Journal for Geometry and Graphics
Volume 17 (2013), No. 1, 53–67.

Amicable Triangles and Perfect Circles

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Abstract. This is a contribution to triangle geometry. Two amicable triangles are inscribed in any circle which is related to a reference triangle ABC. Amicable triangles give rise to some family of circles – so-called perfect circles. In this way it is possible to generalize geometrical objects like the Soddy and Gergonne Line, the Gergonne Point, the Fletcher Point and the points of Eppstein, Griffith, Rigby and Nobbs as well. Amicable triangles and perfect circles have numerous and unusual interesting properties, and only a small part is presented in this article. Some of these results are still lacking of a rigorous mathematical proof; they only have been numerically confirmed.

Key Words: triangle geometry, amicable triangles, perfect circles, Soddy Line, Gergonne Point, Gergonne Line, Nobbs Points

MSC 2010: 51M04

1. Introduction

This paper presents generalizations of several well known geometric objects, whose special cases are described in numerous sources. Despite of an intensive literature recherche, to the author's best knowledge these generalizations seem to be new. This is why references to this article contain only four positions, which I mainly used.

All these considerations take their beginning from two triangles inscribed in any circle accompanying to the reference triangle ABC. These triangles, which I called *amicable triangles*, allowed me to define some family of circles – *perfect circles*. On their base it was possible to build generalizations of geometrical objects like the Soddy and Gergonne Line, the Gergonne Point, the Fletcher Point and the points of Eppstein, Griffith, Rigby and Nobbs as as well as the pair of other points, which do not exist for the incircle. Amicable triangles and perfect circles have numerous and unusual interesting properties. I have been working on this theme for ten years and I put it together on over 200 pages of my elaborations. This article shows only a small part out of these properties.

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2. Amicable Triangles

Given any reference triangle ABC. Let's join each of its vertices A, B and C with corresponding points S_0, T_0 and U_0 lying on the opposite sides. So we obtain the three cevians AS_0, BT_0 and CU_0. These cevians intersect at three points K_0, L_0 and M_0 forming a triangle (cevianic triangle), which in special cases according to Ceva's Theorem degenerates to a single point. Let's now circumscribe the circle $c(O_0, r_0)$ on the triangle $K_0L_0M_0$ (Fig. 1). Thereafter, let's modify the points S_0, T_0 and U_0 along the sides BC, AC and AB such that the circle $c(O_0, r_0)$ becomes the incircle of the triangle ABC. There are exactly two such triples of points S, T and U as well as S_1, T_1 and U_1 for which it is satisfied.

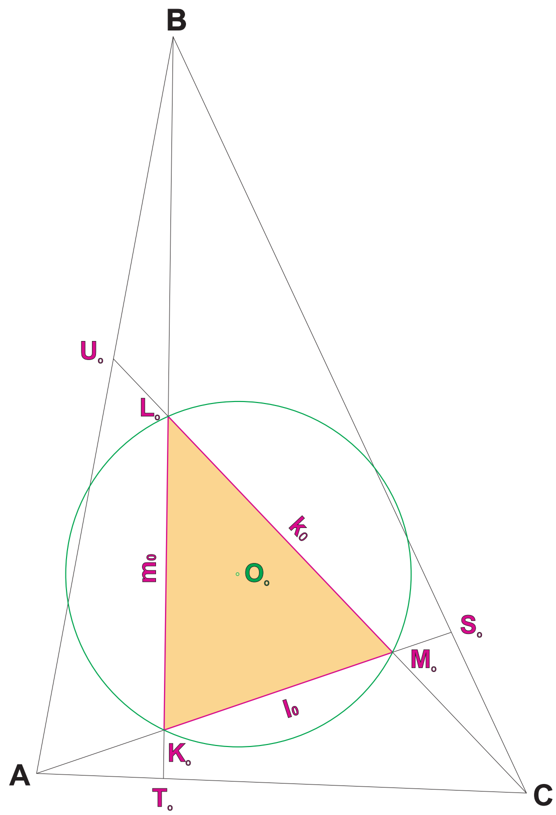

Figure 1: The Golden Theorem

When we set

$$
\begin{aligned}
s &= \frac{|BS|}{|SC|}, \quad & t &= \frac{|CT|}{|TA|}, \quad & u &= \frac{|AU|}{|UB|}, \\
s_1 &= \frac{|BS_1|}{|S_1C|}, \quad & t_1 &= \frac{|CT_1|}{|T_1A|}, \quad & u_1 &= \frac{|AU_1|}{|U_1B|},
\end{aligned}
\tag{1}
$$

and φ for the golden mean we obtain a theorem which I called the *Golden Theorem*.

Theorem 1. Golden Theorem: *The triangles KLM and $K_1L_1M_1$, which sides lie on six cevians passing pairwise through the vertices of the triangle ABC and meeting on the incircle of ABC, define ratios which satisfy*

$$
stu = \frac{1}{s_1 t_1 u_1} = \left(\frac{\pm 1 + \sqrt{5}}{2}\right)^6 = \varphi^{\pm 6}. \tag{2}
$$

The same formula holds also for any ellipse inscribed in the triangle ABC.

