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Archimedean Circles of the Collinear Arbelos and the Skewed Arbelos

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Abstract. Several Archimedean circles of the arbelos can be generalized to the collinear arbelos and the skewed arbelos.

Key Words: arbelos, collinear arbelos, skewed arbelos, Archimedean circles

MSC 2010: 51M04, 51M15, 51N20

1. Introduction

The area surrounded by three mutually touching semicircles with collinear centers constructed on the same side is called an *arbelos*. The radical axis of the two inner semicircles divides the arbelos into two curvilinear triangles with congruent incircles, which are called the *twin circles of Archimedes*. Circles congruent to the twin circles are called *Archimedean circles* of the arbelos.

The arbelos is generalized in several ways, the generalized arbelos of *intersecting type* [10], the generalized arbelos of *non-intersecting type* [9], and the *skewed* arbelos [6, 8, 12]. In [7] both the generalized arbelos in [9] and [10] are unified as the collinear arbelos with an additional generalized arbelos, and its Archimedean circles are defined.

In this paper we give several Archimedean circles of the collinear arbelos. For the skewed arbelos, no definition of Archimedean circles has been given, though several twin circles have been considered. In this paper we define Archimedean circles of the skewed arbelos by generalizing the twin circles of Archimedes of the ordinary arbelos to the skewed arbelos. Then we give several Archimedean circles of the skewed arbelos.

For points P and Q, (PQ) is the circle with diameter PQ and $P(Q)$ is the circle with center P passing through the point Q. For a circle δ, its center is denoted by O_δ.

2. The collinear arbelos

Let $\alpha = (AP)$, $\beta = (BQ)$ and $\gamma = (AB)$ for points P and Q on the line AB. The configuration consisting of the three circles α, β and γ is denoted by (α, β, γ). The point of intersection of

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the radical axis of α and β and the line AB is denoted by O. Let $s = |AQ|/2$, $t = |BP|/2$ and $u = |AB|/2$.

Unless otherwise stated, we use a rectangular coordinate system with origin O such that the points A, B, P, Q have coordinates $(a, 0)$, $(b, 0)$, $(p, 0)$, $(q, 0)$, respectively with $a - b = 2u$. The following relation holds [7]:

$$ta + sb = tq + sp = 0. \tag{1}$$

For points V and W on the line AB, with respective x-coordinates v and w, we write $V < W$ and $V \leq W$ to denote $v < w$ and $v \leq w$, respectively. The perpendicular to AB passing through the point W is denoted by $\mathcal{P}_W$. The following lemma is a slight generalization of the results in [6] and [7]. The proof is similar and therefore omitted.

Lemma 1. The following circles have radius $|AW||BV|/(4u)$ for points V and W on the line AB.

(i) The circles touching the circles γ internally, (AV) externally and the line $\mathcal{P}_W$ from the side opposite to the point B in the case $B < V < A$ and $B \leq W < A$.

(ii) The circles touching γ externally, (AV) internally and $\mathcal{P}_W$ from the side opposite to the point A in the case $V < B \leq W < A$.

(iii) The circles touching γ and (AV) externally and $\mathcal{P}_W$ in the case $A < V$ and $A < W$.

(iv) The circles touching γ and (AV) internally and $\mathcal{P}_W$ in the case $W \leq B$ and $A < V$.

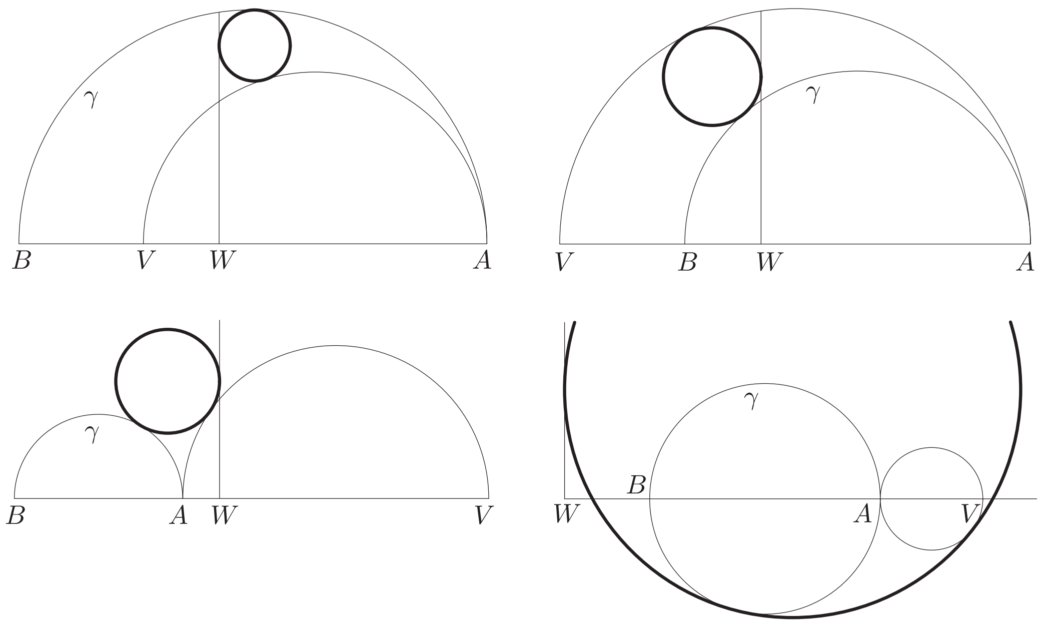

Figure 1: Circles of radius $|AW||BV|/(4u)$

The configuration (α, β, γ) is called a *collinear arbelos* if

(i) $B < Q < P < A$, or

(ii) $B < P \leq Q < A$, or

(iii) $P < B < A < Q$.

In each of the cases the configuration is explicitly denoted by $(BQPA)$, $(BPQA)$ and $(PBAQ)$, respectively. If $P = Q$, then $(BPQA)$ is called a *tangent arbelos*.

For $(BQPA)$ and $(BPQA)$, let δ_α be the circle touching the circles γ internally, α externally in the region $y > 0$ and the line $\mathcal{P}_O$ from the side opposite to B. For $(PBAQ)$, let δ_α be the circle touching γ externally, α internally in the region $y > 0$ and $\mathcal{P}_O$ from the side opposite to A. The circle δ_β is defined similarly (see Fig. 2). The next theorem can also be found in [7].

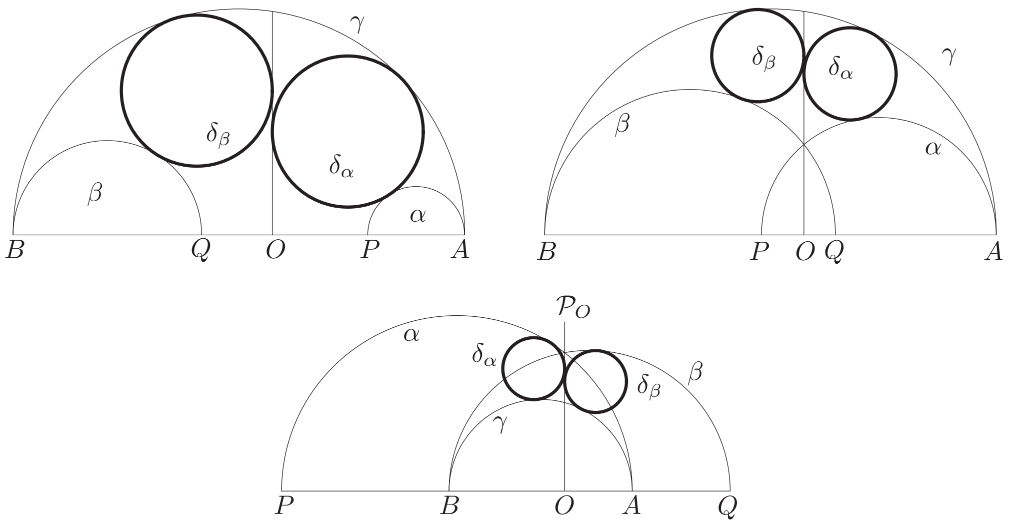

Figure 2: The circles δ_α and δ_β

Theorem 1. For a collinear arbelos (α, β, γ), the circles δ_α and δ_β are congruent with common radius $at/(2u) = st/(s+t)$.

We call the circles δ_α and δ_β the *twin circles* of Archimedes of the collinear arbelos, and denote their common radius by r_{A}. Also circles congruent to the twin circles are said to be Archimedean circles of the collinear arbelos or Archimedean with respect to (α, β, γ).

Let us consider points P_1, P_2, ... and Q_1, Q_2, ... lying on the line AB such that

$$\cdots < P_2 < P_1 < O < Q_1 < Q_2 < \cdots.$$

Figure 3 shows a simple application of Theorem 1, where $A = Q_1$, $B = P_1$, $P = P_2$, $Q = Q_2$, $|P_1P_2| = |P_2P_3| = \cdots = 2t$ and $|Q_1Q_2| = |Q_2Q_3| = \cdots = 2s$. For $i = 1, 2, \ldots$ the line $\mathcal{P}_O$ is the radical axis of the circles $(Q_{i+1}P_i)$ and (Q_iP_{i+1}) by (1). And the following circles have

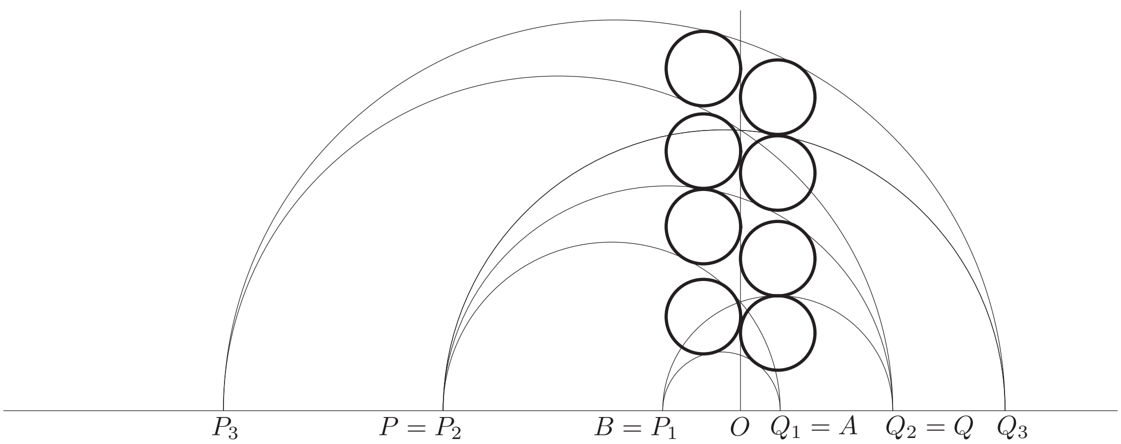

Figure 3: Arbeloi with congruent Archimedean circles

radius r_{A}, (i) the circle touching $(Q_{i+1}P_{i+1})$ internally $(Q_{i+1}P_i)$ externally and $\mathcal{P}_O$ from the side opposite to B, (ii) the circle touching $(Q_{i+1}P_i)$ internally (Q_iP_i) externally and $\mathcal{P}_O$ from the side opposite to B. By exchanging the roles of the points P_i and Q_i, we get similar circles of radius r_{A} touching $\mathcal{P}_O$ from the side opposite to A.

