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Some Concyclicity Properties of Convex Quadrilaterals

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Abstract. In this paper we find some concyclicity properties related to the v-parallelograms and the orthic quadrilaterals of a convex quadrilateral. In particular, we obtain a concyclicity property that allows us to define the second Droz-Farny circle for orthodiagonal quadrilaterals.

Key Words: valtitudes, vaxes, v-parallelogram, orthic quadrilateral, Droz-Farny circles

MSC 2010: 51M04

1. Introduction

In [4] we defined and studied the notions of orthic quadrilateral and principal orthic quadrilateral of a convex quadrilateral $\boldsymbol{Q}$. In [5] we have found some concyclicity properties related to the Varignon parallelogram and the principal orthic quadrilateral. It was defined and studied in particular the notion of the *first Droz-Farny circle* of a cyclic and orthodiagonal quadrilateral, in analogy with the Droz-Farny circles of triangles [2].

In this paper we find some concyclicity properties related to the v-parallelograms [7] and the orthic quadrilaterals of a convex quadrilateral, that generalize the properties of [5]. In particular, in the Sections 2 and 3 we recall some notions and results from [6, 7] and, in the following two sections, we obtain a concyclicity property that allows us to define the second Droz-Farny circle for orthodiagonal quadrilaterals.

2. v-parallelograms, valtitudes and vaxes

Let $A_1A_2A_3A_4$ be a convex quadrilateral, which we will denote by $\boldsymbol{Q}$, with centroid G.

A *v-parallelogram* of $\boldsymbol{Q}$ is any parallelogram with vertices on the sides of $\boldsymbol{Q}$ and sides parallel to the diagonals of $\boldsymbol{Q}$.

To obtain a v-parallelogram of $\boldsymbol{Q}$ we can use the following construction. Fix an arbitrary point V_1 on the segment A_1A_2; draw from V_1 the parallel to the diagonal A_1A_3 and let V_2

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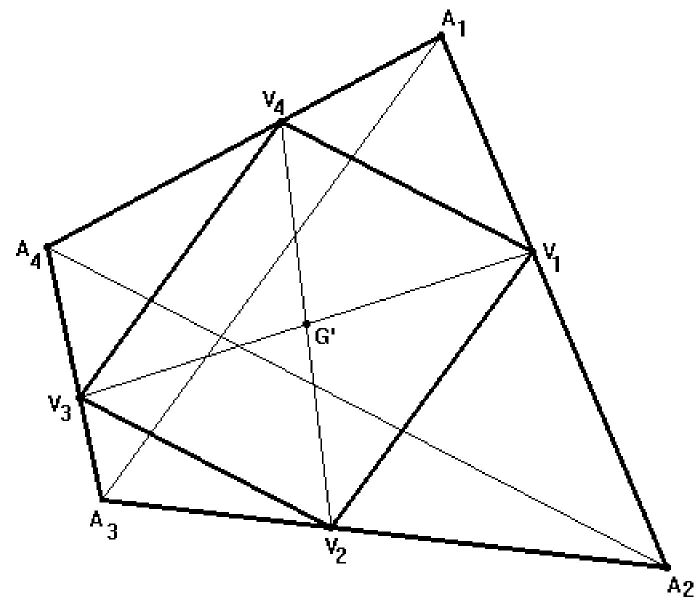

Figure 1: v-parallelograms of $\boldsymbol{Q}$

be the point of intersection of this line with the side A_2A_3; draw from V_2 the parallel to the diagonal A_2A_4 and let V_3 be the point of intersection of this line with the side A_3A_4; finally, draw from V_3 the parallel to the diagonal A_1A_3 and let V_4 be the point of intersection of this line with the segment A_4A_1. The quadrilateral $V_1V_2V_3V_4$ is a v-parallelogram [7] and, by moving V_1 on the segment A_1A_2, we obtain all possible v-parallelograms of $\boldsymbol{Q}$ (see Fig. 1).

In the following, a v-parallelogram of $\boldsymbol{Q}$ shall be denoted by $\boldsymbol{V}$, the vertices on the side A_iA_{i+1} by V_i (with indices taken modulo 4) and the common point of the diagonals of $\boldsymbol{V}$ by G'. Observe that $\boldsymbol{V}$ *is a rectangle if, and only if,* $\boldsymbol{Q}$ *is orthodiagonal.*

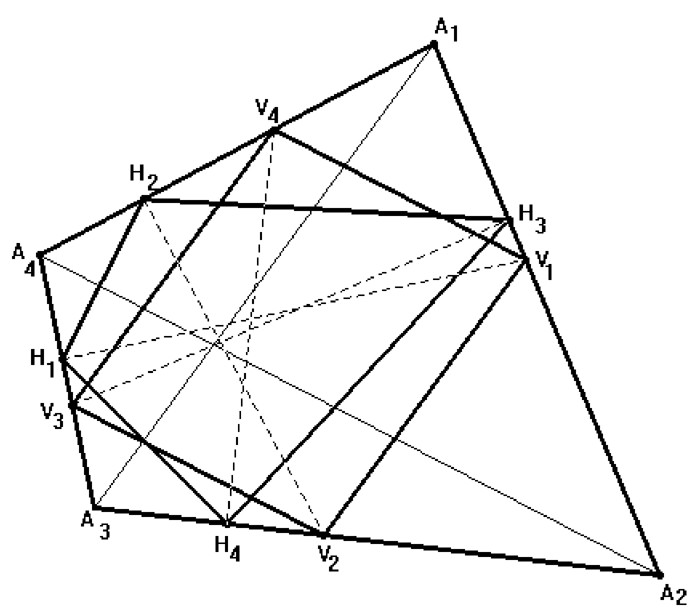

Figure 2: Valtitudes and orthic quadrilaterals of $\boldsymbol{Q}$

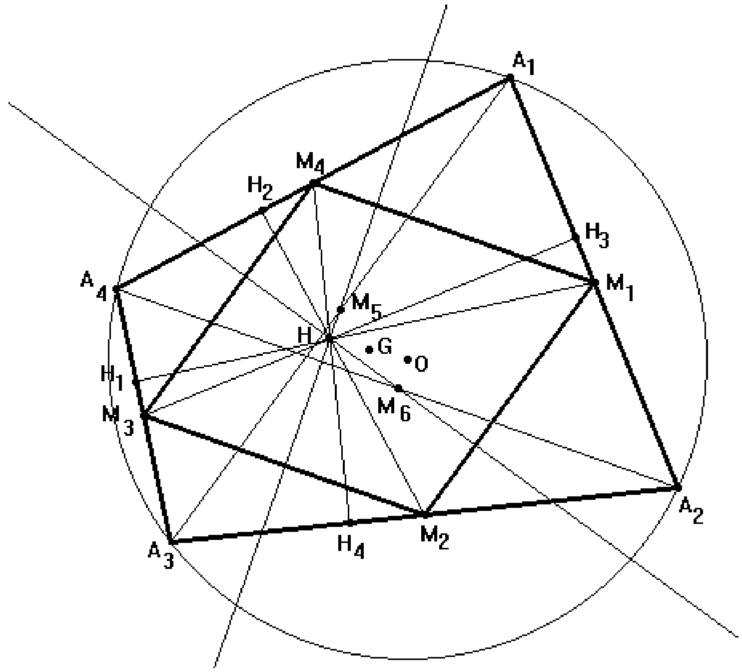

Figure 3: Concurrency of valtitudes if $\boldsymbol{Q}$ is cyclic

The v-parallelogram $\mathrm{M}_1\mathrm{M}_2\mathrm{M}_3\mathrm{M}_4$, with M_i midpoint of the side $\mathrm{A}_i\mathrm{A}_{i+1}$, is the *Varignon parallelogram* of $\boldsymbol{Q}$. In this particular case G′ is the centroid G of $\boldsymbol{Q}$.