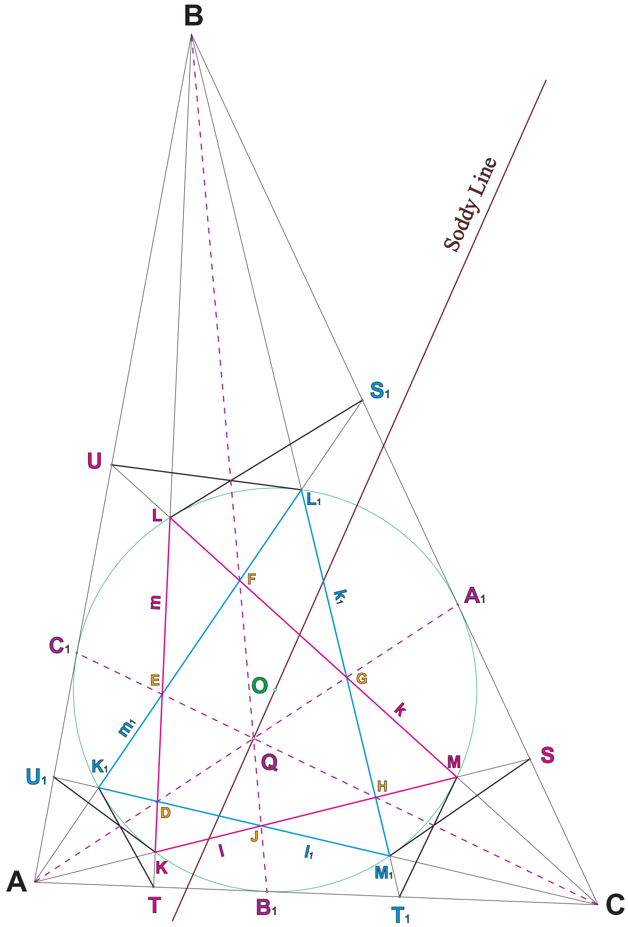

Figure 2: Amicable triangles in the incircle

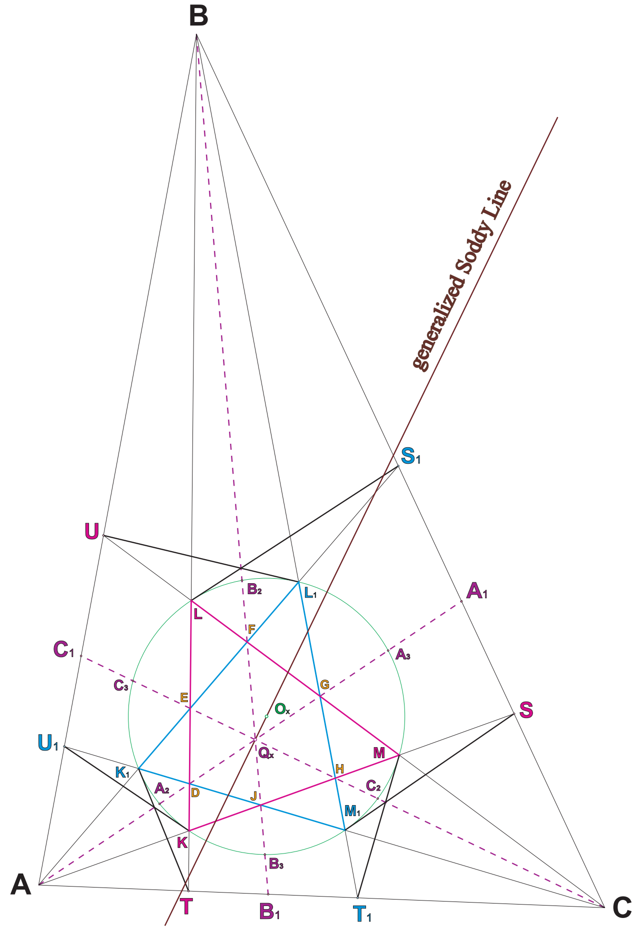

Figure 3: General amicable triangles

The last remark follows by applying an affine transformation.

The two triangles KLM and $K_1L_1M_1$ are called *amicable triangles*. They mutually intersect on the cevians AA_1, BB_1 and CC_1 called *main cevians*, which meet at the Gergonne Point Q (Fig. 2).

There exist at least two proofs of the Golden Theorem (2). One originates from Vladimir Shelomovskii [2] and another from the young Chinese mathematician Yuming Li. The second proof — yet unpublished — is based on Ptolemy's Theorem and on harmonic quadrilaterals.

Based on the Golden Theorem we get for s, t, u, s_1, t_1 and u_1 the following equations:

$$
\begin{aligned}
s &= \frac{3+\sqrt{5}}{2}\cdot\left(\frac{a-b+c}{a+b-c}\right) = \frac{3+\sqrt{5}}{2}\cdot\frac{r_B}{r_C}, & s_1 &= \frac{3-\sqrt{5}}{2}\cdot\left(\frac{a-b+c}{a+b-c}\right) = \frac{3-\sqrt{5}}{2}\cdot\frac{r_B}{r_C},\\
t &= \frac{3+\sqrt{5}}{2}\cdot\left(\frac{b-c+a}{b+c-a}\right) = \frac{3+\sqrt{5}}{2}\cdot\frac{r_C}{r_A}, & t_1 &= \frac{3-\sqrt{5}}{2}\cdot\left(\frac{b-c+a}{b+c-a}\right) = \frac{3-\sqrt{5}}{2}\cdot\frac{r_C}{r_A},\\
u &= \frac{3+\sqrt{5}}{2}\cdot\left(\frac{c-a+b}{c+a-b}\right) = \frac{3+\sqrt{5}}{2}\cdot\frac{r_A}{r_B}, & u_1 &= \frac{3-\sqrt{5}}{2}\cdot\left(\frac{c-a+b}{c+a-b}\right) = \frac{3-\sqrt{5}}{2}\cdot\frac{r_A}{r_B},
\end{aligned}
\tag{3}
$$

where r_A, r_B and r_c are the radii of three mutually tangent vertex circles of the triangle ABC.