2.1. Several Archimedean circles of the collinear arbelos

For the collinear arbelos (α, β, γ), I is the point of intersection of the circle γ and the line $\mathcal{P}_O$ lying in the region $y > 0$, and J_α is the point of intersection of the circle α and the line AI. The point J_β is defined similarly. Let us assume (α, β, γ) is a tangent arbelos.

The circle with center O touching the tangents of the circle β from the point O_α is Archimedean, which is denoted by W_8 in [3] (see Fig. 4). The smallest circle passing through the point J_α and touching $\mathcal{P}_O$ is Archimedean, which is denoted by W_9 in [3]. Also the smallest circle passing through one of the points of intersection of the circles γ and $A(O)$ and touching $\mathcal{P}_O$ is Archimedean, which is denoted by W_{13} in [3]. In this section, we generalize those Archimedean circles to the collinear arbelos. The circle W_8 is generalized as follows.

Theorem 2. For a collinear arbelos (α, β, γ), a circle δ with center O is Archimedean if and only if the external center of similitude of the circles (BP) and δ is the point $O_{(AQ)}$.

Proof: If δ is Archimedean, the external center of similitude of (BP) and δ divides the segment $O_{(BP)}O$ externally in the ratio $t : r_{\mathrm{A}}$. By (1), its coordinates are

$$\left(\frac{-r_{\mathrm{A}}(b+p)/2}{t - r_{\mathrm{A}}},\ 0\right) = \left(\frac{-s(-ta/s - tq/s)}{2t},\ 0\right) = \left(\frac{a+q}{2},\ 0\right),$$

which are also the coordinates of the point $O_{(AQ)}$. Since the correspondence between the radius of δ and the external center of similitude of (BP) and δ is one-to-one, the converse holds. □

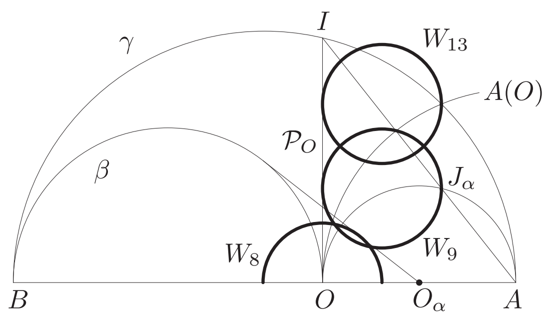

Figure 4: The circles W_8, W_9 and W_{13}

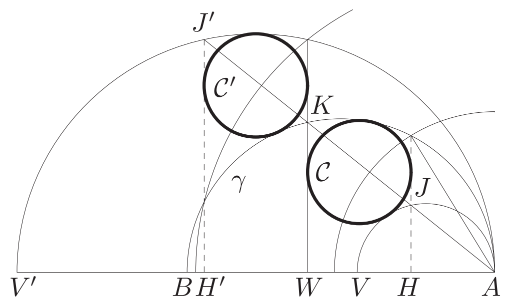

Figure 5: Generalizations of W_9 and W_{13}

Theorem 3. Let V, V' and W be points on the line AB such that $V' < B < V < A$ and $B < W < A$. We denote by K the point of intersection of the circle γ and the line $\mathcal{P}_W$ lying in the region $y > 0$. Let $\mathcal{C}$ (resp. $\mathcal{C}'$) be the circle touching the circles γ internally (resp. externally), (AV) externally (resp. (AV') internally) and the line $\mathcal{P}_W$ from the side opposite to the point B (resp. A). And, let J (resp. J') denote the point of intersection of (AV) (resp. (AV')) and the line AK, and finally H (resp. H') the foot of the perpendicular to AB from the point J (resp. J').

(i) The line $\mathcal{P}_H$ (resp. $\mathcal{P}_{H'}$) touches $\mathcal{C}$ (resp. $\mathcal{C}'$).

(ii) The circle with center A belonging to the pencil of circles determined by the circle γ and the line $\mathcal{P}_H$ (resp. $\mathcal{P}_{H'}$) also belongs to the pencil of circles determined by the circle (AV) (resp. (AV')) and the line $\mathcal{P}_W$.

Proof: We use a rectangular coordinate system with origin W such that the points A, B and V have coordinates $(a, 0)$, $(b, 0)$ and $(v, 0)$, respectively, with $a - b = 2u$ (see Fig. 5). By the similarity $|AH| = a(a - v)/(a - b)$. Hence the x-coordinate of the point H is $a - a(a - v)/(a - b) = a(v - b)/(a - b) = |AW||BV|/(2u)$. This is equal to a diameter of the circle $\mathcal{C}$ by (i) of Lemma 1, i.e., the line $\mathcal{P}_H$ touches $\mathcal{C}$. The rest part of (i) is proved similarly with (ii) of Lemma 1.

The circle γ and the line $\mathcal{P}_H$ have the equations

$$x^2 - (a + b)x + y^2 = -ab \tag{2}$$

and

$$(a - b)x = a(v - b), \tag{3}$$

respectively. Subtracting (3) from (2), we get

$$x^2 - 2ax + y^2 = -av. \tag{4}$$

This is an equation of the circle with center A belonging to the pencil of circles determined by γ and $\mathcal{P}_H$. While the circle (AV) and the line $\mathcal{P}_W$ have the equations

$$x^2 - (a + v)x + y^2 = -av \tag{5}$$

and

$$(-a + v)x = 0, \tag{6}$$

respectively. Adding (5) and (6), we also get (4). Therefore the same circle also belongs to the pencil of circles determined by (AV) and $\mathcal{P}_W$. The rest part of (ii) is proved similarly. □

Let $\mathcal{L}_\alpha$ be the perpendicular to the line AB from the point J_α. The circles W_9 and W_{13} are generalized as follows.

Corollary 1. Let (α, β, γ) be a collinear arbelos.

(i) The line $\mathcal{L}_\alpha$ touches the circle δ_α.

(ii) The circle with center A belonging to the pencil of circles determined by the circle γ and the line $\mathcal{L}_\alpha$ also belongs to the pencil of circles determined by the circle α and the line $\mathcal{P}_O$.

Let $\mathcal{E}$ be the external common tangent of the circles α and β touching the two circles in the region $y > 0$. It touches α and β at the points J_α and J_β, respectively, if $(\alpha, \beta, \gamma) = (BQPA)$ or $(\alpha, \beta, \gamma) = (BPQA)$ [5]. This also holds when $(\alpha, \beta, \gamma) = (PBAQ)$, for the circles α, β and (PQ) form a collinear arbelos, whose Archimedean circles are congruent to those of (α, β, γ) by Theorem 1.

2.2. Infinite Archimedean circles of the collinear arbelos

In [11] we have considered infinite Archimedean circles of the tangent arbelos (α, β, γ): If a circle δ passes through the point O and does not touch the circle β internally, then δ is Archimedean if and only if the external center of similitude of the circles δ and β lies on the circle α. In this section we generalize those circles to the collinear arbelos (see Fig. 6).

Theorem 4. For a collinear arbelos (α, β, γ) and a point S different from the point O, let S_α be the external center of similitude of the circles (OS) and (BP). In the case (α, β, γ) being a tangent arbelos, (OS) does not touch the circle β internally. Then the circle (OS) is Archimedean if and only if the point S_α lies on the circle (AQ) and the vectors $\overrightarrow{OS}$ and $\overrightarrow{O_{(AQ)}S_\alpha}$ are parallel with the same direction.

Proof: If the circle (OS) is Archimedean, its center is expressed by $(r_\mathrm{A}\cos\theta,\ r_\mathrm{A}\sin\theta)$ for a real number θ, and the point S_α divides the segment $O_{(BP)}O_\delta$ in the ratio $t : r_\mathrm{A}$ externally. Therefore its coordinates are

$$\left(\frac{-r_\mathrm{A}(b+p)/2 + tr_\mathrm{A}\cos\theta}{t-r_\mathrm{A}},\ \frac{tr_\mathrm{A}\sin\theta}{t-r_\mathrm{A}}\right) = \left(\frac{a+q}{2} + s\cos\theta,\ s\sin\theta\right)$$

by (1). This implies that S_α lies on the circle (AQ) and the vectors $\overrightarrow{OS}$ and $\overrightarrow{O_{(AQ)}S_\alpha}$ are parallel with the same direction. Since the correspondence between S and S_α is one-to-one, the converse holds. □

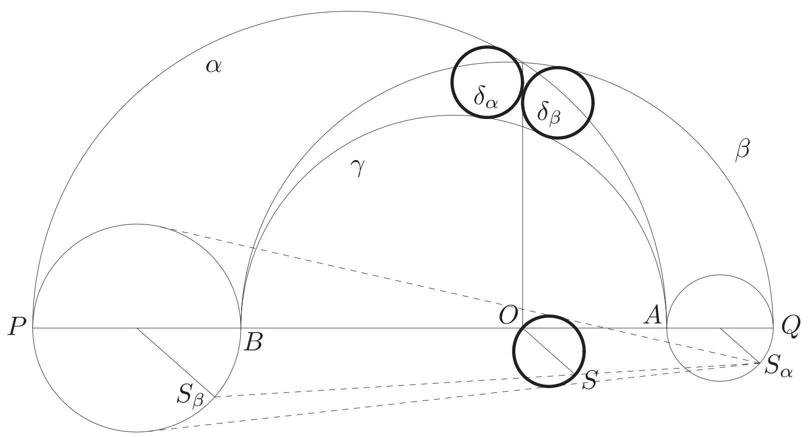

Figure 6: An Archimedean circle (OS)

For a tangent arbelos (α, β, γ), if a circle (OS) touches the circle β internally at the point O, then the external center of similitude of (OS) and β is O, which lies on the circle α, and the vectors $\overrightarrow{OS}$ and $\overrightarrow{O_\alpha O}$ are parallel with the same direction. Therefore this case is excluded in the theorem.

Theorem 5. For a collinear arbelos (α, β, γ) and a point S different from the point O, let S_α and S_β be points on the circles (AQ) and (BP), respectively, such that the vectors $\overrightarrow{O_{(AQ)}S_\alpha}$ and $\overrightarrow{O_{(BP)}S_\beta}$ are parallel to the vector $\overrightarrow{OS}$ with the same direction.

(i) The circle (OS) is Archimedean if and only if the point S divides the segment $S_\alpha S_\beta$ in the ratio $s : t$ internally.

(ii) If S does not lie on the line AB, the circle (OS) is Archimedean if and only if the three points S_α, S_β and S are collinear.