Let $\boldsymbol{V}$ be a v-parallelogram of $\boldsymbol{Q}$ and H_i be the foot of the perpendicular to $\mathrm{A}_{i+2}\mathrm{A}_{i+3}$ from V_i. The quadrilateral $\mathrm{H}_1\mathrm{H}_2\mathrm{H}_3\mathrm{H}_4$ is called the *orthic quadrilateral* of $\boldsymbol{Q}$ relative to $\boldsymbol{V}$ [4], and we will denote it by $\boldsymbol{H}$. The lines $\mathrm{V}_i\mathrm{H}_i$ are called *valtitudes* of $\boldsymbol{Q}$ with respect to $\boldsymbol{V}$ (see Fig. 2).

Observe that $\boldsymbol{H}$ can be a convex, concave or crossed quadrilateral. If $\boldsymbol{V}$ is the Varignon parallelogram, the quadrilateral $\boldsymbol{H}$ relative to $\boldsymbol{V}$ is called *principal orthic quadrilateral* of $\boldsymbol{Q}$ and the lines $\mathrm{M}_i\mathrm{H}_i$ are the *maltitudes* of $\boldsymbol{Q}$.

Given a v-parallelogram $\boldsymbol{V}$ of Q, *if the valtitudes of $\boldsymbol{Q}$ with respect to $\boldsymbol{V}$ are concurrent, then $\boldsymbol{Q}$ is cyclic or orthodiagonal* [7]. Moreover, *if $\boldsymbol{Q}$ is cyclic or orthodiagonal, there is only one v-parallelogram $\boldsymbol{V}$* with respect to whom the valtitudes are concurrent.* To be more precise:

a) If $\boldsymbol{Q}$ is cyclic, $\boldsymbol{V}$* is the Varignon parallelogram of $\boldsymbol{Q}$ and then the concurrent valtitudes are the maltitudes of $\boldsymbol{Q}$. Moreover the point of concurrency of the maltitudes is the anticentre H of $\boldsymbol{Q}$. H is the reflection of O in G and the line containing H, O and G is the Euler line of $\boldsymbol{Q}$ (see Fig. 3). The line through the midpoint M_5 of the diagonal $\mathrm{A}_1\mathrm{A}_3$ of $\boldsymbol{Q}$ and perpendicular to the diagonal $\mathrm{A}_2\mathrm{A}_4$ contains H as well as the line through the midpoint M_6 of $\mathrm{A}_2\mathrm{A}_4$ and perpendicular to $\mathrm{A}_1\mathrm{A}_3$ [3].

b) If $\boldsymbol{Q}$ is orthodiagonal, $\boldsymbol{V}$* is the v-parallelogram determined by the perpendiculars to the sides of $\boldsymbol{Q}$ through the common point K of the diagonals of $\boldsymbol{Q}$. Then K is the point of concurrency of the valtitudes (see Fig. 4).

Given a v-parallelogram $\boldsymbol{V}$ of $\boldsymbol{Q}$, the perpendicular k_i $(i = 1, 2, 3, 4)$ to $\mathrm{A}_i\mathrm{A}_{i+1}$ through V_i will be called *vaxis* of $\boldsymbol{Q}$ with respect to $\boldsymbol{V}$.

Theorem 1. *The vaxes relative to $\boldsymbol{V}$ are concurrent if, and only if, the valtitudes relative to $\boldsymbol{V}$ are concurrent.*

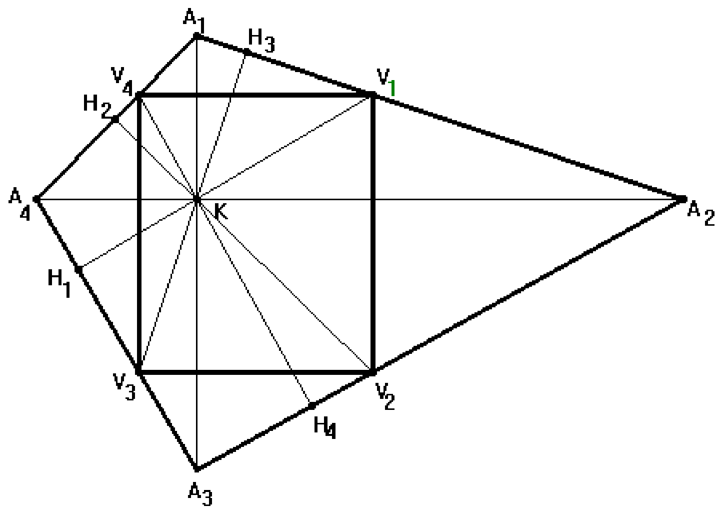

Figure 4: Concurrency of valtitudes if $\boldsymbol{Q}$ is orthodiagonal

Proof: V_i and V_{i+2} are symmetric with respect to G′ (see Fig. 5). Then the vaxis k_i and the line parallel to it passing through V_{i+2}, i.e., the valtitude h_{i+2}, are correspondent in the symmetry with centre G′. It follows that the vaxes k_i are concurrent iff the valtitudes h_i are concurrent. □

Then, from the concurrency properties of valtitudes, it follows that *if the vaxes are concurrent, then* $\boldsymbol{Q}$ *is cyclic or orthodiagonal.* Moreover:

Theorem 2. *If* $\boldsymbol{Q}$ *is cyclic or orthodiagonal, there is only one v-parallelogram* $\boldsymbol{V^*}$ *of* $\boldsymbol{Q}$ *with respect to whom the vaxes are concurrent.*

To be more precise:

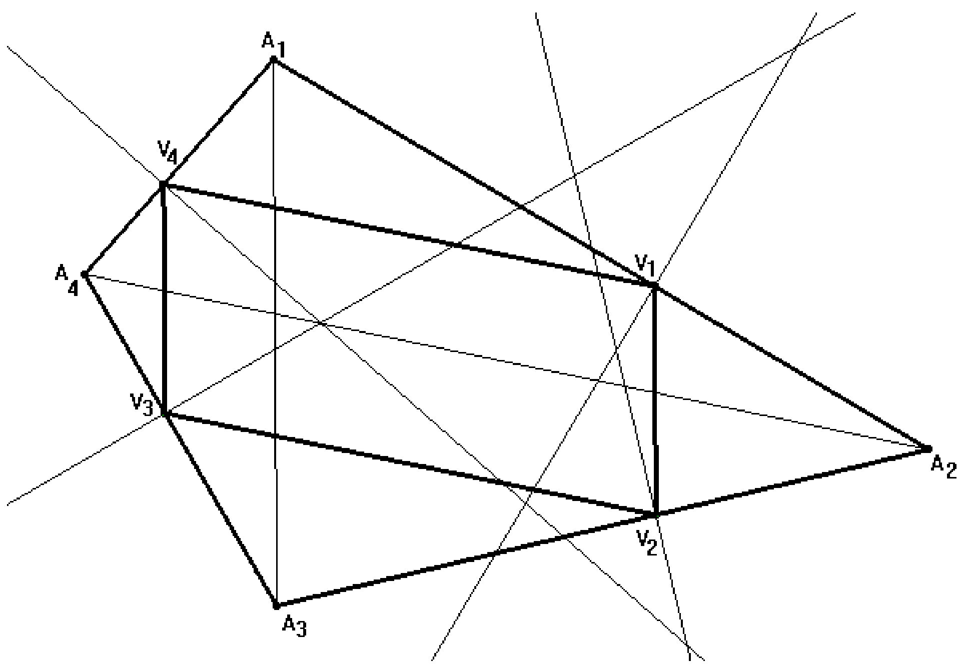

Figure 5: Proof of Theorem 1

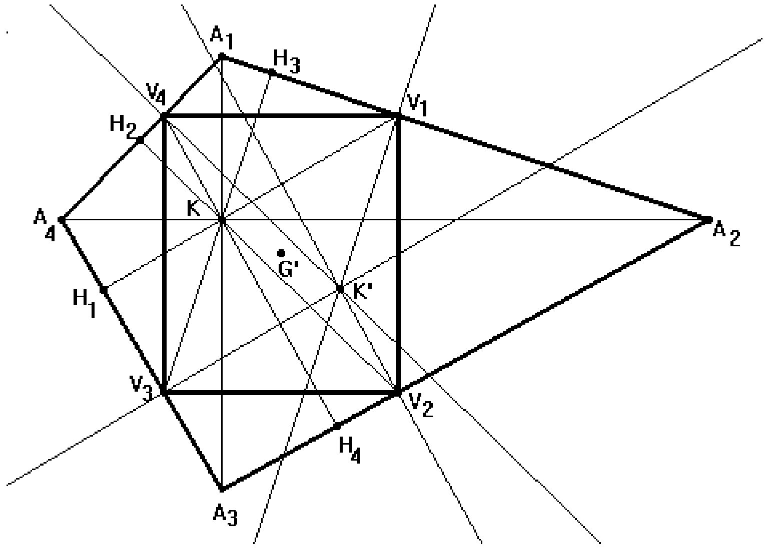

Figure 6: Proof of Theorem 2 if $\boldsymbol{Q}$ is orthodiagonal

a) If $\boldsymbol{Q}$ is cyclic, $\boldsymbol{V^*}$ is the Varignon parallelogram of $\boldsymbol{Q}$, then the concurrent vaxes are the perpendicular bisectors of $\boldsymbol{Q}$ and the point of concurrency is the circumcentre O of $\boldsymbol{Q}$.

b) If $\boldsymbol{Q}$ is orthodiagonal, $\boldsymbol{V^*}$ is the v-parallelogram determined by the perpendiculars to the sides of $\boldsymbol{Q}$ through the common point K of the diagonals of $\boldsymbol{Q}$. Then the point of concurrency of the vaxes is the point K′ symmetric of K with respect to G′ (see Fig. 6).

3. First Droz-Farny circle of a cyclic and orthodiagonal quadrilateral

We recall that the Droz-Farny circles of a triangle are a pair of circles of equal radius obtained by particular geometric constructions [2]. Let $\boldsymbol{T}$ be a triangle of vertices A_1, A_2, A_3; let O and H be its circumcentre and orthocentre, respectively (see Fig. 7). Let H_i be the foot of the height of $\boldsymbol{T}$ at A_i and M_i the midpoint of the side A_iA_{i+1} (with indices taken modulo 3).