We get then the coordinates of the points S, T, U, S_1, T_1, and U_1:

$$\begin{aligned}
\begin{pmatrix} x_S \\ y_S \end{pmatrix} &= \frac{1}{(5+\sqrt{5})a-(1+\sqrt{5})(b-c)}\left[2(a+b-c)\begin{pmatrix} x_B \\ y_B \end{pmatrix}+(3+\sqrt{5})(a-b+c)\begin{pmatrix} x_C \\ y_C \end{pmatrix}\right]\\
\begin{pmatrix} x_T \\ y_T \end{pmatrix} &= \frac{1}{(5+\sqrt{5})b-(1+\sqrt{5})(c-a)}\left[2(b+c-a)\begin{pmatrix} x_C \\ y_C \end{pmatrix}+(3+\sqrt{5})(a+b-c)\begin{pmatrix} x_A \\ y_A \end{pmatrix}\right]\\
\begin{pmatrix} x_U \\ y_U \end{pmatrix} &= \frac{1}{(5+\sqrt{5})c-(1+\sqrt{5})(a-b)}\left[2(a-b+c)\begin{pmatrix} x_A \\ y_A \end{pmatrix}+(3+\sqrt{5})(b+c-a)\begin{pmatrix} x_B \\ y_B \end{pmatrix}\right]
\end{aligned} \tag{4}$$

$$\begin{aligned}
\begin{pmatrix} x_{S_1} \\ y_{S_1} \end{pmatrix} &= \frac{1}{(5-\sqrt{5})a-(1-\sqrt{5})(b-c)}\left[2(a+b-c)\begin{pmatrix} x_B \\ y_B \end{pmatrix}+(3-\sqrt{5})(a-b+c)\begin{pmatrix} x_C \\ y_C \end{pmatrix}\right]\\
\begin{pmatrix} x_{T_1} \\ y_{T_1} \end{pmatrix} &= \frac{1}{(5-\sqrt{5})b-(1-\sqrt{5})(c-a)}\left[2(b+c-a)\begin{pmatrix} x_C \\ y_C \end{pmatrix}+(3-\sqrt{5})(a+b-c)\begin{pmatrix} x_A \\ y_A \end{pmatrix}\right]\\
\begin{pmatrix} x_{U_1} \\ y_{U_1} \end{pmatrix} &= \frac{1}{(5-\sqrt{5})c-(1-\sqrt{5})(a-b)}\left[2(a-b+c)\begin{pmatrix} x_A \\ y_A \end{pmatrix}+(3-\sqrt{5})(b+c-a)\begin{pmatrix} x_B \\ y_B \end{pmatrix}\right]
\end{aligned} \tag{5}$$

and resulting from above the coordinates of the points K, L, M, K_1, L_1, and M_1:

$$\begin{aligned}
\begin{pmatrix} x_K \\ y_K \end{pmatrix} &= \frac{1}{2\left[(7+3\sqrt{5})bc+(3+\sqrt{5})(ab-c^2)+(1+\sqrt{5})a^2(4+2\sqrt{5})b^2+2ac\right]}\cdot\\
&\left[(7+3\sqrt{5})\left[a^2-(b-c)^2\right]\begin{pmatrix} x_A \\ y_A \end{pmatrix}+2\left[b^2-(c-a)^2\right]\begin{pmatrix} x_B \\ y_B \end{pmatrix}+(3+\sqrt{5})\left[c^2-(a-b)^2\right]\begin{pmatrix} x_C \\ y_C \end{pmatrix}\right]\\
\begin{pmatrix} x_L \\ y_L \end{pmatrix} &= \frac{1}{2\left[(7+3\sqrt{5})ac+(3+\sqrt{5})(bc-a^2)+(1+\sqrt{5})b^2(4+2\sqrt{5})c^2+2ab\right]}\cdot\\
&\left[(7+3\sqrt{5})\left[b^2-(c-a)^2\right]\begin{pmatrix} x_B \\ y_B \end{pmatrix}+(3+\sqrt{5})\left[a^2-(b-c)^2\right]\begin{pmatrix} x_A \\ y_A \end{pmatrix}+2\left[c^2-(a-b)^2\right]\begin{pmatrix} x_C \\ y_C \end{pmatrix}\right]\\
\begin{pmatrix} x_M \\ y_M \end{pmatrix} &= \frac{1}{2\left[(7+3\sqrt{5})ab+(3+\sqrt{5})(ac-b^2)+(1+\sqrt{5})c^2(4+2\sqrt{5})a^2+2bc\right]}\cdot\\
&\left[(7+3\sqrt{5})\left[c^2-(a-b)^2\right]\begin{pmatrix} x_C \\ y_C \end{pmatrix}+(3+\sqrt{5})\left[b^2-(c-a)^2\right]\begin{pmatrix} x_B \\ y_B \end{pmatrix}+2\left[a^2-(b-c)^2\right]\begin{pmatrix} x_A \\ y_A \end{pmatrix}\right]
\end{aligned} \tag{6}$$

$$\begin{aligned}
\begin{pmatrix} x_{K_1} \\ y_{K_1} \end{pmatrix} &= \frac{1}{2\left[(7-3\sqrt{5})ab+(3-\sqrt{5})(ac-b^2)-(\sqrt{5}-1)c^2(4-2\sqrt{5})a^2+2bc\right]}\cdot\\
&\left[(7-3\sqrt{5})\left[c^2-(a-b)^2\right]\begin{pmatrix} x_C \\ y_C \end{pmatrix}+(3-\sqrt{5})\left[b^2-(c-a)^2\right]\begin{pmatrix} x_B \\ y_B \end{pmatrix}+2\left[a^2-(b-c)^2\right]\begin{pmatrix} x_A \\ y_A \end{pmatrix}\right]\\
\begin{pmatrix} x_{L_1} \\ x_{L_1} \end{pmatrix} &= \frac{1}{2\left[(7-3\sqrt{5})bc+(3-\sqrt{5})(ab-c^2)-(\sqrt{5}-1)a^2(4-2\sqrt{5})b^2+2ac\right]}\cdot\\
&\left[(7-3\sqrt{5})\left[a^2-(b-c)^2\right]\begin{pmatrix} x_A \\ y_A \end{pmatrix}+(3-\sqrt{5})\left[c^2-(a-b)^2\right]\begin{pmatrix} x_C \\ y_C \end{pmatrix}+2\left[b^2-(a-c)^2\right]\begin{pmatrix} x_B \\ y_B \end{pmatrix}\right]\\
\begin{pmatrix} x_{M_1} \\ x_{M_1} \end{pmatrix} &= \frac{1}{2\left[(7-3\sqrt{5})ac+(3-\sqrt{5})(bc-a^2)(4-2\sqrt{5})c^2-(\sqrt{5}-1)b^2+2ab\right]}\cdot\\
&\left[(7-3\sqrt{5})\left[b^2-(a-c)^2\right]\begin{pmatrix} x_B \\ y_B \end{pmatrix}+(3-\sqrt{5})\left[a^2-(b-c)^2\right]\begin{pmatrix} x_A \\ y_A \end{pmatrix}+2\left[c^2-(a-b)^2\right]\begin{pmatrix} x_C \\ y_C \end{pmatrix}\right]
\end{aligned} \tag{7}$$