Proof: The points S_α and S_β have the coordinates $((a+q)/2 + s\cos\theta, s\sin\theta)$ and $((b+p)/2 + t\cos\theta, t\sin\theta)$ for a real number θ, respectively. If the point S divides the segment $S_\alpha S_\beta$ internally in the ratio $s : t$, its coordinates are

$$\left(\frac{t((a+q)/2 + s\cos\theta) + s((b+p)/2 + t\cos\theta)}{s+t},\ 2r_\mathrm{A}\sin\theta\right) = (2r_\mathrm{A}\cos\theta,\ 2r_\mathrm{A}\sin\theta),$$

since $t(a+q)/2+s(b+p)/2=0$ by (1). Therefore the circle (OS) is Archimedean. Conversely if (OS) is Archimedean, let S' be the point dividing $S_\alpha S_\beta$ in the ratio $s:t$ internally. Then $S'=S$ as just proved. The part (ii) follows from (i). □

2.3. Power's Archimedean circles

Frank Power has found a new type of Archimedean circles [13]: For the tangent arbelos (α,β,γ), if two congruent circles touch at one of the farthest points on the circle α from the line AB and also touch the circle γ, they are Archimedean. In this section we generalize those circles to the collinear arbelos.

If two congruent circles of radius r touching at a point D also touch a given circle $\mathcal{C}$ at points different from D, we say that D *generates* circles of radius r with $\mathcal{C}$. And the two circles are said to *be generated by D with $\mathcal{C}$*. If the generated circles are Archimedean, we call them *Power type Archimedean circles*. In [1], [2], [4] and [13], only one case is considered, in which the two congruent circles touch a given circle internally. But we do not exclude the case in which the two circles touch a given circle externally.

Lemma 2. For a circle δ of radius r a point D generates circles of radius $|r^2-|DO_\delta|^2|/(2r)$ with δ.

Proof: If x is the radius of the generated circles, then $(r\pm x)^2=|DO_\delta|^2+x^2$. □

Lemma 3. For points V and W on the line AB, if the circle (AV) and the line $\mathcal{P}_W$ intersect, the points of intersection generate circles of radius $|AW||BV|/(2u)$ with the circle γ.

Proof: Let K be one of the points of intersection, $d=|KO_\gamma|$, $w=|AW|$ and $v=|BV|$. There are three cases to be considered,

(i) $B<V<A$,
(ii) $V<B$ and
(iii) $A<V$ (see Fig. 1).

In (iii) we get

$$d^2=|KW|^2+|WO_\gamma|^2=|AW||WV|+|WO_\gamma|^2=w(v-2u-w)+(u+w)^2=u^2+wv.$$

Similarly we get $d^2=u^2\pm wv$ in the other cases. Therefore by Lemma 2, the radius of the generated circles is $wv/(2u)$. □

Each of the three cases in the proof corresponds to (i), (ii) and (iii) of Lemma 1, respectively, in which the radius of the touching circles in Lemma 1 is half the size of the radius of the generated circles in Lemma 3. Power's result is generalized to the collinear arbelos by this fact and Theorem 1 (see Fig. 7).

Corollary 2. The points of intersection of the circle α and the perpendicular bisector of the segment AO generate Archimedean circles with the circle γ for a collinear arbelos (α,β,γ).

If (α,β,γ) is a tangent arbelos, the point O generates circles of radius $2r_\mathrm{A}$ with the circle γ [12]. The fact is also generalized by Theorem 1 and Lemmas 1 and 3. *If $(\alpha,\beta,\gamma)=(BPQA)$ or $(\alpha,\beta,\gamma)=(PBAQ)$, the points of intersection of the circles α and β generate circles of radius $2r_\mathrm{A}$ with each of the circles γ and (PQ).*

If $(\alpha,\beta,\gamma)=(BPQA)$ or $(\alpha,\beta,\gamma)=(PBAQ)$, the circle $(AO_{(BP)})$ and the line $\mathcal{P}_O$ intersect. If $(\alpha,\beta,\gamma)=(BQPA)$, let L_1 and L_2 be the limiting points of the pencil of circles determined by the circles α and β such that $L_1<L_2$. Since $B<L_1<O<P<L_2$, we get

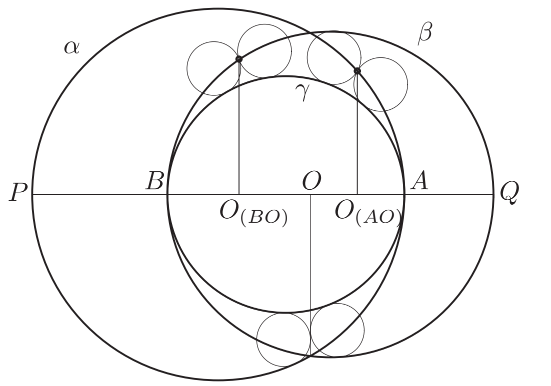

Figure 7: A generalization of Power's circles

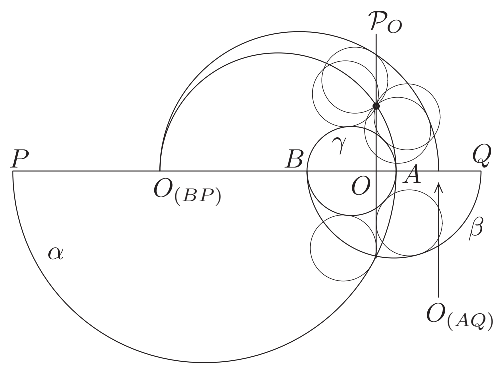

Figure 8: A generalization of Bui's circles

$b+p<0$, i.e., $a(b+p)/2<0$. Therefore $(AO_{(BP)})$ and $\mathcal{P}_O$ also intersect in this case. Quang Tuan Bui has found several Power type Archimedean circles for the tangent arbelos, one of which is as follows: The points of intersection of the circle (AO_β) and the line $\mathcal{P}_O$ generate Archimedean circles with the circle γ for a tangent arbelos (α,β,γ) [2]. This can also be generalized (see Fig. 8).

Corollary 3. The points of intersection of the circle $(AO_{(BP)})$ and the line $\mathcal{P}_O$ generate Archimedean circles with each of the circles γ and $(O_{(BP)}O_{(AQ)})$ for a collinear arbelos (α,β,γ).

Proof: The points of intersection generate Archimedean circles with γ by Theorem 1 and Lemma 3. If $(\alpha,\beta,\gamma)=(BQPA)$ or $(\alpha,\beta,\gamma)=(BPQA)$, the radius of the circles generated by the points with $(O_{(BP)}O_{(AQ)})$ is

$$\frac{(a-(a+q)/2)(-(b+p)/2)}{(a+q)/2-(b+p)/2}=\frac{s(a+q)t/s}{a+q+(a+q)t/s}=\frac{st}{s+t}\,.$$

by Lemma 3 and (1). The rest of the corollary is proved similarly. □

Theorem 6. If a circle δ of radius r touches a circle $\mathcal{C}$ and has a point in common with the circle $(O_{\mathcal{C}}O_\delta)$, the point generates circles of radius r with $\mathcal{C}$, one of which is δ itself.

Proof: Let T be the tangent point of the circles δ and $\mathcal{C}$. The theorem holds if $\delta=(TO_{\mathcal{C}})$. Let us assume that the circles δ and $(O_{\mathcal{C}}O_\delta)$ intersect and the line $TO_{\mathcal{C}}$ intersects δ and the common chord of δ and $(O_{\mathcal{C}}O_\delta)$ at points S and H, respectively. Notice $|HS||HT|=|HO_{\mathcal{C}}||HO_\delta|$. Let $|HT|=x$ and let s be the radius of $\mathcal{C}$. If δ touches $\mathcal{C}$ externally, $(2r-x)x=(s+x)(r-x)$, which implies $x=rs/(r+s)$. Hence the points of intersection generate circles of radius r with $\mathcal{C}$ by Lemma 3. Two of the four generated circles coincide with δ, since δ has radius r and passes through the points of intersection of δ and $(O_{\mathcal{C}}O_\delta)$. The rest of the theorem is proved similarly. □

Any Archimedean circle δ touching a circle $\mathcal{C}$ and intersecting the circle $(O_\delta O_{\mathcal{C}})$ can be regarded as a Power type Archimedean circle generated by the points of intersection of δ and $(O_\delta O_{\mathcal{C}})$ with $\mathcal{C}$ by the theorem. Also we can get two more Power type Archimedean circles touching δ in this case.

2.4. Another Bui's Archimedean circles

Quang Tuan Bui has also found that the points of intersection of the circles $A(O)$ and $B(I)$ generate Archimedean circles with the circle γ for a tangent arbelos (α,β,γ) [1]. In this section we generalize this.

Theorem 7. Let V, W be points on the segment AB such that $|VW| = 2r > 0$ and let M be the midpoint of the segment VW. Let the circle γ intersect the lines $\mathcal{P}_V$ and $\mathcal{P}_W$ in the points J and K, respectively. Let $\mathcal{C}$ be the circle touching $\mathcal{P}_V$, $\mathcal{P}_W$ and γ.

(i) The circles $A(J)$, $B(K)$ and the line $\mathcal{P}_M$ belong to the same pencil of circles.

(ii) Let δ be a circle with center on the line AB and touching $\mathcal{C}$. If the pencil is intersecting and the points of intersection and $\mathcal{C}$ lie on the same side of γ, the points generate circles of radius r with each of the circles γ and δ.

Proof: Let v and w be the x-coordinates of the points V and W, respectively (see Fig. 9). Notice that

$$|AJ|^2 = |VJ|^2 + |AV|^2 = (a-v)(2u-(a-v)) + (a-v)^2 = 2u(a-v).$$

Therefore the circle $A(J)$ has the equation

$$(x-a)^2 + y^2 = 2u(a-v). \tag{7}$$

Similarly the circle $B(K)$ has the equation

$$(x-b)^2 + y^2 = 2u(w-b). \tag{8}$$

Subtracting (7) from (8) with $a - b = 2u$, we get $x = (v+w)/2$. This proves part (i).

In order to prove (ii), let R be one of the points of intersection of $A(J)$ and $B(K)$, and let S and T be the points on the line AB such that $\delta = (ST)$ and $T < S$. There are four cases to be considered:

(C1) the circle $\mathcal{C}$ touches the circles γ and δ internally,
(C2) $\mathcal{C}$ touches γ internally δ externally,
(C3) $\mathcal{C}$ touches γ externally δ internally,
(C4) $\mathcal{C}$ touches γ and δ externally.

Let us assume (C1). Then $W < V$, since R lies inside of γ. Therefore

$$v - w = 2r. \tag{9}$$

Let s and t be the x-coordinates of S and T, respectively. By the Pythagorean theorem,

$$\left(\frac{s-t}{2} - \frac{v-w}{2}\right)^2 - \left(\frac{v+w}{2} - \frac{s+t}{2}\right)^2 = \left(\frac{a-b}{2} - \frac{v-w}{2}\right)^2 - \left(\frac{a+b}{2} - \frac{v+w}{2}\right)^2.$$

Simplifying this, we get

$$aw + bv + st - ab = sw + tv. \tag{10}$$

Let h be the distance between R and AB. From (7), $h^2 = (a-b)(a-v) - ((v+w)/2 - a)^2$. Therefore by (9) and (10),

$$|RO_\delta|^2 - \left(\frac{s-t}{2}\right)^2 = h^2 + \left(\frac{v+w}{2} - \frac{s+t}{2}\right)^2 - \left(\frac{s-t}{2}\right)^2$$
$$= \quad aw + bv + st - ab - \frac{(s+t)(v+w)}{2} = sw + tv - \frac{(s+t)(v+w)}{2} = -(s-t)r.$$

Hence R generates circles of radius r with δ by Lemma 2. Since γ satisfies the same condition as $\mathcal{C}$, R also generates circles of radius r with γ. The rest of the part (ii) is proved similarly. □

The circle γ and the line $\mathcal{L}_\alpha$ do not always intersect if $(\alpha,\beta,\gamma) = (PBAQ)$. If they intersect, we denote one of the points of intersection by L_α. The perpendicular to the line AB passing through the center of the circle δ_α is denoted by $\mathcal{K}_\alpha$. If $\mathcal{L}_\alpha$ touches the circle γ for $(PBAQ)$, the circle $B(L_\alpha)$ degenerates to the point B. Therefore the circle $B(L_\alpha)$ and the line $\mathcal{K}_\alpha$ do not intersect in this case.