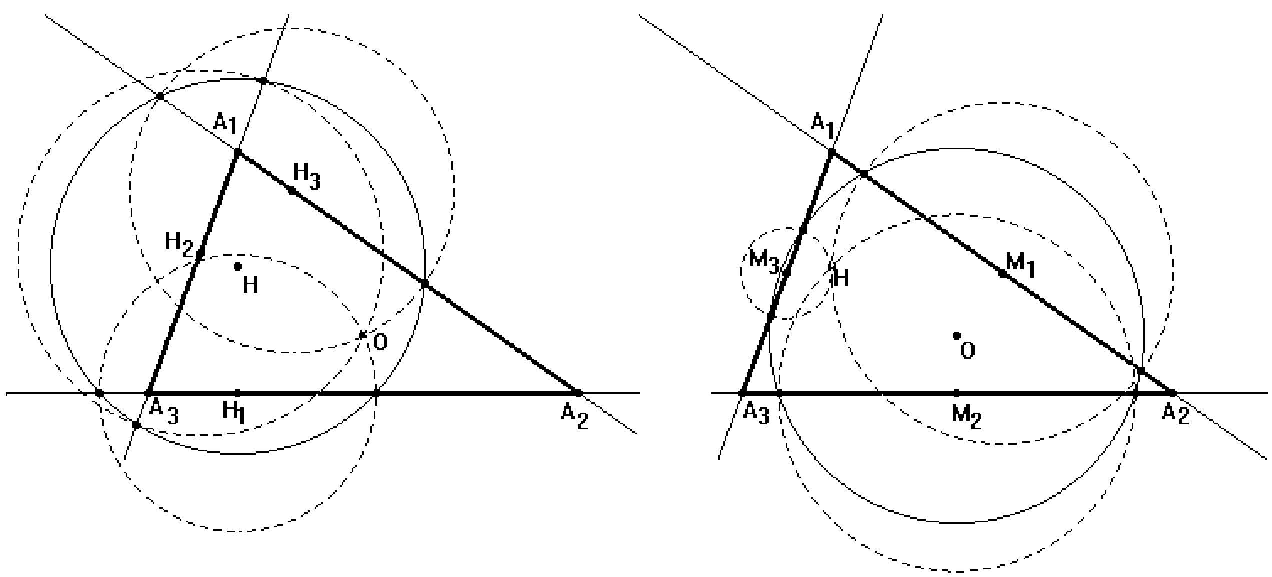

Figure 7: Droz-Farny circles of a triangle

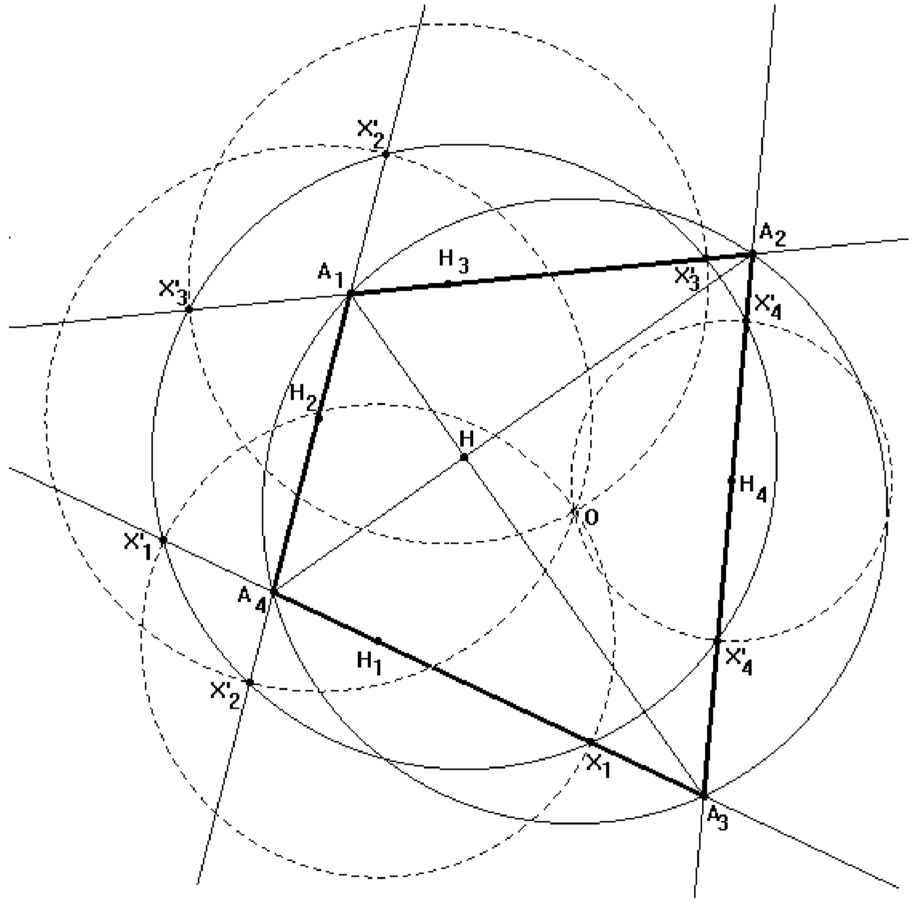

Figure 8: First Droz-Farny circle of $\boldsymbol{Q}$

a) If we consider the points of intersection of the circle with centre H_i and radius $\mathrm{H}_i\mathrm{O}$ with the line $\mathrm{A}_{i+1}\mathrm{A}_{i+2}$, then we obtain six points which lie all on a circle with centre H *(first Droz-Farny circle)*.

b) If we consider the points of intersection of the circle with centre M_i and radius $\mathrm{M}_i\mathrm{H}$ with the line $\mathrm{A}_i\mathrm{A}_{i+1}$, then we obtain six points which lie all on a circle with centre O *(second Droz-Farny circle)*.

In [5] it has been studied if and how the same concyclicity properties hold for convex quadrilaterals. Here we report some results.

Let $\boldsymbol{Q}$ be a cyclic quadrilateral and let O and H be its circumcentre and anticentre, respectively. Consider the principal orthic quadrilateral $\mathrm{H}_1\mathrm{H}_2\mathrm{H}_3\mathrm{H}_4$ of $\boldsymbol{Q}$.

a) Let X_i and X'_i be the points of intersection of the circle with centre H_i and radius $\mathrm{H}_i\mathrm{O}$ with the line $\mathrm{A}_{i+2}\mathrm{A}_{i+3}$. Then, we obtain the eight points X_i, X'_i $(i = 1, 2, 3, 4)$.

Theorem 3. *The eight points* X_i, X'_i *lie all on a circle if, and only if,* $\boldsymbol{Q}$ *is orthodiagonal.*

As for the triangle case, if $\boldsymbol{Q}$ is cyclic and orthodiagonal, the circle containing the eight points X_i, X'_i with centre H, is called the *first Droz-Farny circle* of $\boldsymbol{Q}$ (see Fig. 8). Its radius is equal to the circumradius of $\boldsymbol{Q}$.

b) Let Y_i and Y'_i be the points of intersection of the circle with centre M_i and radius $\mathrm{M}_i\mathrm{H}$ with the line $\mathrm{A}_i\mathrm{A}_{i+1}$. Then, we obtain eight points Y_i, Y'_i $(i = 1, 2, 3, 4)$.

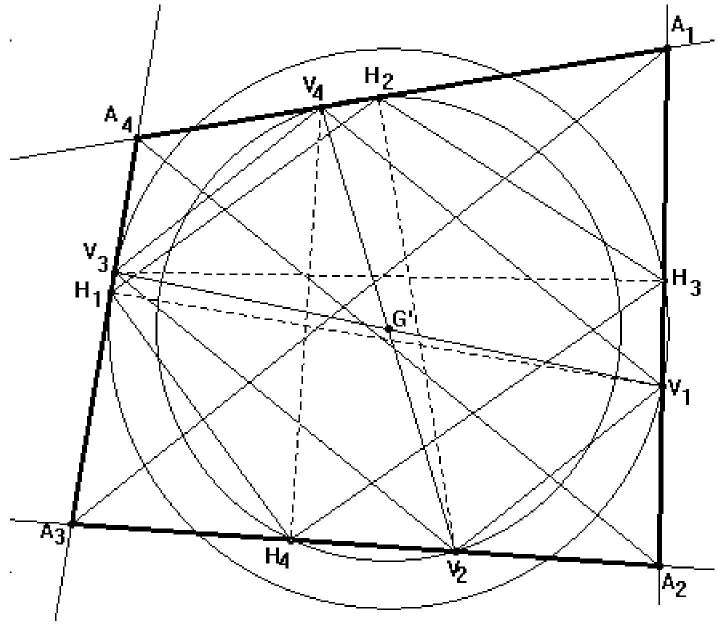

Figure 9: Proof of Theorem 5

Theorem 4. *The eight points* Y_i, Y_i' *all lie on a circle if, and only if,* $\boldsymbol{Q}$ *is orthodiagonal.*

However, if $\boldsymbol{Q}$ is orthodiagonal, the points Y_i and Y_i' are the vertices A_i of $\boldsymbol{Q}$ and the circle that contains them is the circumcircle of $\boldsymbol{Q}$.