The amicable triangles based on the incircle have many interesting properties. Here are two of them:

$$\frac{k^2+l^2+m^2}{P_{KLM}} = \frac{k_1^2+l_1^2+m_1^2}{P_{K_1L_1M_1}}, \qquad \frac{1}{k^2}+\frac{1}{l^2}+\frac{1}{m^2} = \frac{1}{k_1^2}+\frac{1}{l_1^2}+\frac{1}{m_1^2} \tag{8}$$

where k, l, m, k_1, l_1, m_1 are the respective lengths of the sides of the triangles KLM and $K_1L_1M_1$ while P_{KLM} and $P_{K_1L_1M_1}$ are the surface areas of these triangles. This result has been proved by V. Shelomovskii [2]. The above equations hold also for generalized amicable triangles.

The segments A_1S, A_1S_1, B_1T, B_1T_1, C_1U, and C_1U_1 have the following property — among numerous other properties:

$$\begin{gathered}\frac{1}{|A_1S|}+\frac{1}{|B_1T|}+\frac{1}{|C_1U|}=\frac{1}{|A_1S_1|}+\frac{1}{|B_1T_1|}+\frac{1}{|C_1U_1|}\\ \frac{1}{|A_1S|^2}+\frac{1}{|B_1T|^2}+\frac{1}{|C_1U|^2}=\frac{1}{|A_1S_1|^2}+\frac{1}{|B_1T_1|^2}+\frac{1}{|C_1U_1|^2}\end{gathered}\tag{9}$$

and resulting from these equations:

$$\frac{|A_1S|+|B_1T|+|C_1U|}{|A_1S|+|B_1T|+|C_1U|}=\frac{|A_1S|\cdot|B_1T|\cdot|C_1U|}{|A_1S|\cdot|B_1T|\cdot|C_1U|}\tag{10}$$

I called the pairs of triples of numbers satisfying above and similar equations *quadratic tertionals*. Quadratic tertionals are the form of multigrade equations with negative powers.

3. Perfect circles

The tangent lines to the circle $c(O,r)$ circle at the points K, L, M, K_1, L_1, and M_1 intersect the sides of the triangle ABC at the points U_1, S_1, T_1, T, U, and S, respectively. This property distinguishes the family of circles with radii $0 \le r_x \le R$, where R denotes the radius of the circumcircle of the triangle ABC (Fig. 4). I called these circles *perfect circles*. The smallest of them is the Fermat Point and the biggest one the circumcircle of ABC.

The incircle is obviously the perfect circle (Fig. 4). Based on these circles we can generalize the Soddy Line, Circles and Points, the Gergonne Point and Line, Griffiths, Rigby and Nobbs Points. We can also define the other points, which don't exist for the incircle of the triangle ABC.

The perfect circle can be also defined through following equations:

$$\frac{s}{s_1}=\frac{t}{t_1}=\frac{u}{u_1}=\Psi=(s\cdot t\cdot u)^{\frac{2}{3}}\tag{11}$$

The coordinates of the vertices of the both amicable triangles based on perfect circles can be presented using the cubic root of the product of stu and all three radii of the generalized vertex circles r_{A_p}, r_{B_p} and r_{C_p}. The cubic root of stu appears here in the first and second power.

The following equations have been calculated using the program Mathematica.

$$\begin{aligned}
\begin{pmatrix} x_K \\ y_K \end{pmatrix} &= \frac{1}{\sqrt[3]{(stu)^2}\, r_{B_p} r_{C_p} + r_{A_p} r_{C_p} + \sqrt[3]{stu}\, r_{A_p} r_{B_p}} \cdot \\
&\quad \left[\sqrt[3]{(stu)^2}\, r_{B_p} r_{C_p} \begin{pmatrix} x_A \\ y_A \end{pmatrix} + r_{A_p} r_{C_p} \begin{pmatrix} x_B \\ y_B \end{pmatrix} + \sqrt[3]{stu}\, r_{A_p} r_{B_p} \begin{pmatrix} x_C \\ y_C \end{pmatrix} \right] \\
\begin{pmatrix} x_L \\ y_L \end{pmatrix} &= \frac{1}{\sqrt[3]{(stu)^2}\, r_{A_p} r_{C_p} + r_{A_p} r_{B_p} + \sqrt[3]{stu}\, r_{B_p} r_{C_p}} \cdot \\
&\quad \left[\sqrt[3]{(stu)^2}\, r_{A_p} r_{C_p} \begin{pmatrix} x_B \\ y_B \end{pmatrix} + r_{A_p} r_{B_p} \begin{pmatrix} x_C \\ y_C \end{pmatrix} + \sqrt[3]{stu}\, r_{B_p} r_{C_p} \begin{pmatrix} x_A \\ y_A \end{pmatrix} \right] \\
\begin{pmatrix} x_M \\ y_M \end{pmatrix} &= \frac{1}{\sqrt[3]{(stu)^2}\, r_{A_p} r_{B_p} + r_{B_p} r_{C_p} + \sqrt[3]{stu}\, r_{A_p} r_{C_p}} \cdot \\
&\quad \left[\sqrt[3]{(stu)^2}\, r_{A_p} r_{B_p} \begin{pmatrix} x_C \\ y_C \end{pmatrix} + r_{B_p} r_{C_p} \begin{pmatrix} x_A \\ y_A \end{pmatrix} + \sqrt[3]{stu}\, r_{A_p} r_{C_p} \begin{pmatrix} x_B \\ y_B \end{pmatrix} \right]
\end{aligned} \tag{12}$$