Corollary 4. Let the circle γ and the line $\mathcal{L}_\alpha$ intersect for a collinear arbelos (α,β,γ).

(i) The circles $A(L_\alpha)$, $B(I)$ and the line $\mathcal{K}_\alpha$ belong to the same pencil of circles. If the pencil is intersecting, the points of intersection generate Archimedean circles with each of the circles γ and α.

(ii) The circles $A(I)$, $B(L_\alpha)$ and the line $\mathcal{K}_\alpha$ belong to the same pencil of circles. If the pencil is intersecting, the points of intersection generate Archimedean circles with the circle γ.

If (α,β,γ) is a tangent arbelos, $A(O) = A(L_\alpha)$ holds. Hence Theorem 7 and Corollary 4 are generalizations of Bui's result.

2.5. The external common tangents of the circles α and β

Let us recall that $\mathcal{E}$ is the external common tangent of the circles α and β touching the two circles in the region $y > 0$. For a tangent arbelos (α,β,γ), the maximal circle touching the line $\mathcal{E}$ and the minor circular arc of the circle γ cut by $\mathcal{E}$ internally is Archimedean, which is called *Bankoff quadruplet circle* and denoted by W_4 in [3].

In this section we generalize this circle and give several new Power type Archimedean circles related to $\mathcal{E}$. Let E be the point of intersection of the lines $\mathcal{E}$ and $O_\gamma I$ for the collinear arbelos (α,β,γ). The tangent of the circle γ at the point I is denoted by $\mathcal{I}$.

Lemma 4. The lines $\mathcal{E}$ and $O_\gamma I$ are perpendicular.

Proof: Let K_α and K_β be the points on the circles α and β, respectively, such that the vectors $\overrightarrow{O_\alpha K_\alpha}$, $\overrightarrow{O_\beta K_\beta}$ and $\overrightarrow{O_\gamma I} = (-(a+b)/2,\ \sqrt{-ab})$ are parallel with the same direction.

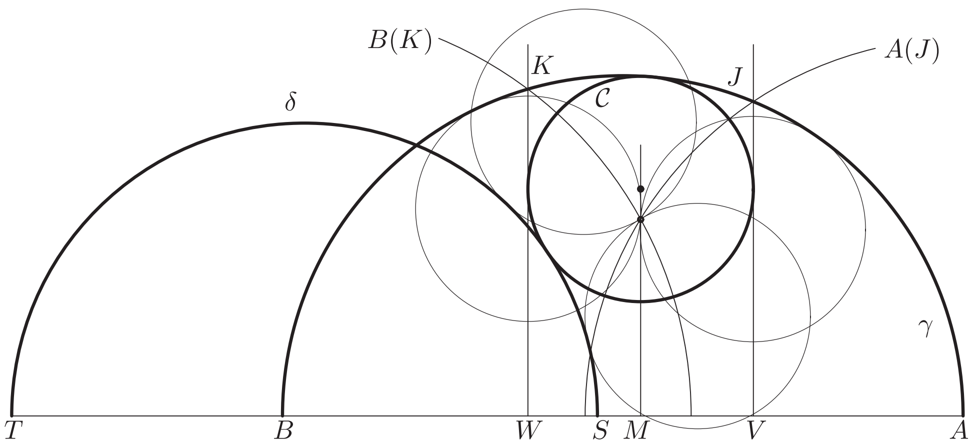

Figure 9: A generalization of Bui's Archimedean circles

We denote their coordinates by

$$\left(\frac{1}{2}(a+p)+\frac{1}{2}(a-p)\cos\theta,\ \frac{1}{2}(a-p)\sin\theta\right) \text{ and } \left(\frac{1}{2}(b+q)+\frac{1}{2}(q-b)\cos\theta,\ \frac{1}{2}(q-b)\sin\theta\right),$$

respectively, where $\cos\theta=-(a+b)/(a-b)$ and $\sin\theta=2\sqrt{-ab}/(a-b)$. Since $\overrightarrow{K_\alpha K_\beta}=\frac{1}{2}\big(b+q-a-p+(q-b-a+p)\cos\theta,\ (q-b-a+p)\sin\theta\big)$, the inner product of the vectors $\overrightarrow{K_\alpha K_\beta}$ and $\overrightarrow{O_\gamma I}$ is

$$-\left(\frac{b+q-a-p}{2}-\frac{(q-b-a+p)(a+b)}{2(a-b)}\right)\frac{(a+b)}{2}+\frac{(q-b-a+p)2\sqrt{-ab}}{2(a-b)}\sqrt{-ab}=\frac{ap-bq}{2},$$

which equals 0. Therefore the lines $K_\alpha K_\beta$ and $O_\gamma I$ are perpendicular. Hence the lines $O_\alpha K_\alpha$ and $O_\beta K_\beta$ are perpendicular to $K_\alpha K_\beta$. This implies that the lines $K_\alpha K_\beta$ and $\mathcal{E}$ coincide. Therefore the lines $\mathcal{E}$ and $O_\gamma I$ are perpendicular. □

Since the point K_α in the proof is the tangent point of the circle α and the line $\mathcal{E}$, it coincides with the point J_α defined in 2.1. Similarly the point K_β coincides with the point J_β. By the proof, the respective coordinates of the points J_α and J_β are

$$\left(\frac{a(-b+p)}{a-b},\ \frac{(a-p)\sqrt{-ab}}{a-b}\right) \text{ and } \left(\frac{b(-a+q)}{b-a},\ \frac{(b-q)\sqrt{-ab}}{b-a}\right).$$

Theorem 8. The line $\mathcal{E}$ has the equation

$$(a+b)x-2\sqrt{-ab}y-ab=ap. \tag{11}$$

Proof: By Lemma 4, the line $\mathcal{E}$ has an equation $(a+b)x-2\sqrt{-ab}y+d=0$ for some real number d. But it passes through the point J_α. Therefore we get $d=-a(b+p)$. □

The right side of (11) is the power of the point O with respect to the circle α, which also equals the power of the point O with respect to the circle β. The remaining external common tangent of the circles α and β has the equation $(a+b)x+2\sqrt{-ab}y-ab=ap$.

Theorem 9. Let (α,β,γ) be a collinear arbelos.

(i) The lines $\mathcal{E}$ and $\mathcal{I}$ are parallel, and the circle (EI) is Archimedean.

(ii) If IJ is a diameter of the circle γ, the points of intersection of the circle (EJ) and the perpendicular bisector of IJ generate Archimedean circles with γ.

(iii) If $(\alpha,\beta,\gamma)=(BQPA)$ or $(\alpha,\beta,\gamma)=(BPQA)$ (resp. $(\alpha,\beta,\gamma)=(PBAQ)$), the points of intersection of the circle $(O_\gamma I)$ (resp. $(O_\gamma E)$) and the line $\mathcal{E}$ (resp. $\mathcal{I}$) generate Archimedean circles with γ.

Proof: That the lines $\mathcal{E}$ and $\mathcal{I}$ are parallel follows from Lemma 4 (see Figs. 10 and 11). By Theorem 1 and (11), the distance between the point $I(0,\sqrt{-ab})$ and the line $\mathcal{E}$ is

$$\frac{|-2\sqrt{-ab}\sqrt{-ab}-a(b+p)|}{\sqrt{(a+b)^2+(-2\sqrt{-ab})^2}}=\frac{a|b-p|}{a-b}=\frac{at}{u}=2r_{\mathrm{A}}.$$

This proves (i).

The part (ii) follows from Lemma 3. To prove (iii), let K be one of the points of intersection of the circle $(O_\gamma E)$ and the line $\mathcal{I}$ in the case $(\alpha,\beta,\gamma)=(PBAQ)$. Then

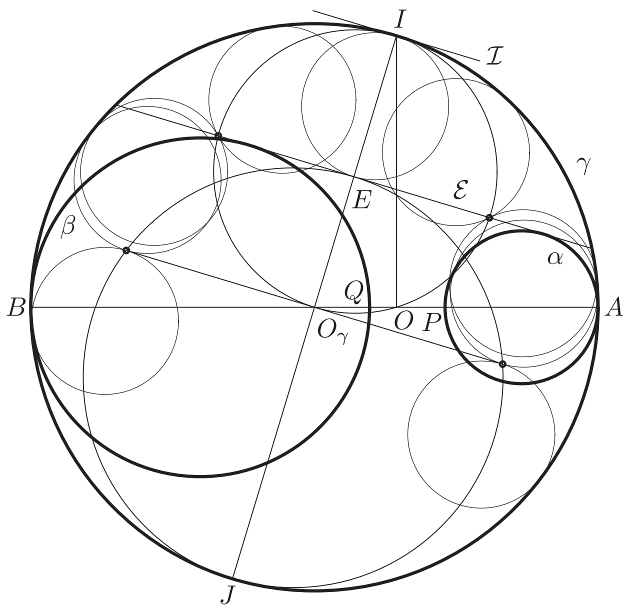

Figure 10: $(BQPA)$

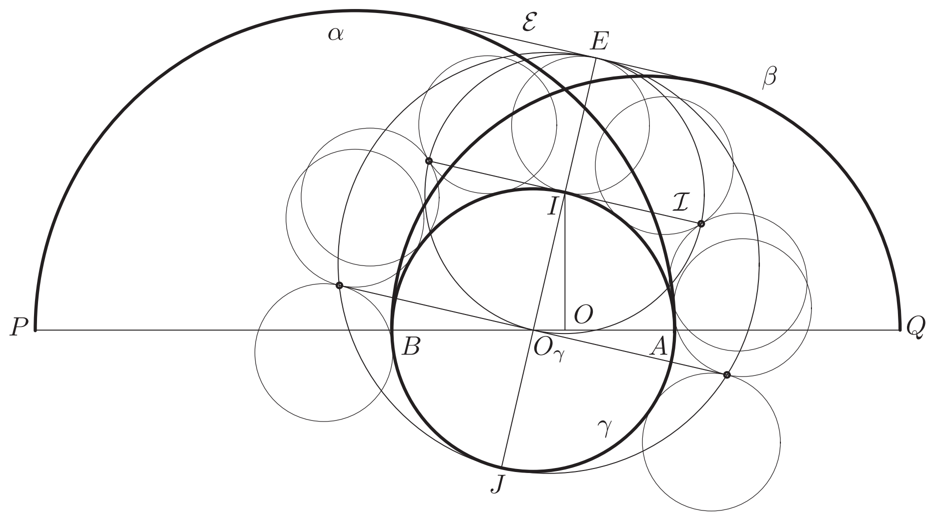

Figure 11: $(PBAQ)$

$|KO_\gamma|^2 - u^2 = |IK|^2 = 2ur_{\rm A}$ by (i). Hence by Lemma 2, the point K generates circles of radius $r_{\rm A}$ with γ. The rest part of (iii) follows from Lemma 3. □

The part (i) of Theorem 9 is a generalization of the circle W_4 of the tangent arbelos. For a tangent arbelos (α, β, γ), the circle touching the line $\mathcal{E}$ at the point E and the line parallel to $\mathcal{E}$ passing through the point O is Archimedean, which is denoted by W_5 in [3]. This circle is also generalized to the collinear arbelos. Let (α, β, γ) be a collinear arbelos. Since the points J_α and J_β lie on the lines AI and BI, respectively, $\angle J_\beta I J_\alpha = \pi/2$. If K is the point of intersection of the lines $\mathcal{P}_O$ and $\mathcal{E}$, then K is the midpoint of the segment $J_\alpha J_\beta$. Hence the point I lies on the circle $(J_\alpha J_\beta)$ with center K. Let L be the point of intersection of the lines $J_\alpha P$ and $J_\beta Q$. Then $J_\alpha I J_\beta L$ is a rectangle and L is the remaining point of intersection of the circle $(J_\alpha J_\beta)$ and the line $\mathcal{P}_O$. Therefore reflecting the Archimedean circle (EI) and the line $\mathcal{I}$ in the line $\mathcal{E}$, we get one more Archimedean circle touching $\mathcal{E}$ at the point E and the line parallel to $\mathcal{E}$ passing through the point L. This is a generalization of the Archimedean circle

W_5 of the tangent arbelos.

