, pp. 602, 603], [1, pp. 168, 169] and [10, pp. 181, 264]). A very nice description of EULER's proof is available on the Internet (see [12]). Some recent contributions to the Fermat geometric problem are in [2], [3], [4], [5] and [6].

In the 3-dimensional Euclidean space, the (semi)circle is replaced by a sphere Σ, the rectangle $ABB'A'$ by the rectangular hexahedron (or the rectangular box or the rectangular parallelepiped) $ABCDA'B'C'D'$ whose top and bottom faces $T = ABCD$ and $T' = A'B'C'D'$ as well as all four other faces $ABB'A'$, $BCC'B'$, $CDD'C'$ and $DAA'D'$ are rectangles with T inscribed into the equator ε of Σ and the points A_P and B_P by four intersections A_P, B_P, C_P, D_P of the plane π_T determined by T with the lines PA', PB', PC', PD' for any point P not on the plane $\pi_{T'}$ determined by T'. Let r be the radius of Σ and let $e = |A'A| = |B'B| = |C'C| = |D'D|$. Here is a version of the Fermat geometric problem in the 3-dimensional space (see Fig. 2).

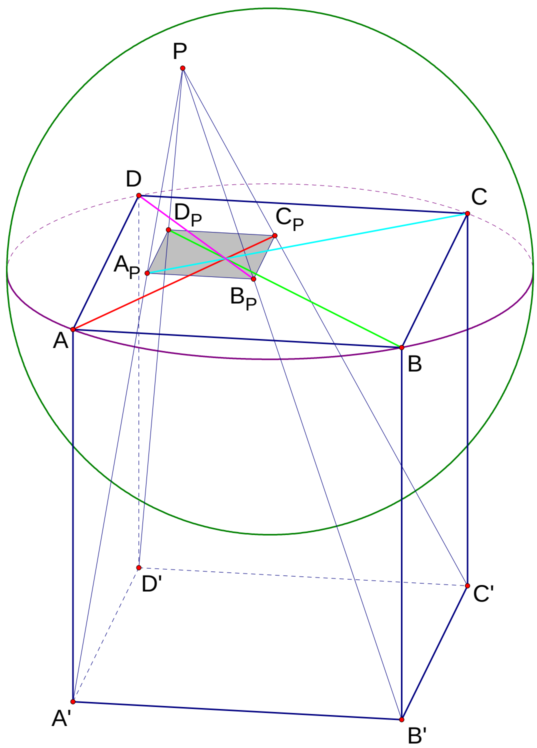

Figure 2: The space version of Fermat's problem

Theorem 1. *For any $P \in \Sigma \setminus (\varepsilon \cup \pi_{T'})$, the following statements are equivalent:*

(a) *The ratio $\frac{2\,r}{e}$ of the sphere's diameter and the hexahedron's height is $\sqrt{2}$.*

(b) *The sum $\sigma_T = |AC_P|^2 + |BD_P|^2 + |CA_P|^2 + |DB_P|^2$ has the value $8\,r^2$.*

Proof: We use analytic geometry, which offers a simple proof. Let the origin of the rectangular coordinate system be the center O of the sphere Σ so that the points A, B, C and D have the coordinates $(u, v, 0)$, $(-u, v, 0)$, $(-u, -v, 0)$ and $(u, -v, 0)$, where $u = \frac{r\,a_-}{a_+}$, $v = \frac{2\,r\,a}{a_+}$ and $a_\pm = 1 \pm a^2$ for some real number a. The equation of the sphere Σ is a standard $x^2 + y^2 + z^2 = r^2$. Its equator ε is a circle in the (x, y)-plane with the center in the origin and the radius r.