$$\begin{aligned}
\begin{pmatrix} x_{K_1} \\ y_{K_1} \end{pmatrix} &= \frac{1}{\sqrt[3]{(stu)^{-2}}\, r_{A_p} r_{B_p} + r_{B_p} r_{C_p} + \sqrt[3]{(stu)^{-1}}\, r_{A_p} r_{C_p}} \cdot \\
&\quad \left[\sqrt[3]{(stu)^{-2}}\, r_{A_p} r_{B_p} \begin{pmatrix} x_C \\ y_C \end{pmatrix} + r_{B_p} r_{C_p} \begin{pmatrix} x_A \\ y_A \end{pmatrix} + \sqrt[3]{(stu)^{-1}}\, r_{A_p} r_{C_p} \begin{pmatrix} x_B \\ y_B \end{pmatrix} \right] \\
\begin{pmatrix} x_{L_1} \\ y_{L_1} \end{pmatrix} &= \frac{1}{\sqrt[3]{(stu)^{-2}}\, r_{B_p} r_{C_p} + r_{A_p} r_{C_p} + \sqrt[3]{(stu)^{-1}}\, r_{A_p} r_{B_p}} \cdot \\
&\quad \left[\sqrt[3]{(stu)^{-2}}\, r_{B_p} r_{C_p} \begin{pmatrix} x_A \\ y_A \end{pmatrix} + r_{A_p} r_{C_p} \begin{pmatrix} x_B \\ y_B \end{pmatrix} + \sqrt[3]{(stu)^{-1}}\, r_{A_p} r_{B_p} \begin{pmatrix} x_C \\ y_C \end{pmatrix} \right] \\
\begin{pmatrix} x_{M_1} \\ y_{M_1} \end{pmatrix} &= \frac{1}{\sqrt[3]{(stu)^{-2}}\, r_{A_p} r_{C_p} + r_{A_p} r_{B_p} + \sqrt[3]{(stu)^{-1}}\, r_{B_p} r_{C_p}} \cdot \\
&\quad \left[\sqrt[3]{(stu)^{-2}}\, r_{A_p} r_{C_p} \begin{pmatrix} x_B \\ y_B \end{pmatrix} + r_{A_p} r_{B_p} \begin{pmatrix} x_C \\ y_C \end{pmatrix} + \sqrt[3]{(stu)^{-1}}\, r_{B_p} r_{C_p} \begin{pmatrix} x_A \\ y_A \end{pmatrix} \right]
\end{aligned} \tag{13}$$

For each perfect circle the points S, T, U, S_1, T_1, and U_1 are located on the ellipse, whose foci lie on the generalized Soddy Line. Any line passing through the generalized Gergonne Point intersects this ellipse in two points. The both tangent lines to this ellipse in those points intersect on the generalized Gergonne Line.

The generalized Soddy Line is always perpendicular to the generalized Gergonne Line and cuts it at the generalized Fletcher Point.

Each perfect circle can still be defined in a different way – without being based on amicable triangles. Let's draw three pairs of cevians coming out of all three vertices of the triangle ABC and tangent to the circle $c(O,r)$ on the left and on the right. Let's denote the six tangents as follows: t_{A_l}, t_{A_r}, t_{B_l}, t_{B_r}, t_{C_l}, and t_{C_r}. These tangents intersect mutually in twelve points. On each cevian there are four points of intersection with other cevians, but only every second and third of these points are significant for us.

Let's denote these points (intersections) $I_{A_lB_r} = C_4$, $I_{A_lC_r} = B_5$, $I_{B_lC_r} = A_4$, $I_{B_lA_r} = C_5$, $I_{C_lA_r} = B_4$, and $I_{C_lB_r} = A_5$. The points $I_{C_lB_r}$ and $I_{B_lC_r}$ lie on the line α, the points $I_{A_lC_r}$ and $I_{C_lA_r}$ on the line β, and the points $I_{B_lA_r}$ and $I_{A_lB_r}$ on the line γ. Let's move now the circle $c(O,r)$ circle so that the lines α, β and γ pass through the vertices A, B and C, respectively (Fig. 5). We need only two coincidences of them. At this moment the circle $c(O,r)$ becomes a perfect circle $c(O_x,r_x)$. The lines α, β and γ meet at one point Q_x, the generalized Gergonne Point. The tangent lines t_{A_l}, t_{A_r}, t_{B_l}, t_{B_r}, t_{C_l}, and t_{C_r} passing through the vertices A, B and C are tangent to the perfect circle $c(O_x,r_x)$ at the points A_L, A_R, B_L, B_R, C_L, and C_R.