3. The skewed arbelos

We now consider another kind of generalization of the arbelos. From now on α and β are the circles (AO) and (BO), respectively for a point O on the segment AB. We also redefine a and b as the radii of α and β, respectively. We now use a rectangular coordinate system with origin O so that the points A and B have coordinates $(2a, 0)$ and $(-2b, 0)$, respectively. We consider various circles touching the two circles at points different from O. Such a circle is expressed by the equation

$$\left(x - \frac{b-a}{z^2-1}\right)^2 + \left(y - \frac{2z\sqrt{ab}}{z^2-1}\right)^2 = \left(\frac{a+b}{z^2-1}\right)^2 \tag{12}$$

for a real number $z \neq \pm 1$ [12]. The circle is denoted by γ_z. It touches α and β internally if $|z| < 1$ and externally if $|z| > 1$. The external common tangents of α and β have the equations $(a-b)x \mp 2\sqrt{ab}y + 2ab = 0$, which are denoted by $\gamma_{\pm 1}$, where the double-signs correspond. The configuration consisting of the three circles α, β and γ_z is also denoted by $(\alpha, \beta, \gamma_z)$, and called a *skewed arbelos*. We redefine r_{A} as the common radius of Archimedean circles of the tangent arbelos $(\alpha, \beta, \gamma_0)$, i.e., $r_{\mathrm{A}} = ab/(a+b)$.

3.1. Basic properties of the skewed arbelos

We summarize several results in [8] with additional basic properties of the skewed arbelos. Let A_z be the tangent point of the circles α and γ_z. The point B_z is defined similarly (see Figs. 12 and 13). Let $c = \sqrt{a/b}$, $c' = \sqrt{b/a}$, $\varphi = \varphi(a, b) = 2ab/(az^2 + b)$, $\psi = \psi(a, b) = cz\varphi$, $\varphi' = \varphi(b, a)$, and $\psi' = \psi(b, a)$. Then the coordinates of A_z and B_z are (φ, ψ) and $(-\varphi', \psi')$, respectively.

Let δ_z^α be the circle different from the circle β touching the circle α and the tangents of β from the point A_z. It touches α internally if $|z| < 1$, and externally if $|z| > 1$. The circle δ_z^β is defined similarly. They are congruent with common radius $|1 - z^2|r_{\mathrm{A}}$ and their centers have the coordinates $(r_{\mathrm{A}}(1 + z^2),\ 2r_{\mathrm{A}}cz)$ and $(-r_{\mathrm{A}}(1 + z^2),\ 2r_{\mathrm{A}}c'z)$, respectively.

Let T_β be the tangent point of the circles β and δ_z^β and let $\mathcal{T}_\alpha$ be their common tangent at T_β. The point T_α and the line $\mathcal{T}_\beta$ are defined similarly. The points T_α and T_β have the coordinates $(z^2\varphi',\ c^2\psi')$ and $(-z^2\varphi,\ c'^2\psi)$, respectively. The two points lie on the line A_zB_z. The tangents $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ have the equations

$$(-az^2 + b)x + 2z\sqrt{ab}y - 2abz^2 = 0 \quad \text{and} \quad (-a + bz^2)x + 2z\sqrt{ab}y - 2abz^2 = 0, \tag{13}$$

respectively. They are perpendicular to the lines $A_zO_{\gamma_z}$ and $B_zO_{\gamma_z}$, respectively. The distance between the center of the circle δ_z^α and the line $\mathcal{T}_\alpha$ is $|1 - z^2|r_{\mathrm{A}}$. Therefore $\mathcal{T}_\alpha$ also touches δ_z^α. Similarly $\mathcal{T}_\beta$ touches δ_z^β.

Let W_α be the tangent point of $\mathcal{T}_\alpha$ and δ_z^α. The point W_β is defined similarly. Their coordinates are $(z^2\varphi, z^2\psi)$ and $(-z^2\varphi', z^2\psi')$, respectively. Hence they lie on the lines A_zO and B_zO, respectively. The lines $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ intersect at the point $(0, z\sqrt{ab})$, which is denoted by O_z. Since it is the midpoint of the segment $O_{\delta_z^\alpha}O_{\delta_z^\beta}$, the lines $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ are the internal common tangents of the circles δ_z^α and δ_z^β. The points T_α, T_β, W_α, W_β and O lie on the circle

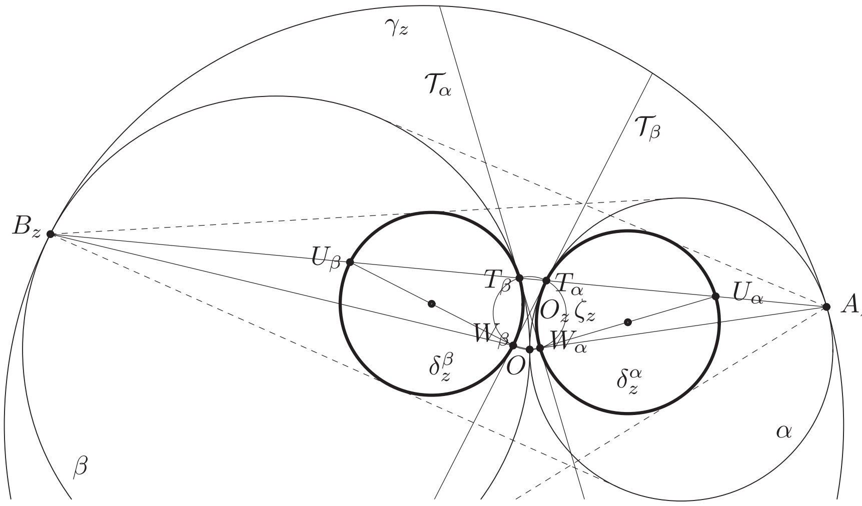

Figure 12: A skewed arbelos $(\alpha, \beta, \gamma_z)$ for $|z| < 1$

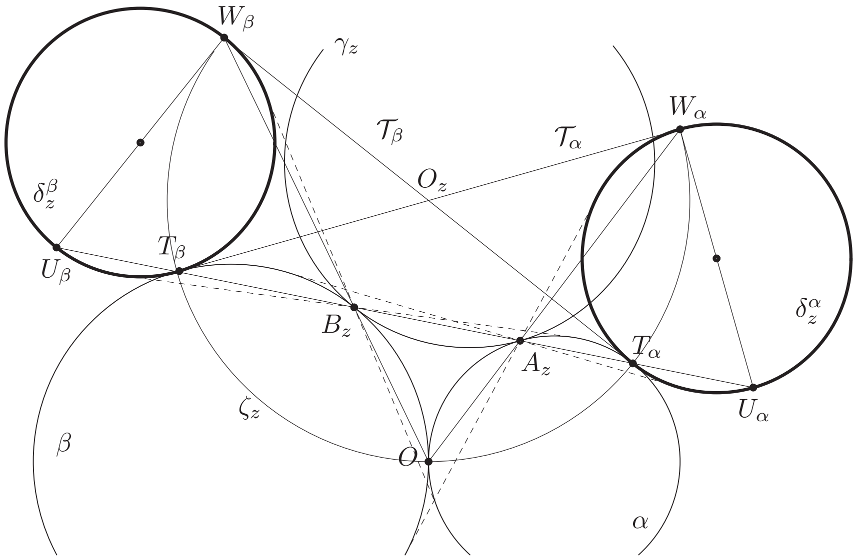

Figure 13: A skewed arbelos $(\alpha, \beta, \gamma_z)$ for $|z| > 1$

with center O_z and radius $|z|\sqrt{ab}$, which is denoted by ζ_z. It is orthogonal to the circles α, β, δ_z^α and δ_z^β.

Let U_α be the reflected image of the point W_α in the center of the circle δ_z^α. The point U_β is defined similarly. If $\phi = \phi(a, b) = 2r_{\mathrm{A}}(1 + z^2) - z^2\varphi$, $\rho = \rho(a, b) = 4r_{\mathrm{A}}cz - z^2\psi$, the coordinates of U_α and U_β are (ϕ, ρ) and $(-\phi', \rho')$, respectively, where $\phi' = \phi(b, a)$ and $\rho' = \rho(b, a)$. The point U_α also lies on the line A_zB_z because

$$U_\alpha = \frac{2ab + b^2 + (a - b)az^2 + abz^4}{(a + b)^2} A_z + \frac{a^2 - (a - b)az^2 - abz^4}{(a + b)^2} B_z.$$

Similarly the point U_β lies on the same line. Let F_α be the foot of perpendicular to the line $\mathcal{T}_\alpha$ from the point A_z. The distance between A_z and $\mathcal{T}_\alpha$ is

$$|A_zF_\alpha| = \varphi|1 - z^2|. \tag{14}$$

3.2. A generalization of the twin circles of Archimedes

In this section we generalize the twin circles of Archimedes of the tangent arbelos to the skewed arbelos. If $|z| < 1$, the tangent $\mathcal{T}_\alpha$ of the circle β intersects the circle γ_z at two points, since β is contained in γ_z. Hence there are two circles touching γ_z internally, α externally and $\mathcal{T}_\alpha$ from the side opposite to the point B (see Fig. 14). If $z > 1$, the line $\mathcal{T}_\alpha$ and the circle α have no points in common. For if $z = 1$, $\mathcal{T}_\alpha$ is the external common tangent of the circles α and β touching in the region $y > 0$, and the coordinates of the point T_β shows that it moves on β counter clockwise when the value of z increases. The fact is also true if $z < -1$, for reflecting the skewed arbelos $(\alpha, \beta, \gamma_z)$ in the line AB, we get $(\alpha, \beta, \gamma_{-z})$. The line $\mathcal{T}_\alpha$ touches γ_z from the side opposite to the point A_z if and only if $|A_zF_\alpha| = 2(a+b)/(z^2-1)$ by (12). Solving the equation for z with (14), we get $z = \pm z_\alpha$, where

$$z_\alpha = \sqrt{1 + \frac{a + \sqrt{a(a+4b)}}{2r_{\mathrm{A}}}}.$$

Similarly z_β is defined. Since the radius of γ_z is a monotone decreasing function of z^2 when $z^2 > 1$, while T_β moves on β counter clockwise when the value of z increases, $\mathcal{T}_\alpha$ and γ_z have no points in common, if $|z| > z_\alpha$. Therefore there are two circles touching γ_z, α internally and $\mathcal{T}_\alpha$, if $|z| > z_\alpha$ (see Figs. 15 and 16). While $\mathcal{T}_\alpha$ coincides with γ_z and touches α and β when $|z| = 1$. Therefore it intersects γ_z at two points if $1 < |z| < z_\alpha$. Hence there is no circles touching γ_z, α internally and $\mathcal{T}_\alpha$ in this case (see Fig. 13).