We assume that the coordinates of the points A', B', C' and D' are (u, v, m), $(-u, v, m)$, $(-u, -v, m)$ and $(u, -v, m)$ for some real number m. An arbitrary point P on the sphere has the coordinates $\left(\frac{2\,r\,s\,t_-}{s_+\,t_+}, \frac{4\,r\,s\,t}{s_+\,t_+}, \frac{r\,s_-}{s_+}\right)$ for some real numbers s and t. Recall that the point $(x, y, 0)$ in the plane π_T lies on the line through the different points (b, c, d) and (e, f, g) with $g \neq d$ provided $x = \frac{b\,g - d\,e}{g - d}$ and $y = \frac{c\,g - d\,f}{g - d}$. If we apply this to the pairs of points (A, P), (B, P), (C, P) and (D, P), we easily find that the coordinates of the intersections A_P, B_P, C_P and D_P are

$$\left(\frac{r\,U^-}{W}, \frac{2\,r\,V^-}{W}, 0\right), \left(\frac{r\,U^+}{W}, \frac{2\,r\,V^-}{W}, 0\right), \left(\frac{r\,U^+}{W}, \frac{2\,r\,V^+}{W}, 0\right), \text{ and } \left(\frac{r\,U^-}{W}, \frac{2\,r\,V^+}{W}, 0\right),$$

respectively, where

$$U^\pm = 2\,a_+\,m\,s\,t_- \pm a_-\,r\,s_-\,t_+, \quad V^\pm = 2\,a_+\,m\,s\,t \pm a\,r\,s_-\,t_+, \quad \text{and } W = a_+\,t_+(m\,s_+ - r\,s_-).$$

The above equivalence is the consequence of the identity $\sigma_T - 8\,r^2 = \frac{4\,r^2\,s_-^2\,(2\,r^2 - m^2)}{W^2}$. □

Remark 1. Note that the quadrangle $A_P B_P C_P D_P$ is a rectangle.

Remark 2. The 2-dimensional formula $|AB_P|^2 + |BA_P|^2 = |AB|^2$ is a degenerate consequence of the above 3-dimensional formula

$$|AC_P|^2 + |BD_P|^2 + |CA_P|^2 + |DB_P|^2 = 8\,r^2$$

by letting the hexahedron degenerate to a vertical rectangle having its top side a diameter of the (hemi)sphere, and taking the point P on the (semi)circle formed by intersecting the (hemi)sphere with the plane of the rectangle.

2. Invariants of Fermat's space configuration

Our first goal is to introduce several statements similar to (b) that could be added to Theorem 1. In other words, we explore what other relationships in the above space configuration remain invariant as the point P changes position on the sphere. Each of the conditions (c) – (f) below is equivalent to the condition (a) above.

We begin with the vertices of the bottom rectangle T' replacing the vertices of the top rectangle T.

(c) *The sum* $\sigma_{T'} = |A'C_P|^2 + |B'D_P|^2 + |C'A_P|^2 + |D'B_P|^2$ *has the value* $16\,r^2$.

Proof of (c): Since $16\,r^2 - \sigma_{T'} = \dfrac{4\,m\,s_+\,(2\,r^2 - m^2)\,(m\,s_+ - 2\,r\,s_-)}{W^2}$, the conclusion holds for all points $P \in \Sigma$ with the third coordinate different from m and $\dfrac{m}{2}$.

Of course, Pythagoras' theorem might be useful in computing the sum $\sigma_{T'}$. We have $A'C_P^2 = A'A^2 + AC_P^2$, $B'D_P^2 = B'B^2 + BD_P^2$, $C'A_P^2 = C'C^2 + CA_P^2$, $D'B_P^2 = D'D^2 + DB_P^2$. By summing up these four identities, we get $\sigma_{T'} = 4\,e^2 + \sigma_T = 8\,r^2 + 8\,r^2 = 16\,r^2$.

This method will also yield generalizations. Take points A_*, B_*, C_*, D_*, A_{P*}, B_{P*}, C_{P*}, D_{P*} such that $\mathbf{AA}_* = \lambda\mathbf{AA}'$, $\mathbf{BB}_* = \lambda\mathbf{BB}'$, $\mathbf{CC}_* = \lambda\mathbf{CC}'$, $\mathbf{DD}_* = \lambda\mathbf{DD}'$, $\mathbf{AC}_{P*} = \mu\mathbf{AC}_P$, $\mathbf{BD}_{P*} = \mu\mathbf{BD}_P$, $\mathbf{CA}_{P*} = \mu\mathbf{CA}_P$, $\mathbf{DB}_{P*} = \mu\mathbf{DB}_P$. Then it follows easily that the sum $|A_*C_{P*}|^2 + |B_*D_{P*}|^2 + |C_*A_{P*}|^2 + |D_*B_{P*}|^2$ has the value $8\,r^2\,(\lambda^2 + \mu^2)$. In this case too, some points on the sphere must be excluded. □

Let A_P^*, B_P^*, C_P^*, D_P^* be the midpoints of the segments AA_P, BB_P, CC_P, DD_P. Note that $A_P^*B_P^*C_P^*D_P^*$ is a rectangle. Let O, N and S be the center, the north pole and the south pole of the sphere Σ. For $Y = O, N$, let $\sigma_{Y*} = |YA_P^*|^2 + |YB_P^*|^2 + |YC_P^*|^2 + |YD_P^*|^2$.