In this paper, if $\boldsymbol{Q}$ is orthodiagonal (not cyclic), we will find a circle that we can assume to be the analogue of the second Droz-Farny circle of a triangle.

4. Some more concyclicity properties of a convex quadrilateral

Let $\boldsymbol{V}$ be a v-parallelogram of $\boldsymbol{Q}$ and $\boldsymbol{H}$ the orthic quadrilateral relative to $\boldsymbol{V}$. Let G' be the common point of the diagonals of $\boldsymbol{V}$.

Theorem 5. *The points* H_1, V_1, H_3, V_3 *as well as the points* H_2, V_2, H_4, V_4 *are concyclic. Both circles are centred at* G'.

Proof: The circle with diameter $\mathrm{V}_1\mathrm{V}_3$ passes through H_1 and H_3, because $\angle\mathrm{V}_1\mathrm{H}_1\mathrm{V}_3$ and $\angle\mathrm{V}_1\mathrm{H}_3\mathrm{V}_3$ are right angles (see Fig. 9). Analogously, the circle with diameter $\mathrm{V}_2\mathrm{V}_4$ passes through H_2 and H_4. □

Corollary 1. *The eight points* V_i, H_i $(i \in \{1,2,3,4\})$ *lie on one circle centred at* G' *if, and only if,* $\boldsymbol{Q}$ *is orthodiagonal.*

Proof: The two circles containing all the points V_i and H_i coincide if, and only if, $\mathrm{V}_1\mathrm{V}_3 = \mathrm{V}_2\mathrm{V}_4$, that is, if, and only if, $\boldsymbol{V}$ is a rectangle, i.e., if, and only if, $\boldsymbol{Q}$ is orthodiagonal (see Fig. 10). □

If $\boldsymbol{Q}$ is orthodiagonal, the circle containing the vertices of the Varignon parallelogram and those of the principal orthic quadrilateral is the *eight-point circle* of $\boldsymbol{Q}$ [1].

Suppose now that $\boldsymbol{Q}$ is cyclic and let O be its circumcentre. Let B_i be the common point of the valtitudes $\mathrm{V}_i\mathrm{H}_i$ and $\mathrm{V}_{i+1}\mathrm{H}_{i+1}$ and let G_i be the midpoint of the segment OB_i. Let M_i be the midpoint of the side $\mathrm{A}_i\mathrm{A}_{i+1}$.

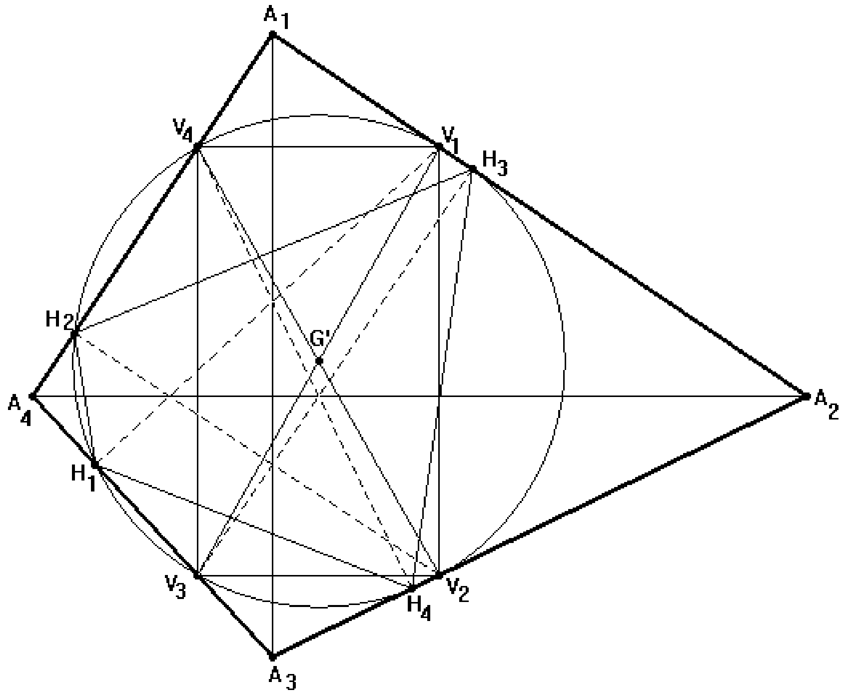

Figure 10: Proof of Corollary 1

Theorem 6. *If* $\boldsymbol{Q}$ *is cyclic and orthodiagonal, the eight points* M_i, M_{i+1}, H_{i+2}, H_{i+3} *lie on a circle with centre* G_{i+2}.

Proof: Let us prove the theorem for $i = 1$ (see Fig. 11).
The triangle $\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3$ is similar to the triangle $\mathrm{H}_3\mathrm{A}_2\mathrm{H}_4$ [4, p. 86] and it is also similar to the triangle $\mathrm{M}_1\mathrm{A}_2\mathrm{M}_2$. It follows then that the triangles $\mathrm{H}_3\mathrm{A}_2\mathrm{H}_4$ and $\mathrm{M}_1\mathrm{A}_2\mathrm{M}_2$ are similar. In