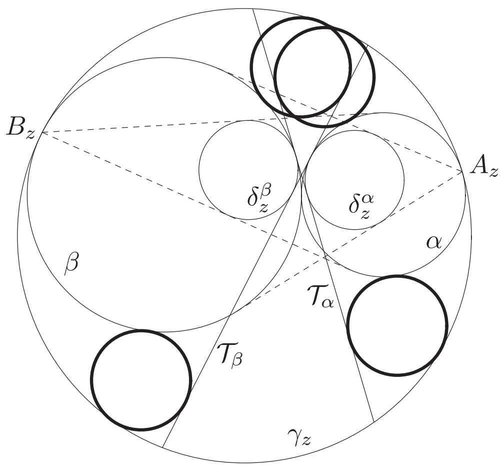

Figure 14: $|z| < 1$

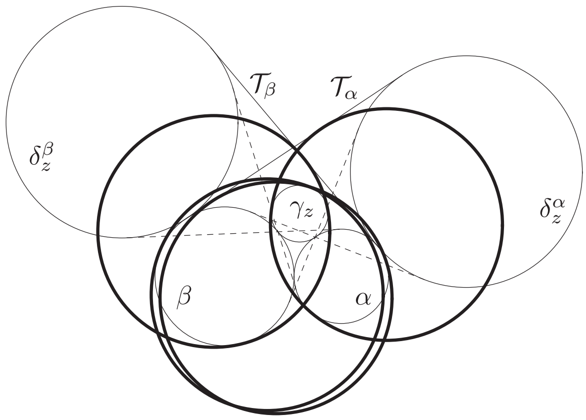

Figure 15: $|z| > \max(z_\alpha, z_\beta)$

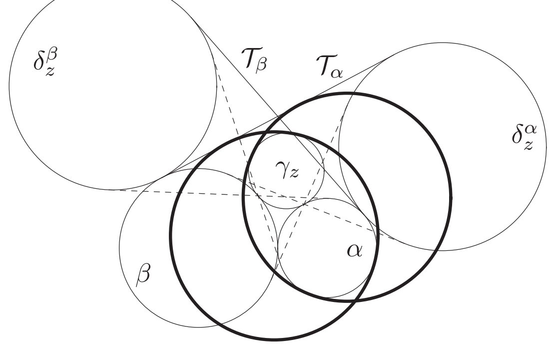

Figure 16: $\min(z_\alpha, z_\beta) < z < \max(z_\alpha, z_\beta)$

Theorem 10. Let $(\alpha, \beta, \gamma_z)$ be a skewed arbelos.

(i) If $|z| < 1$, the two circles touching the circles γ_z internally, α externally and the line $\mathcal{T}_\alpha$ from the side opposite to the point B and the two circles touching the circles γ_z internally, β externally and the line $\mathcal{T}_\beta$ from the side opposite to the point A are congruent with common radius $(1 - z^2)r_{\mathrm{A}}$.

(ii) If $|z| \geq z_\alpha$, the radius of the two circles touching γ_z and α internally and $\mathcal{T}_\alpha$ is $(z^2 - 1)r_{\mathrm{A}}$. Also if $|z| \geq z_\beta$, the radius of the two circles touching γ_z and β internally and $\mathcal{T}_\beta$ is $(z^2 - 1)r_{\mathrm{A}}$.

Proof: Let G and H be the remaining points of intersection of the circles α and γ_z with the line A_zF_α, respectively. If $|z| < 1$, then $|GH| = 2(a + b)/(1 - z^2) - 2a = 2(az^2 + b)/(1 - z^2)$ by (12). Therefore by (i) of Lemma 1 with (14), the radius of the circles touching α is

$$\frac{|GH| \cdot |A_zF_\alpha|}{2|A_zH|} = \frac{2(az^2 + b)}{1 - z^2} \cdot \frac{2ab(1 - z^2)}{az^2 + b} \bigg/ \frac{4(a + b)}{(1 - z^2)} = (1 - z^2)r_{\mathrm{A}}.$$

Similarly the circles touching β have radius $(1 - z^2)r_{\mathrm{A}}$. The part (ii) is proved similarly by (iv) of Lemma 1. □

Theorem 10 is a generalization of the twin circles of Archimedes of the tangent arbelos. Hence it may be appropriate to call circles of radius $|1 - z^2|r_{\mathrm{A}}$ Archimedean circles of the skewed arbelos $(\alpha, \beta, \gamma_z)$. If $|z| < 1$ or $|z| > \max(z_\alpha, z_\beta)$, the four Archimedean circles in Theorem 10 exist (see Figs. 14 and 15). If $\min(z_\alpha, z_\beta) < |z| < \max(z_\alpha, z_\beta)$, there are only two Archimedean circles among the four (see Fig. 16). If $1 < |z| < \min(z_\alpha, z_\beta)$, no such circle exists (see Fig. 13). But Archimedean circles are still defined as circles of radius $(z^2 - 1)r_{\mathrm{A}}$ in the last case. If $z = z_\alpha$, the two Archimedean circles touching α, γ_z and $\mathcal{T}_\alpha$ coincide. If $|z| = 1$, the four Archimedean circles degenerate to the points A_z and B_z.

3.3. Perpendicular case

We consider the case in which the lines $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ are perpendicular for the skewed arbelos $(\alpha, \beta, \gamma_z)$. By (13), the two lines are perpendicular if and only if $z = \pm z_1$ or $z = \pm z_2$, where $z_1 = (a + b + \sqrt{a^2 + 6ab + b^2})/(2\sqrt{ab})$ and $z_2 = 1/z_1$. Since $z_1 > \max(z_\alpha, z_\beta)$, the four Archimedean circles in Theorem 10 exist in this case (see Figs. 17 and 18).

Theorem 11. For a skewed arbelos $(\alpha, \beta, \gamma_z)$, the following four statements are equivalent.

(i) The lines $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ are perpendicular.

(ii) One of the Archimedean circles touching the circles γ_z, α and $\mathcal{T}_\alpha$ coincides with one of the Archimedean circles touching the circles γ_z, β and $\mathcal{T}_\beta$.

(iii) One of the Archimedean circles touching γ_z, α and $\mathcal{T}_\alpha$ touches α and δ_z^α at the point T_α.

(iv) The circle ζ_z is Archimedean.

Let $\mathcal{A}_1^\alpha$ and $\mathcal{A}_2^\alpha$ (resp. $\mathcal{A}_1^\beta$ and $\mathcal{A}_2^\beta$) be the Archimedean circles touching γ_z, α and $\mathcal{T}_\alpha$ (resp. β and $\mathcal{T}_\beta$) such that $\mathcal{A}_1^\alpha = \mathcal{A}_1^\beta$ when (ii) holds. In this case the following statements are true.

(v) Each of the distances of the line AB from the points $O_{\mathcal{A}_1^\alpha}$, $O_{\mathcal{A}_2^\alpha}$, $O_{\mathcal{A}_2^\beta}$ and O_{γ_z} is $2r_{\mathrm{A}}$.

(vi) The points O_{γ_z}, $O_{\mathcal{A}_2^\alpha}$ and $O_{\mathcal{A}_2^\beta}$ lie on a line parallel to AB.

(vii) There is an Archimedean circle touching the external common tangents of $\mathcal{A}_2^\alpha$ and $\mathcal{A}_2^\beta$ and the circles α and β.

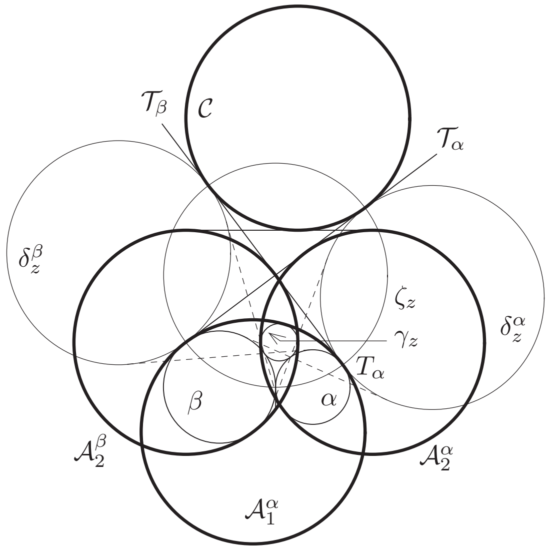

Figure 17: $z = z_1$

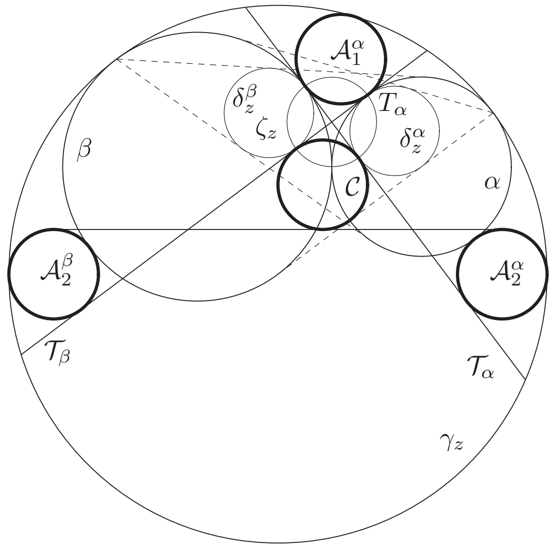

Figure 18: $z = z_2$

(viii) Let $\mathcal{C}$ be the circle obtained by reflecting $\mathcal{A}_1^\alpha$ in the point O_z. If $|z| < 1$, then $\mathcal{C}$ is the incircle of the triangle made by $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$ and the common external tangent of $\mathcal{A}_2^\alpha$ and $\mathcal{A}_2^\beta$ nearer to the point O. If $|z| > 1$, then $\mathcal{C}$ is one of the excircles of the triangle made by $\mathcal{T}_\alpha$, $\mathcal{T}_\beta$ and the external common tangent of $\mathcal{A}_2^\alpha$ and $\mathcal{A}_2^\beta$ farther from O.