(d) *The sums* σ_{N*} *and* σ_{O*} *have the values* $6\,r^2$ *and* $2\,r^2$.

Proof of (d): The differences $\sigma_{N*} - 6\,r^2$ and $\sigma_{O*} - 2\,r^2$ are both equal to $\dfrac{r^2\,s_-^2\,(2\,r^2 - m^2)}{W^2}$. □

For different points X and Y in the space and a real number $k \neq -1$, let $Z = X(k)Y$ denote the point that divides the segment XY in the ratio k (i.e., $XZ : ZY = k : 1$). For $Y = O, N, S$, let σ_{Yk} denote the sums

$$|YN(k)A_P^*|^2 + |YN(k)B_P^*|^2 + |YN(k)C_P^*|^2 + |YN(k)D_P^*|^2.$$

(e) *The sums* σ_{Ok}, σ_{Nk} *and* σ_{Sk} *have the constant values* $\dfrac{2(k^2+2)r^2}{(k+1)^2}$, $\dfrac{6\,k^2\,r^2}{(k+1)^2}$ *and* $\dfrac{2(3\,k^2 + 8\,k + 8)\,r^2}{(k+1)^2}$.

Proof of (e): The differences $\sigma_{Sk} - \dfrac{2(3\,k^2 + 8\,k + 8)\,r^2}{(k+1)^2}$, $\sigma_{Ok} - \dfrac{2(k^2+2)r^2}{(k+1)^2}$ and $\sigma_{Nk} - \dfrac{6\,k^2\,r^2}{(k+1)^2}$ are all equal to $\dfrac{k^2\,r^2\,s_-^2\,(2\,r^2 - m^2)}{(k+1)^2\,W^2}$. □

Let A'_P, B'_P, C'_P and D'_P be the midpoints of $A'A_P$, $B'B_P$, $C'C_P$ and $D'D_P$. Let $\sigma_{O'}$ be the sum $|OA'_P|^2 + |OB'_P|^2 + |OC'_P|^2 + |OD'_P|^2$.

(f) *The sum* $\sigma_{O'}$ *has the constant value* $4\,r^2$.

Proof of (f): Since $4\,r^2 - \sigma_{O'} = \dfrac{m\,s_+\,(2\,r^2 - m^2)\,(m\,s_+ - 2\,r\,s_-)}{W^2}$, the conclusion holds for all points $P \in \Sigma$ with the third coordinate different from m and $\dfrac{m}{2}$. □

3. The ratio $\frac{2\,r}{e}$ equal $\frac{\sqrt{2}}{3}$

Let X denote the center of a square on either segments A_PC_P or B_PD_P perpendicular to the plane π_T. Our last result shows that some other values for the ratio $\dfrac{2\,r}{e}$ besides the familiar $\sqrt{2}$ can appear in the Fermat space configuration.

Theorem 2. *For any $P \in \Sigma \setminus (\varepsilon \cup \pi_{T'})$, the following statements are equivalent:*

(i) *The ratio* $\frac{2\,r}{e}$ *of the sphere's diameter and the hexahedron's height is* $\frac{\sqrt{2}}{3}$.

(ii) *The sum* $\sigma_X = |XA_P^*|^2 + |XB_P^*|^2 + |XC_P^*|^2 + |XD_P^*|^2$ *is equal to* $2\,r^2$.

Proof: Since the point X is $\left(\frac{2\,m\,r\,s\,t_-}{t_+\,W}, \frac{4\,m\,r\,s\,t}{t_+\,W}, \frac{\pm 2\,r^2\,s_-}{W}\right)$, this follows from the identity $\sigma_X - 2\,r^2 = \frac{r^2\,s_-^2\,(18\,r^2 - m^2)}{W^2}$. □

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