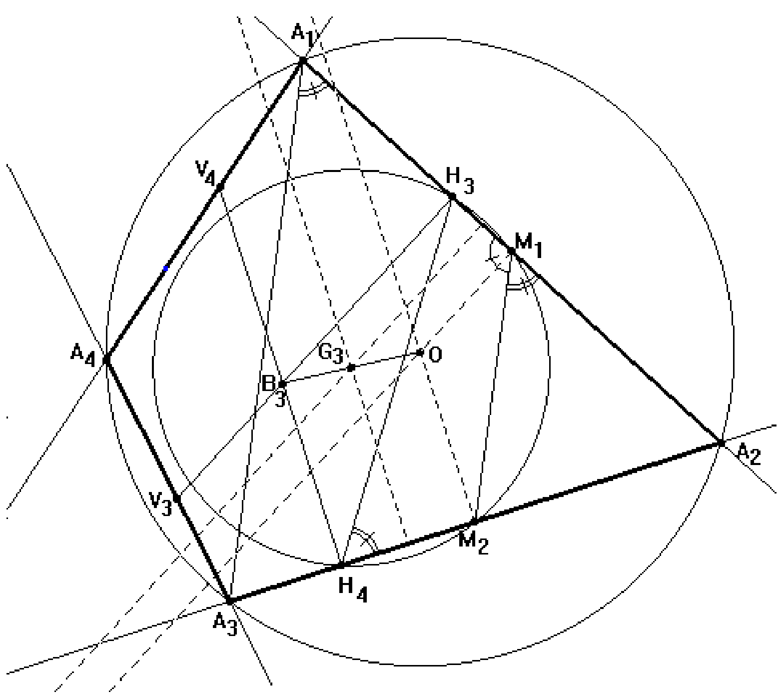

Figure 11: Proof of Theorem 6

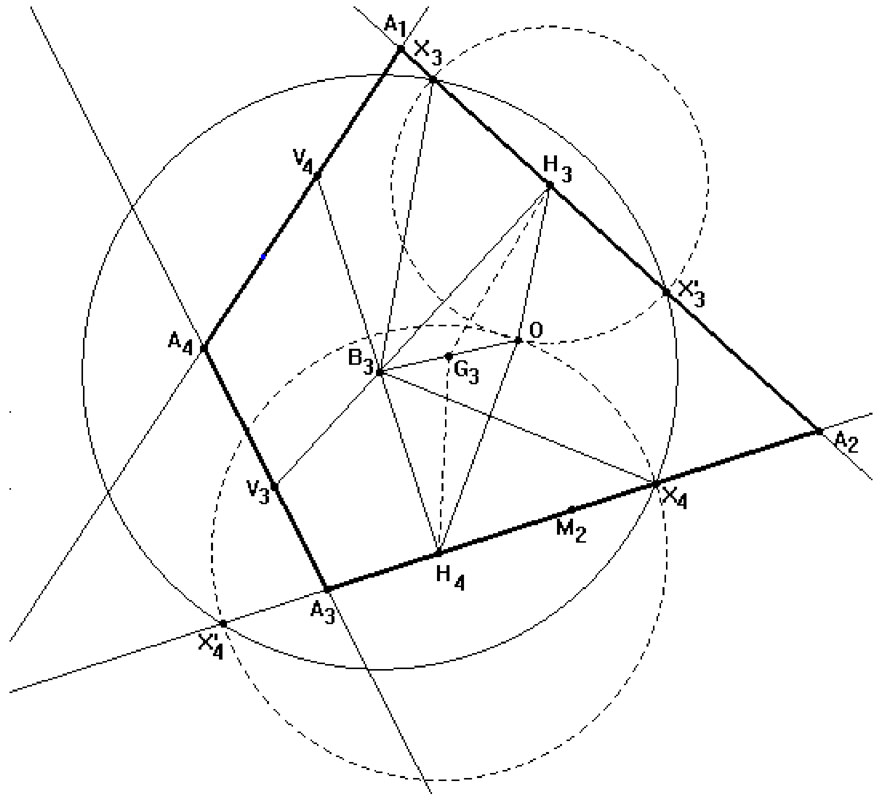

Figure 12: Proof of Theorem 7

particular, $\angle A_2H_4H_3 = \angle A_2M_1M_2$, then $\angle M_2H_4H_3$ and $\angle H_3M_1M_2$ are supplementary angles and the quadrilateral $M_1M_2H_4H_3$ is cyclic. Let us prove now that the circumcentre of this quadrilateral is G_3. The perpendiculars to the line A_1A_2 through M_1 and H_3 pass through O and B_3, respectively. Then, the perpendicular bisector of the segment M_1H_3 passes through the midpoint of OB_3, that is G_3. Analogously, the perpendicular bisector of the segment M_2H_4 has to pass through G_3, then the circumcentre of the quadrilateral $M_1M_2H_4H_3$ is G_3. Analogously for $i = 2, 3, 4$. □

Let X_i and X'_i be the points of intersection of the circle with centre H_i and radius H_iO with the line $A_{i+2}A_{i+3}$. Then, we obtain eights points X_i, X'_i $(i = 1, 2, 3, 4)$.

Theorem 7. *If* ***Q*** *is cyclic and orthodiagonal, the points* X_i *and* X'_i *that are on the lines containing two consecutive sides of* ***Q*** *lie on a circle with centre* B_i.

Proof: Let us prove that the points X_3, X'_3, X_4, X'_4, lie on a circle with centre B_3 (see Fig. 12). Analogously in the other cases.

Since B_3 lies on the perpendicular bisector of the segment $X_3X'_3$, it is $B_3X_3 = B_3X'_3$; analogously, $B_3X_4 = B_3X'_4$. Moreover, since X_3 lies on the circle with centre H_3 and radius OH_3, it is $H_3X_3 = H_3O$. Then, by applying Pythagoras' theorem to the triangle $B_3H_3X_3$ and the property of the median G_3H_3 of the triangle B_3H_3O.[1] Thus we have:

$$\overline{B_3X_3}^2 = \overline{H_3X_3}^2 + \overline{B_3H_3}^2 = \overline{OH_3}^2 + \overline{B_3H_3}^2 = 2\,\overline{H_3G_3}^2 + \frac{1}{2}\,\overline{OB_3}^2.$$

[1] The property of the median AD of a triangle ABC, that is the consequence of the law of cosines ([2, p. 71]), states that $\overline{AB}^2 + \overline{AC}^2 = 2\,\overline{AD}^2 + \frac{1}{2}\,\overline{BC}^2$.

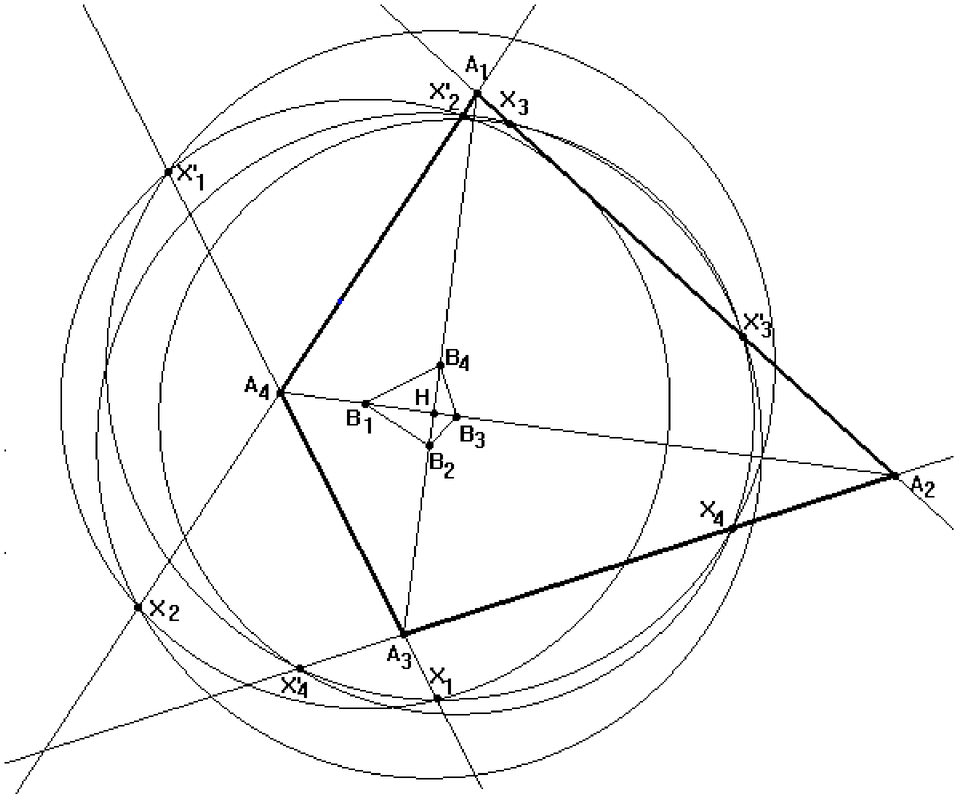

Figure 13: Conciclicity of X_i, X'_i on four circles

Analogously, we find:

$$\overline{\mathrm{B}_3\mathrm{X}_4}^2 = 2\,\overline{\mathrm{H}_4\mathrm{G}_3}^2 + \frac{1}{2}\,\overline{\mathrm{OB}_3}^2.$$