Proof: Let us assume (i). The reflected image of the circle δ_z^α in $\mathcal{T}_\beta$ also touches $\mathcal{T}_\alpha$ and the circle α. Therefore it coincides with one of the Archimedean circles touching α. Also the reflected image of the circle δ_z^β in $\mathcal{T}_\alpha$ coincides with one of the Archimedean circles touching the circle β. Since the two images of the reflections coincide, (ii) holds. Let us assume (ii). If $|z| < 1$, the circle $\mathcal{A}_1^\alpha$ touches $\mathcal{T}_\beta$ from the side opposite to α. Since it also touches α, the tangent point coincides with the point T_α. If $|z| > 1$, then $\mathcal{A}_1^\alpha$ touches α internally and touches $\mathcal{T}_\beta$. Therefore it also touches $\mathcal{T}_\beta$ at T_α. While δ_z^α touches α at this point in both the cases. Therefore $\mathcal{A}_1^\alpha$ touches δ_z^α at T_α. This implies (iii). The part (iii) implies (i) obviously. The equivalence of the parts (i) and (iv) is also obvious.

Let us assume (ii), and let $z = z_1$. Since T_α is the midpoint of the segment $O_{\mathcal{A}_1^\alpha}O_{\delta_z^\alpha}$, the y-coordinate of the point $O_{\mathcal{A}_1^\alpha}$ is $2c^2\psi' - 2r_{\mathrm{A}}cz$, which equals $-2r_{\mathrm{A}}$. Since the points $O_{\mathcal{A}_1^\alpha}$ and $O_{\mathcal{A}_2^\alpha}$ are symmetric in the point O_α, the y-coordinate of $O_{\mathcal{A}_2^\alpha}$ is $2r_{\mathrm{A}}$, which also equals the y-coordinate of $O_{\mathcal{A}_2^\beta}$. While $2z\sqrt{ab}/(z^2-1) = 2r_{\mathrm{A}}$ holds by (iv). This proves (v) and (vi) by (12). The Archimedean circle in (vii) is obtained by reflecting $\mathcal{A}_1^\alpha$ in the line AB. By the proof of (v), the y-coordinate of the point $O_{\mathcal{C}}$ is $2z\sqrt{ab} - (-2r_{\mathrm{A}}) = 2z\sqrt{ab} + 2r_{\mathrm{A}}$. Therefore the distance between $O_{\mathcal{C}}$ and the line $O_{\mathcal{A}_2^\alpha}O_{\mathcal{A}_2^\beta}$ is $2z\sqrt{ab}$. This proves (viii), since $z\sqrt{ab}$ is the radius of the Archimedean circles by (iv). The case $z = z_2$ is proved similarly. The rest of the case is proved by reflecting the two cases in the line AB. □

The part (v) of Theorem 11 shows that the smallest circles passing through each of the four points and touching AB are Archimedean circles of the tangent arbelos $(\alpha, \beta, \gamma_0)$. Let z and w be real numbers different from ± 1. The product of the radius of the circle γ_z and the common radius of the Archimedean circles of the skewed arbelos $(\alpha, \beta, \gamma_z)$ equals ab by (12). Therefore the circle γ_z has radius a if and only if the Archimedean circles of $(\alpha, \beta, \gamma_z)$ have radius b. Also γ_z has radius $a + b$ (resp. r_{A}) if and only if the Archimedean circles of $(\alpha, \beta, \gamma_z)$ have radius r_{A} (resp. $a + b$). The circle γ_z is congruent to the Archimedean circles of the skewed arbelos $(\alpha, \beta, \gamma_w)$ if and only if the Archimedean circles of $(\alpha, \beta, \gamma_z)$ are congruent to the circle γ_w. In this case $(\alpha, \beta, \gamma_z)$ and $(\alpha, \beta, \gamma_w)$ are said to be complement to each other. It is equivalent to

$$|1-z^2||1-w^2| = (a+b)/r_{\mathrm{A}}. \tag{15}$$

In the rest of this section the two lines $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ of the skewed arbelos $(\alpha, \beta, \gamma_z)$ are explicitly denoted by $\mathcal{T}_z^\alpha$ and $\mathcal{T}_z^\beta$, respectively. Since $z_1 z_2 = 1$ and $|1-z_i^2| r_{\mathrm{A}} = |z_i|\sqrt{ab}$ $(i=1,2)$ by (iv) of Theorem 11, (15) holds if $z = z_1$ and $w = z_2$. Hence the skewed arbeloi $(\alpha, \beta, \gamma_{z_1})$ and $(\alpha, \beta, \gamma_{z_2})$ are complement to each other. Indeed the Archimedean circle $\mathcal{A}_1^\alpha$ of $(\alpha, \beta, \gamma_{z_1})$ (resp. $(\alpha, \beta, \gamma_{z_2})$) coincides with the circle γ_{z_2} (resp. γ_{z_1}) (see Fig. 19). The lines $\mathcal{T}_{z_1}^\alpha$ and $\mathcal{T}_{z_2}^\alpha$ are perpendicular. The circle ζ_{z_2} (resp. ζ_{z_1}) is orthogonal to the circle γ_{z_1} (resp. γ_{z_2}). Similar properties also hold for the skewed arbeloi $(\alpha, \beta, \gamma_{-z_1})$ and $(\alpha, \beta, \gamma_{-z_2})$.

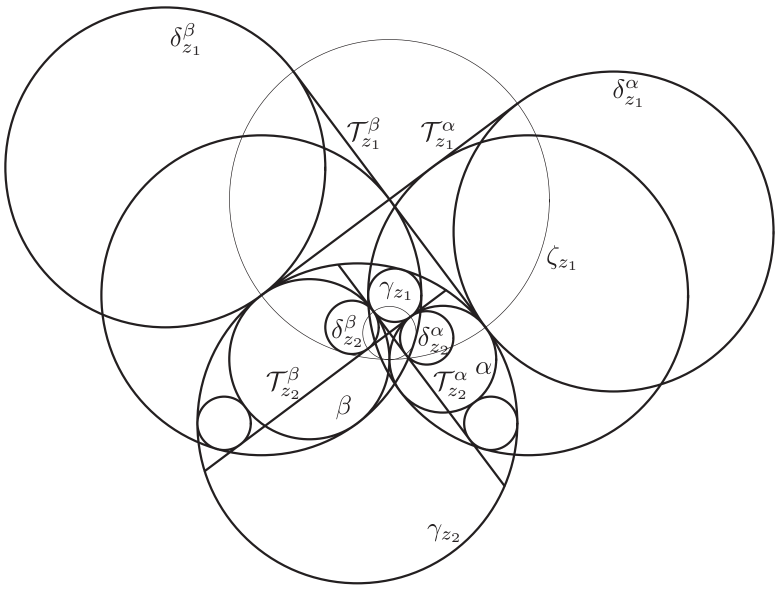

Figure 19: $(\alpha, \beta, \gamma_{z_1})$ and $(\alpha, \beta, \gamma_{z_2})$

3.4. Several Archimedean circles and Power type Archimedean circles

In this section we generalize the Archimedean circles of the tangent arbelos in 2.1, 2.3 and 2.4 to the skewed arbelos $(\alpha, \beta, \gamma_z)$. For a circle δ, the center of similitude of the circles α and δ is defined as the external (resp. internal) center of similitude of the two circles if $|z| < 1$ (resp. $|z| > 1$). The same notion of center of similitude applies to the circles β and δ. The circle W_8 in 2.1 is generalized as follows.

Theorem 12. For the skewed arbelos $(\alpha, \beta, \gamma_z)$ $(z \neq \pm 1)$, let δ be a circle with center O_z. The circle δ is Archimedean if and only if the center of similitude of the circles β and δ is the center of the circle $(A_z T_\beta)$.

Proof: We assume $|z| < 1$. If δ is Archimedean, the external center of similitude of β and δ divides the segment $O_\beta O_z$ externally in the ratio $b : (1-z^2) r_{\mathrm{A}}$. Therefore its coordinates are

$$\left(\frac{-(1-z^2) r_{\mathrm{A}}(-b)}{b-(1-z^2) r_{\mathrm{A}}}, \frac{bz\sqrt{ab}}{b-(1-z^2) r_{\mathrm{A}}} \right) = \left(\frac{ab(1-z^2)}{az^2+b}, \frac{(a+b)z\sqrt{ab}}{az^2+b} \right).$$

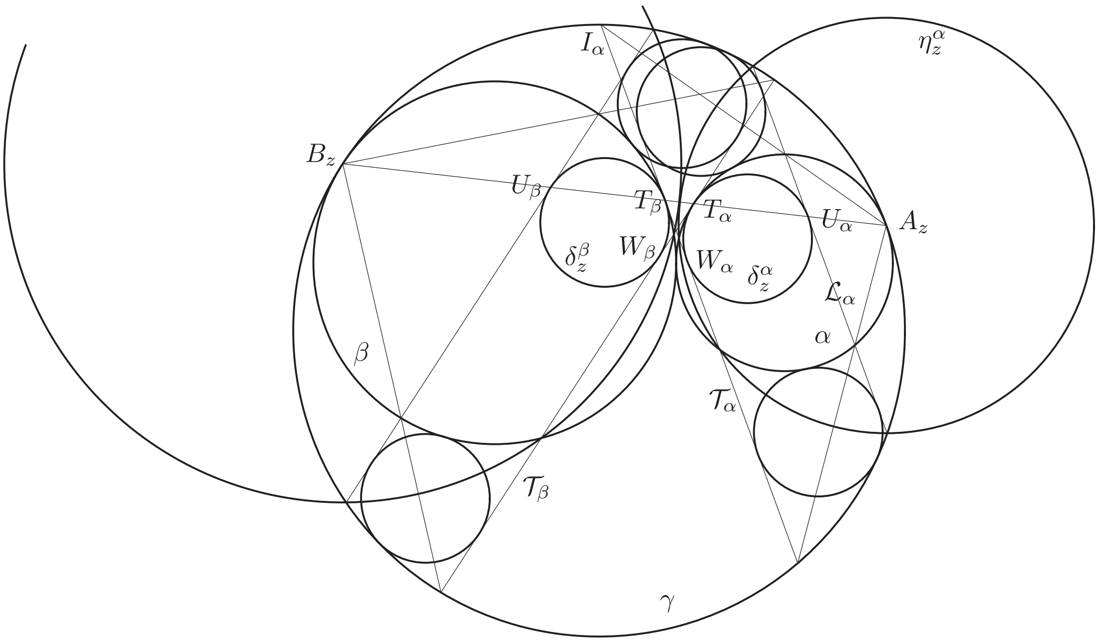

Figure 20: Generalizations of the circles W_9 and W_{13} for $|z| < 1$

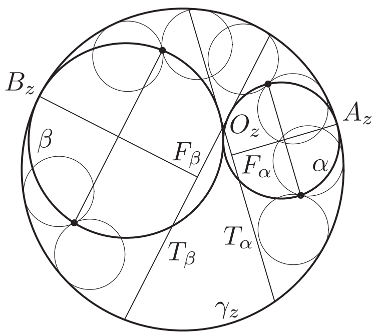

Figure 21: Power's Archimedean circles for $|z| < 1$

These are also the coordinates of the center of the circle (A_zT_β). Since the correspondence between δ and the external center of similitude of β and δ is one-to-one, the converse holds. The other case is proved similarly. □

The circle (A_zT_β) passes through the point O and has the radius $\sqrt{ab\,(a+bz^2)\,/(az^2+b)}$. In the limiting cases $z \to \pm 1$, the Archimedean circles obtained by Theorem 12 can be regarded as the points $(0, \pm\sqrt{ab})$.