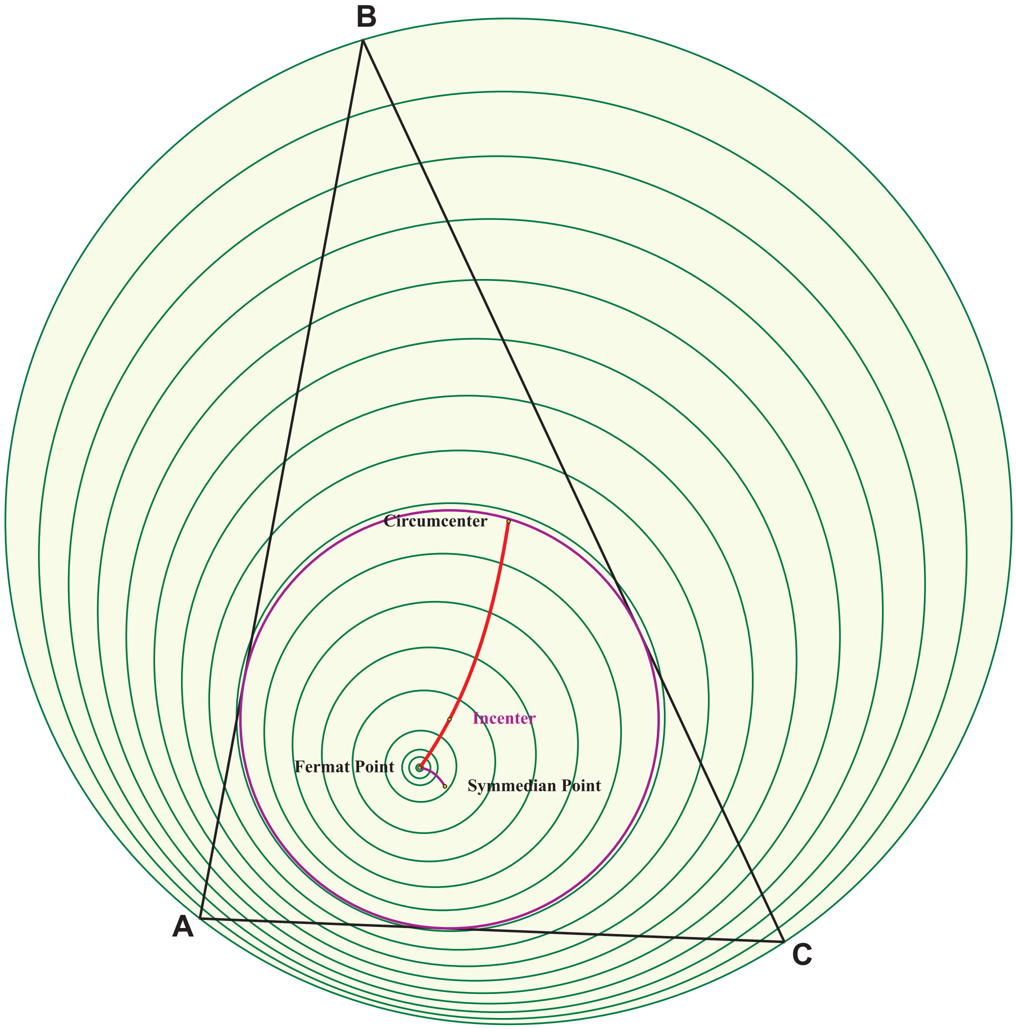

Figure 4: Locus of incenters of all perfect circles

So we obtain six triples of collinear points: $\{A_1, B_L, C_L\}$, $\{A_1, C_R, B_R\}$, $\{B_1, C_L, A_L\}$, $\{B_1, A_R, C_R\}$, $\{C_1, A_L, B_L\}$ and $\{C_1, B_R, A_R\}$ (Fig. 5). Denoting the lengths of the line segments connecting the vertices with the points of their tangency to the circle $c(O_x, r_x)$ by $r_{A_L} = r_{A_R} = r_{A_p}$, $r_{B_L} = r_{B_R} = r_{B_p}$, $r_{C_L} = r_{C_R} = r_{C_p}$ we get:

$$\begin{gathered} s\,\frac{r_{C_p}}{r_{B_p}} = t\,\frac{r_{A_p}}{r_{C_p}} = u\,\frac{r_{B_p}}{r_{A_p}} = \sqrt[3]{stu} \\ \frac{r_{B_p}}{r_{A_p}} = \sqrt{ss_1}, \quad \frac{r_{C_p}}{r_{A_p}} = \sqrt{tt_1}, \quad \frac{r_{A_p}}{r_{B_p}} = \sqrt{uu_1}. \end{gathered} \tag{14}$$

The trilinear coordinates of the generalized Gergonne Point Q_x are then:

$$\alpha : \beta : \gamma = \frac{bc}{r_{A_p}} : \frac{ca}{r_{B_p}} : \frac{ab}{r_{C_p}} \tag{15}$$

and using the Cartesian coordinates we get:

$$\begin{pmatrix} x_{Q_x} \\ y_{Q_x} \end{pmatrix} = \frac{1}{r_{B_p}r_{C_p} + r_{A_p}r_{C_p} + r_{A_p}r_{B_p}} \left[r_{B_p}r_{C_p}\begin{pmatrix} x_A \\ y_A \end{pmatrix} + r_{A_p}r_{C_p}\begin{pmatrix} x_B \\ y_B \end{pmatrix} + r_{A_p}r_{B_p}\begin{pmatrix} x_C \\ y_C \end{pmatrix} \right] \tag{16}$$