However, for Theorem 6, we have $\mathrm{H}_3\mathrm{G}_3 = \mathrm{H}_4\mathrm{G}_3$, then $\mathrm{B}_3\mathrm{X}_3 = \mathrm{B}_3\mathrm{X}_4$. □

Theorem 7 states that there exist four circles each of which passes through the points X_i, X'_i that belong to the lines containing two consecutive sides of $\boldsymbol{Q}$ (see Fig. 13).

Note that the four circles coincide if, and only if, all the points B_i coincide, that is if, and only if, the valtitudes are concurrent. Since $\boldsymbol{Q}$ is cyclic, this happens if, and only if, the valtitudes are maltitudes and $\boldsymbol{V}$ is the Varignon parallelogram. In this case the circle containing the eight points X_i, X'_i is the first Droz-Farny circle of $\boldsymbol{Q}$ [5].

5. Second Droz-Farny circle of an orthodiagonal quadrilateral

Let $\boldsymbol{Q}$ be any convex quadrilateral and let K be the common point of its diagonals. Let $\boldsymbol{V}$ be a v-parallelogram of $\boldsymbol{Q}$, k_i be the vaxes relative to $\mathrm{A}_i\mathrm{A}_{i+1}$ and O_i the common point of the vaxes k_i and k_{i+1}.

Let Y_i and Y'_i be the points of intersection of the circle with centre V_i and radius $\mathrm{V}_i\mathrm{K}$ with the line $\mathrm{A}_i\mathrm{A}_{i+1}$. Then, we obtain eight points Y_i, Y'_i $(i = 1, 2, 3, 4)$.

Theorem 8. *If $\boldsymbol{Q}$ is orthodiagonal, the points Y_i and Y'_i that are on the lines containing two consecutive sides of $\boldsymbol{Q}$ lie on a circle with centre O_i.*

Proof: Let us prove that the points Y_1, Y'_1, Y_2, Y'_2 lie on a circle with centre O_1 (see Fig. 14). The same arguments hold in the other cases.