Let us assume $|z| < 1$ (see Fig. 20). If β' is the remaining circle touching the circle γ_z and passing through the points of intersection of the circle α and the line $\mathcal{T}_\alpha$, then $(\alpha, \beta', \gamma_z)$ can be regarded as a collinear arbelos $(BPQA)$ with some notational changes, and the Archimedean circles of $(\alpha, \beta, \gamma_z)$ and $(\alpha, \beta', \gamma_z)$ are congruent. Hence some results of the collinear arbelos can be applied to $(\alpha, \beta, \gamma_z)$.

Since the circle δ_z^α touches $\mathcal{T}_\alpha$ at the point W_α, the point U_α is also the tangent point of δ_z^α and the remaining external common tangent of the two Archimedean circles touching α. Let $\mathcal{L}_\alpha$ be this tangent and let I_α be one of the points of intersection of γ_z and $\mathcal{T}_\alpha$. By (i) of Corollary 1, *the lines $\mathcal{L}_\alpha$, A_zI_α and the circle α intersect at a point.* Let η_z^α be the circle with center A_z and passing through the points of intersection of α and $\mathcal{T}_\alpha$. By (ii) of

Corollary 1, *the circle* η_z^α *passes through the points of intersection of* γ_z *and* $\mathcal{L}_\alpha$. The two facts are generalizations of the circles W_9 and W_{13} in 2.1 to the skewed arbelos.

Power's Archimedean circles in 2.3 are generalized to a skewed arbelos by Corollary 2: *If* $|z| < 1$, *the points of intersection of the circle* α *and the perpendicular bisector of the segment* A_zF_α *generate Archimedean circles with the circle* γ_z (see Fig. 21). Let G and H be as in the proof of Theorem 10. Bui's Archimedean circles in 2.3 are generalized by Corollary 3: *The points of intersection of the circle* $(A_zO_{(GH)})$ *and the line* $\mathcal{T}_\alpha$ *generate Archimedean circles with* γ_z *if* $|z| < 1$. Also Bui's Archimedean circles in 2.4 are generalized by (i) of Corollary 4 (see Fig. 20): *If* $|z| < 1$, *the circles* η_α, $H(I_\alpha)$ *and the line parallel to* $\mathcal{T}_\alpha$ *passing through the center of the circle* δ_z^α *belong to the same intersecting pencil of circles, and the points of intersection generate Archimedean circles with each of* γ_z *and* α.

3.5. Infinite Archimedean circles of the skewed arbelos

In this section we generalize the infinite Archimedean circles of the tangent arbelos in 2.2 to the skewed arbelos $(\alpha, \beta, \gamma_z)$. Let ϵ_z^α be the circle of radius az^2 with center $(az^2, 0)$. Also ϵ_z^β is the circle of radius bz^2 with center $(-bz^2, 0)$. For different points S and T, the circle (TS) is said to touch ϵ_z^α appropriately at T, if (TS) touches ϵ_z^α externally (resp. internally) at T when $0 < |z| < 1$ (resp. $|z| > 1$) and $T = O$ when $z = 0$ (see Figs. 22 and 23). For a point S_α on α, the vector $\overrightarrow{O_\alpha S_\alpha}$ is said to be parallel to $\overrightarrow{TS}$ appropriately, if the two vectors are parallel with the same (resp. opposite) direction if $|z| < 1$ (resp. $|z| > 1$). The same notions of appropriate tangency and appropriately parallel vector apply to the circles (TS) and ϵ_z^β and the vector $\overrightarrow{O_\beta S_\beta}$ for a point S_β on β, respectively.

Theorem 13. For a skewed arbelos $(\alpha, \beta, \gamma_z)$ $(z \neq \pm 1)$, let (TS) be a circle touching the circle ϵ_z^α appropriately at the point T, where it does not touch β internally. Let S_α be the center of similitude of the circles β and (TS). Then the circle (TS) is Archimedean if and only if the point S_α lies on the circle α and the vector $\overrightarrow{O_\alpha S_\alpha}$ is parallel to $\overrightarrow{TS}$ appropriately.

Proof: We assume $|z| < 1$. If the circle (TS) is Archimedean, its center is expressed by $(az^2 + l\cos\theta, l\sin\theta)$ for a real number θ, where $l = az^2 + (1 - z^2)r_\mathrm{A}$. Since the point S_α divides the segment $O_\beta O_\delta$ in the ratio $b : (1 - z^2)r_\mathrm{A}$ externally, it has the coordinates

$$\left(\frac{-(1 - z^2)r_\mathrm{A}(-b) + b(az^2 + l\cos\theta)}{b - (1 - z^2)r_\mathrm{A}},\ \frac{bl\sin\theta}{b - (1 - z^2)r_\mathrm{A}}\right) = (a(1 + \cos\theta),\ a\sin\theta)\,.$$

Hence S_α lies on the circle α and the vector $\overrightarrow{O_\alpha S_\alpha}$ is parallel to $\overrightarrow{TS}$ with the same direction. The converse holds, since the correspondence between (TS) and S_α is one-to-one. The other case is proved similarly. □

By the theorem, we get infinite Archimedean circles touching the circle ϵ_z^α. Exchanging the roles of the circles α and β, we get one more infinite set of Archimedean circles. In the limiting cases $z \to \pm 1$, the infinite Archimedean circles obtained by Theorem 13 can be regarded as the points on α except the point O.

Theorem 14. For a skewed arbelos $(\alpha, \beta, \gamma_z)$ $(z \neq \pm 1)$, let (TS) be a circle touching the circle ϵ_z^α appropriately at the point T. Let S_α and S_β be the points on the circles α and β, respectively, such that the vectors $\overrightarrow{O_\alpha S_\alpha}$ and $\overrightarrow{O_\beta S_\beta}$ are parallel to the vector $\overrightarrow{TS}$ appropriately.

(i) The circle (TS) is Archimedean if and only if the point S divides the segment $S_\alpha S_\beta$ in the ratio $a|1 - z^2| : (az^2 + b)$ internally in the case $|z| < 1$ and externally in the case $|z| > 1$.

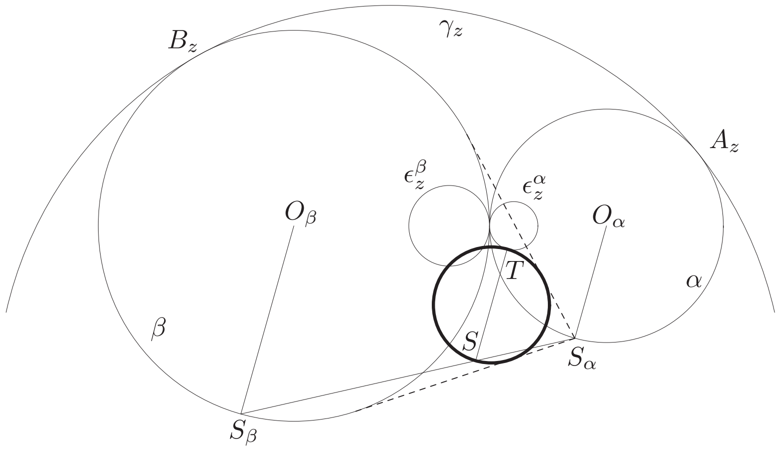

Figure 22: An Archimedean circle (TS) for $|z| < 1$

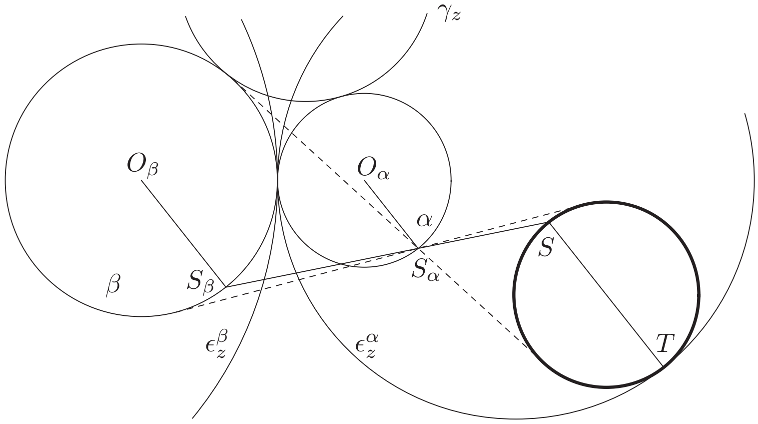

Figure 23: An Archimedean circle (TS) for $|z| > 1$

(ii) If S does not lie on the line AB, then (TS) is Archimedean if and only if the three points S_α, S_β and S are collinear.

Proof: We assume $|z| < 1$. Let $(a(1 + \cos\theta), a\sin\theta)$ and $(b(-1 + \cos\theta), b\sin\theta)$ be the coordinates of the points S_α and S_β for a real number θ, respectively. If the point S divides the segment $S_\alpha S_\beta$ internally in the ratio $a(1 - z^2) : (az^2 + b)$, its coordinates are

$$\begin{aligned}
&\left(\frac{(az^2+b)a(1+\cos\theta)+a(1-z^2)b(-1+\cos\theta)}{a(1-z^2)+az^2+b}, \frac{(az^2+b)a\sin\theta+a(1-z^2)b\sin\theta}{a(1-z^2)+az^2+b}\right)\\
=\ &(az^2+(az^2+2(1-z^2)r_{\mathrm{A}})\cos\theta, (az^2+2(1-z^2)r_{\mathrm{A}})\sin\theta).
\end{aligned}$$

Hence $|TS| = 2(1 - z^2)r_{\mathrm{A}}$, i.e., the circle (TS) is Archimedean. Conversely, if (TS) is Archimedean, let S' be the point dividing $S_\alpha S_\beta$ internally in the same ratio. Then $S' = S$ as just proved. The case $|z| > 1$ is proved similarly. The part (ii) follows from (i). □

The line $\mathcal{T}_\alpha$ touches the circle ϵ_z^α at the point W_α, which is the tangent point of $\mathcal{T}_\alpha$ and the circle δ_z^α. Therefore ϵ_z^α, δ_z^α and $\mathcal{T}_\alpha$ touch at this point. The circle ζ_z is also orthogonal to the circles ϵ_z^α and ϵ_z^β.

4. Conclusion

We have obtained several kinds of Archimedean circles of the collinear arbelos and the skewed arbelos by generalizing Archimedean circles of the ordinary arbelos. For the collinear arbelos (α, β, γ), the circles (AQ) and (BP) act like the circles α and β of the tangent arbelos, respectively. For the skewed arbelos $(\alpha, \beta, \gamma_z)$, the internal common tangents of the circles δ_z^α and δ_z^β correspond to the radical axis of the circles α and β of the tangent arbelos and play important roles. It seems that the Archimedean circles of those generalized arbeloi could be main topics on the generalized arbeloi as in the case of the tangent arbelos.

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