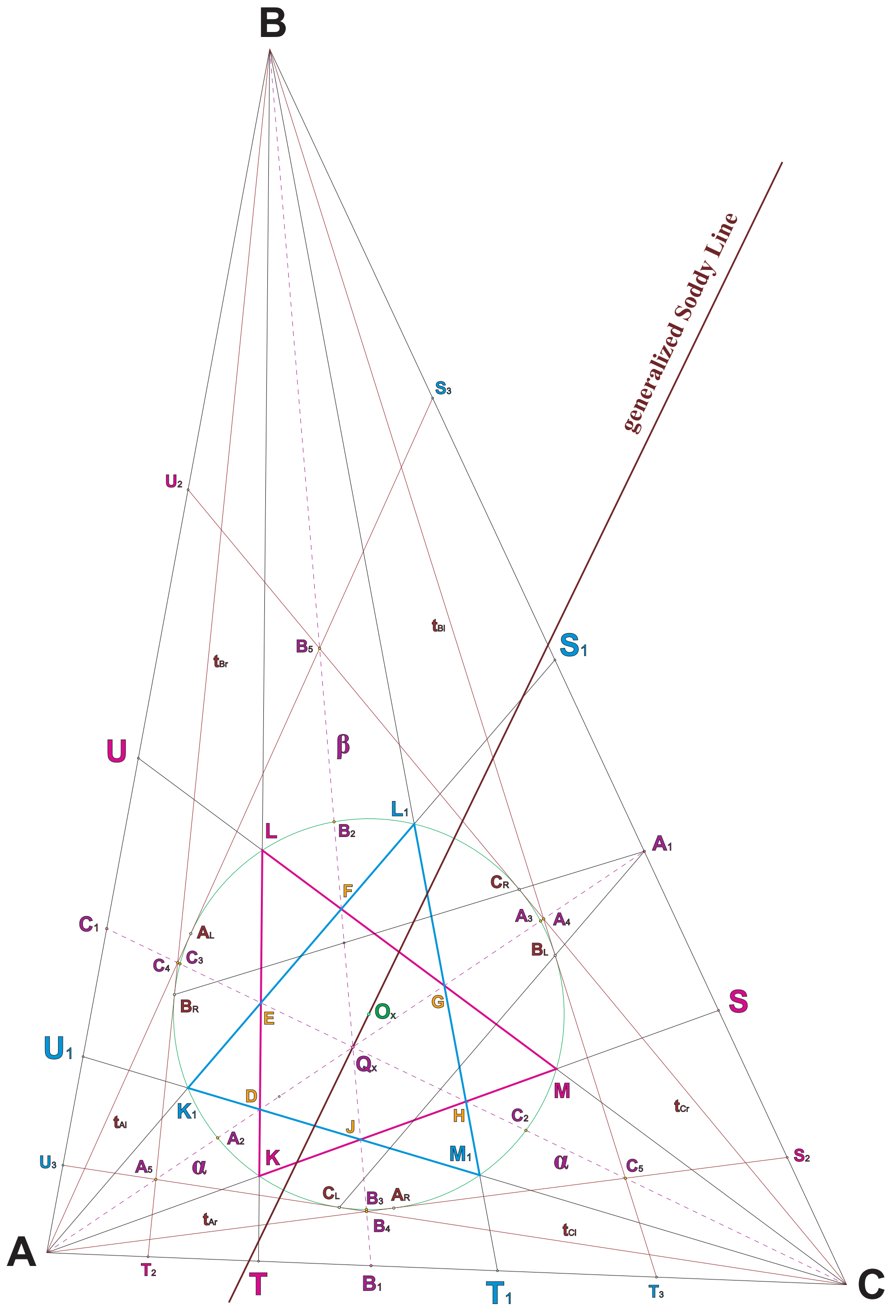

Figure 5: Alternate definition of perfect circles

The centers O_x of the perfect circles $c(O_x, r_x)$ belonging to the same family lie on the curve connecting the Fermat Point F_P ($r_x = 0$) with the circumcenter ($r_x = R$) (Fig. 5). The locus of the generalized Gergonne Points Q_x is the curve beginning at the Fermat Point and ending at the Symmedian Point S_P.

The lines α, β and γ intersect the circle $c(O_x, r_x)$ at two points each: α at A_2 and A_3, β at B_2 and B_3 and γ at C_2 and C_3. Let's define the following three cross ratios: $[A, A_2, A_3, A_1]_c$, $[B, B_2, B_3, B_1]_c$ and $[C, C_2, C_3, C_1]_c$ as $c_{rn} = |NN_3| \cdot |N_2N_1|/(|NN_2| \cdot |N_3N_1|)$. The values of these ratios are not only equal but they are also maximal from all values of cross ratios based on the circle $c(O_x, r_x)$ circle and on the sides a, b and c of the reference triangle ABC (Fig. 5). All other cross ratios in this paper are used in the same form as c_{rn}. It is worth noting, that for any triangle KLM (not necessarily inscribed in a perfect circle) we get:

$$[A, K, M, S]_c = [B, L, K, T]_c = [C, M, L, U]_c = stu. \tag{17}$$

Each perfect circle has an interesting property:

$$stu = \left(\frac{s}{s_1}\right)^{\frac{3}{2}} = \left(\frac{t}{t_1}\right)^{\frac{3}{2}} = \left(\frac{u}{u_1}\right)^{\frac{3}{2}}. \tag{18}$$

4. Further properties

Let's draw the ellipse e_A tangent to the sides b and c of the triangle ABC at the points T and U_1 and simultaneously passing through the point A_1. It turns out that the ellipse e_A is also tangent to the side a at this point. The remaining two ellipses e_B and e_C have the same property. However it happens only if the circle $c(O_x, r_x)$ circle is the perfect circle in the triangle ABC.

The tangent line t_{A_l} intersects the ellipses e_A, e_B and e_C at the points $\{A_{LA_1}, A_{LA_2}\}$, $\{A_{LB_1}, A_{LB_2}\}$ and $\{A_{LC_1}, A_{LC_2}\}$. We also get $\{A_{RA_1}, A_{RA_2}\}$, $\{A_{RB_1}, A_{RB_2}\}$ and $\{A_{RC_1}, A_{RC_2}\}$ for t_{A_r}. For both remaining vertices there are: $\{B_{LA_1}, B_{LA_2}\}$, $\{B_{LB_1}, B_{LB_2}\}$ and $\{B_{LC_1}, B_{LC_2}\}$ for t_{B_l}, $\{B_{RA_1}, B_{RA_2}\}$, $\{B_{RB_1}, B_{RB_2}\}$ and $\{B_{RC_1}, B_{RC_2}\}$ for t_{B_r}, $\{C_{LA_1}, C_{LA_2}\}$, $\{C_{LB_1}, C_{LB_2}\}$ and $\{C_{LC_1}, C_{LC_2}\}$ for t_{C_l} and $\{C_{RA_1}, C_{RA_2}\}$, $\{C_{RB_1}, C_{RB_2}\}$ and $\{C_{RC_1}, C_{RC_2}\}$ for t_{C_r}.

The points B_{RA_2} and B_{LA_2} lie on the line AS and the points C_{LA_2} and C_{RA_2} on the line AS_1. We find same properties for the lines BT, BT_1, CU and CU_1.
Among all circles $c(O_x, r_x)$ there are at least five, which can be defined as special cases:

- Fermat Point ($r_x = 0$, $stu = 1$),
- Circle I ($stu = 2 + \sqrt{5}$),
- Circle II ($stu = 8$),
- Incircle ($r_x = r$, $stu = \varphi^6$),
- Circumcircle ($r_x = R$, $stu = \infty$).