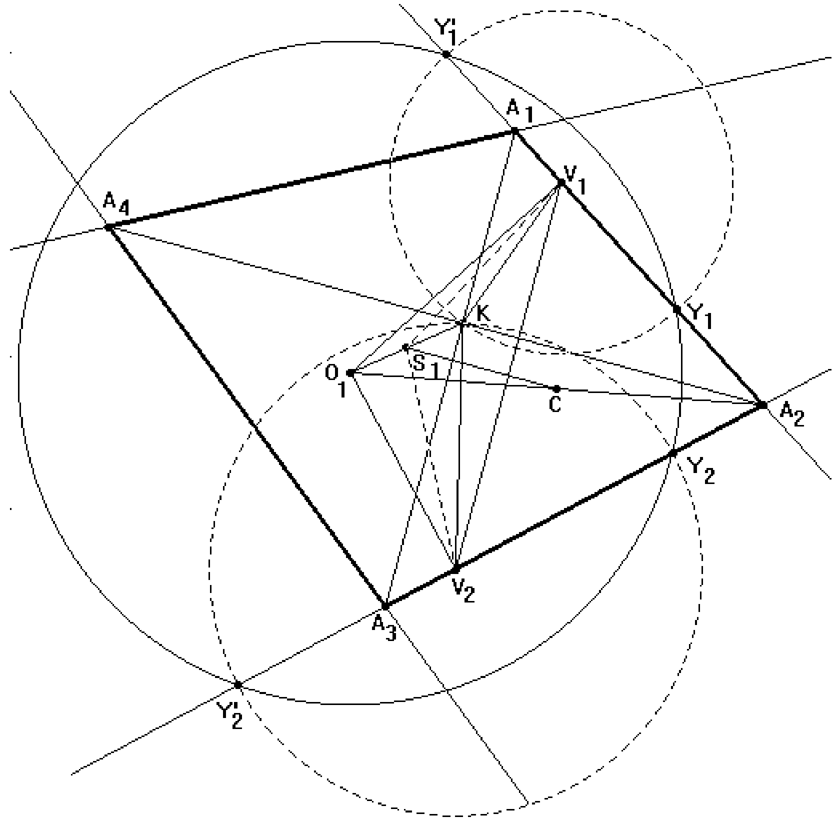

Figure 14: Proof of Theorem 8

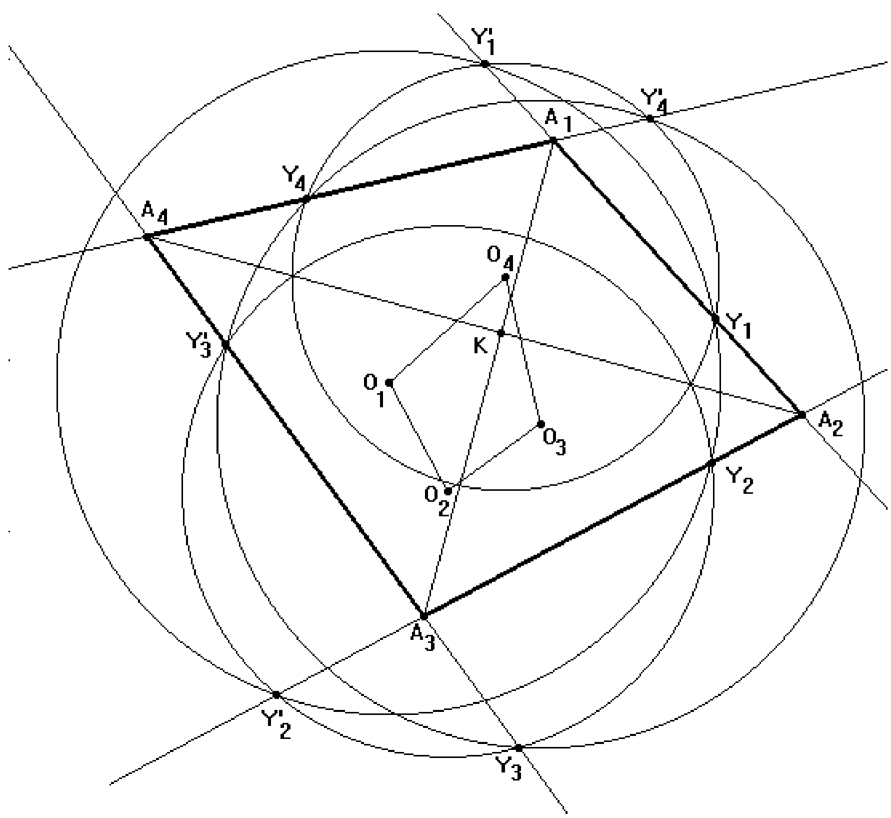

Figure 15: Concyclicity of Y_i, Y'_i on four circles

Since O_1 lies on the perpendicular bisector of the segment Y_1Y_1', thus we have $O_1Y_1 = O_1Y_1'$; analogously, we have $O_1Y_2 = O_1Y_2'$. Moreover, since Y_1 lies on the circle with centre V_1 and radius V_1K, thus we have $Y_1V_1 = KV_1$. Let S_1 be the midpoint of the segment KO_1. Then by applying Pythagoras' theorem to triangle $O_1V_1Y_1$ and the property of the median V_1S_1 of the triangle O_1V_1K, we have

$$\overline{O_1Y_1}^2 = \overline{Y_1V_1}^2 + \overline{O_1V_1}^2 = \overline{V_1K}^2 + \overline{O_1V_1}^2 = 2\,\overline{V_1S_1}^2 + \frac{1}{2}\,\overline{KO_1}^2.$$

Analogously, we find

$$\overline{O_1Y_2}^2 = 2\,\overline{V_2S_1}^2 + \frac{1}{2}\,\overline{KO_1}^2.$$

Therefore, if we prove that $V_1S_1 = V_2S_1$, it follows that $O_1Y_1 = O_1Y_2$. The quadrilateral $V_1A_2V_2O_1$ is cyclic, because the angles in V_1 and V_2 are right angles, and the circumcentre of this quadrilateral is the midpoint C of the segment O_1A_2. In the triangle KA_2O_1, the segment S_1C joins the midpoints of the sides O_1K and O_1A_2, therefore it is parallel to KA_2. Thence, since $\boldsymbol{Q}$ is orthodiagonal, the line S_1C is perpendicular to V_1V_2 and thus we have the perpendicular bisector of the segment V_1V_2. It follows that $V_1S_1 = V_2S_1$. □

Theorem 8 states that there exist four circles each of which passes through the points Y_i, Y_i' that belong to the lines containing two consecutive sides of $\boldsymbol{Q}$ (see Fig. 15).

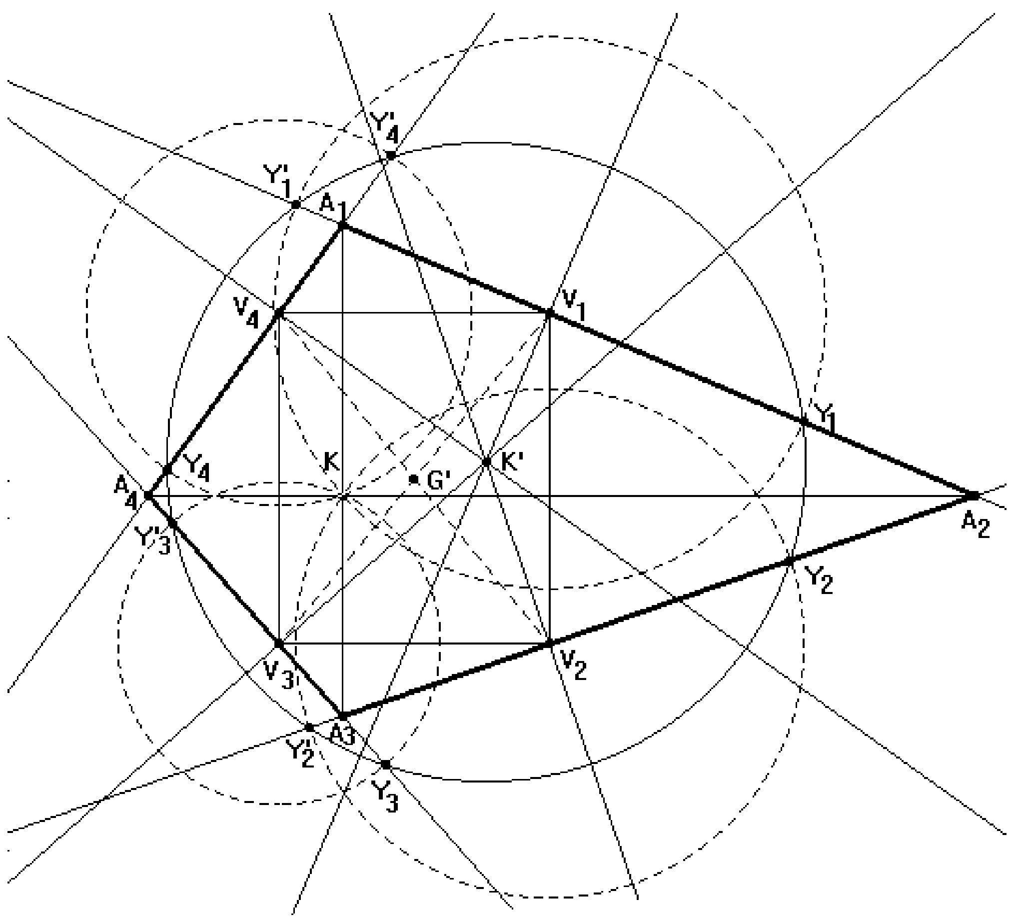

Figure 16: Second Droz-Farny circle of $\boldsymbol{Q}$

Note that the four circles coincide if, and only if, all the points O_i coincide, that is if, and only if, the vaxes k_i are concurrent. For Theorem 2, if $\boldsymbol{Q}$ is cyclic or orthodiagonal, there exists only one v-parallelogram $\boldsymbol{V^*}$ of $\boldsymbol{Q}$ relative to whom the vaxes are concurrent. Moreover, if $\boldsymbol{Q}$ is orthodiagonal, $\boldsymbol{V^*}$ is the v-parallelogram determined by the perpendiculars to the sides of $\boldsymbol{Q}$ passing through the common point K of the diagonals of $\boldsymbol{Q}$ and the point of concurrency of the vaxes is the point K' that is the reflection of K in G' (see Fig. 16). Therefore the following holds true:

Theorem 9. *If $\boldsymbol{Q}$ is orthodiagonal, the points Y_i and Y_i' obtained with respect to $\boldsymbol{V^*}$ lie on a circle with centre K'.*

We call this circle *second Droz-Farny circle* of $\boldsymbol{Q}$.

Observe that if $\boldsymbol{Q}$ is also cyclic, $\boldsymbol{V^*}$ is the Varignon parallelogram, K' is the circumcentre of $\boldsymbol{Q}$ and the second Droz-Farny circle is the circumcircle of $\boldsymbol{Q}$ [5].

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