The Fermat Point, the incircle and circumcircle do not require any comment, but also the Circles I and II (Fig. 6) have very interesting properties. For the Circle I there are:

- the lines ST, TU and US are tangent to the circle $c(O_x, r_x)$ at the points M_1, K_1 and L_1;
- the lines S_1T_1, T_1U_1 and U_1S_1 are tangent to $c(O_x, r_x)$ at the points M, K and L;
- the quadruplets of points $\{A_1, M, M_1, B_1\}$, $\{B_1, K, K_1, C_1\}$, $\{C_1, L, L_1, A_1\}$ and the sextuplets of points $\{T, C_L, J, G, C_R, S_1\}$, $\{U, A_L, E, J, A_R, T_1\}$, $\{S, B_L, G, E, B_R, U_1\}$ are collinear;

and for the Circle II (Fig. 6):

- the lines SB_1, TC_1, UA_1, S_1C_1, T_1A_1 and U_1B_1 are tangent to the circle $c(O_x, r_x)$ at the points M_1, K_1, L_1, L, M and K;
- the cross ratios $[A, U_3, U_2, B]_c$, $[B, S_3, S_2, C]_c$, $[C, T_3, T_2, A]_c$ are all $17 + 12\sqrt{2}$;
- the ellipse e_A tangent to the triangle ABC triangle at the points T, U_1 and A_1 is tangent to the circle $c(O_x, r_x)$ circle at the points L and M_1;
- the ellipse e_B tangent to ABC at the points U, $S1$ and $B1$ is tangent to $c(O_x, r_x)$ at the points M and K_1;

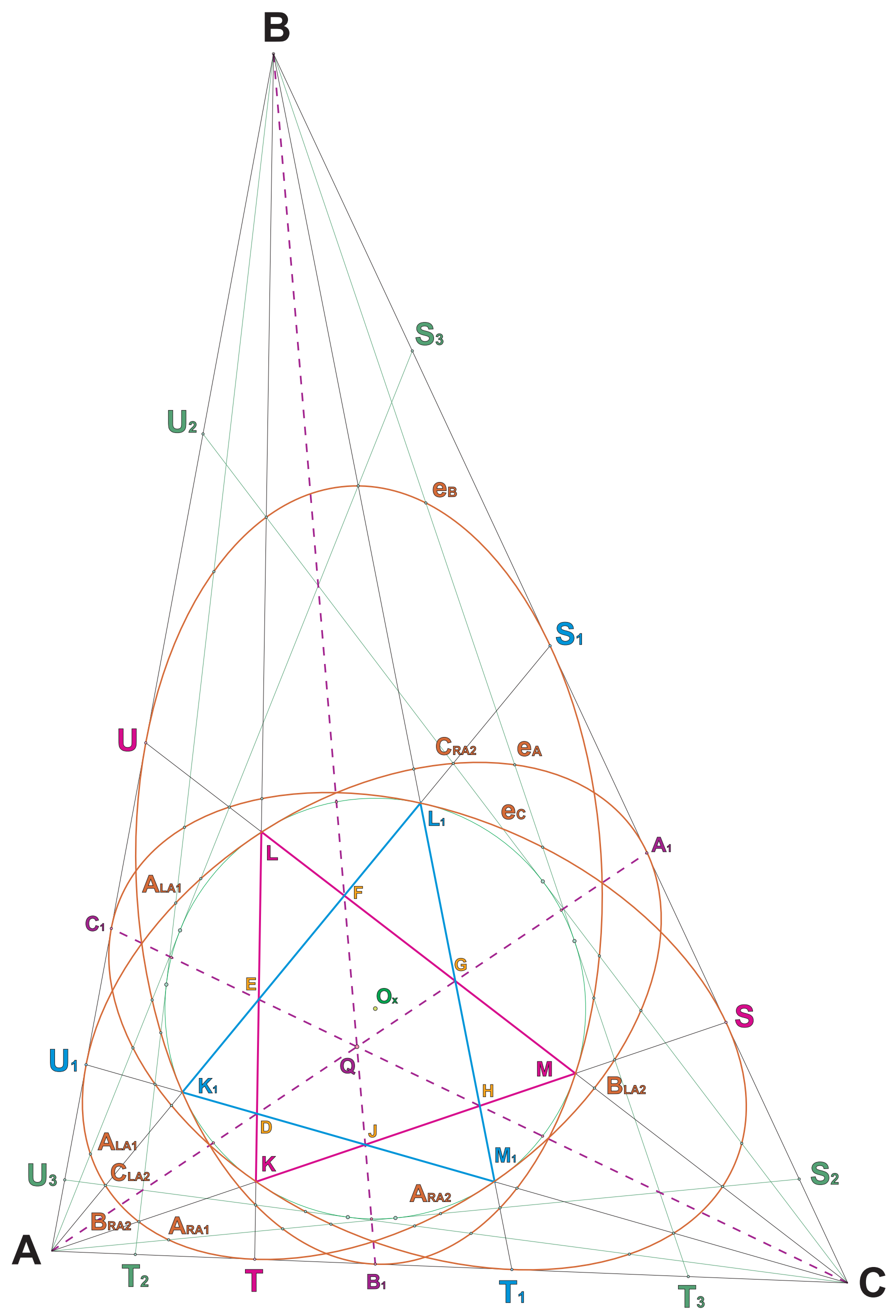

Figure 6: Special case of amicable triangles

- the ellipse e_C tangent to ABC at the points S, $T1$ and $C1$ is tangent to $c(O_x, r_x)$ at the points K and L_1;
- the cross ratios satisfy

$$[A, B_{RA_2}, B_{LA_2}, S]_c = [A, C_{LA_2}, C_{RA_2}, S1]_c = 17 + 12\sqrt{2}$$
$$= [A, A_{RA_1}, A_{RA_2}, S_2]_c^2 = [A, A_{LA_1}, A_{LA_2}, S_3]_c^2\,;$$

  similarly for the other vertices B and C;
- the ellipse tangent to the sides a, b and c at the points A_1, B_1 and C_1 is also tangent to the lines TU_1, US_1, ST_1 and passes through 15 other special points in the triangle ABC;

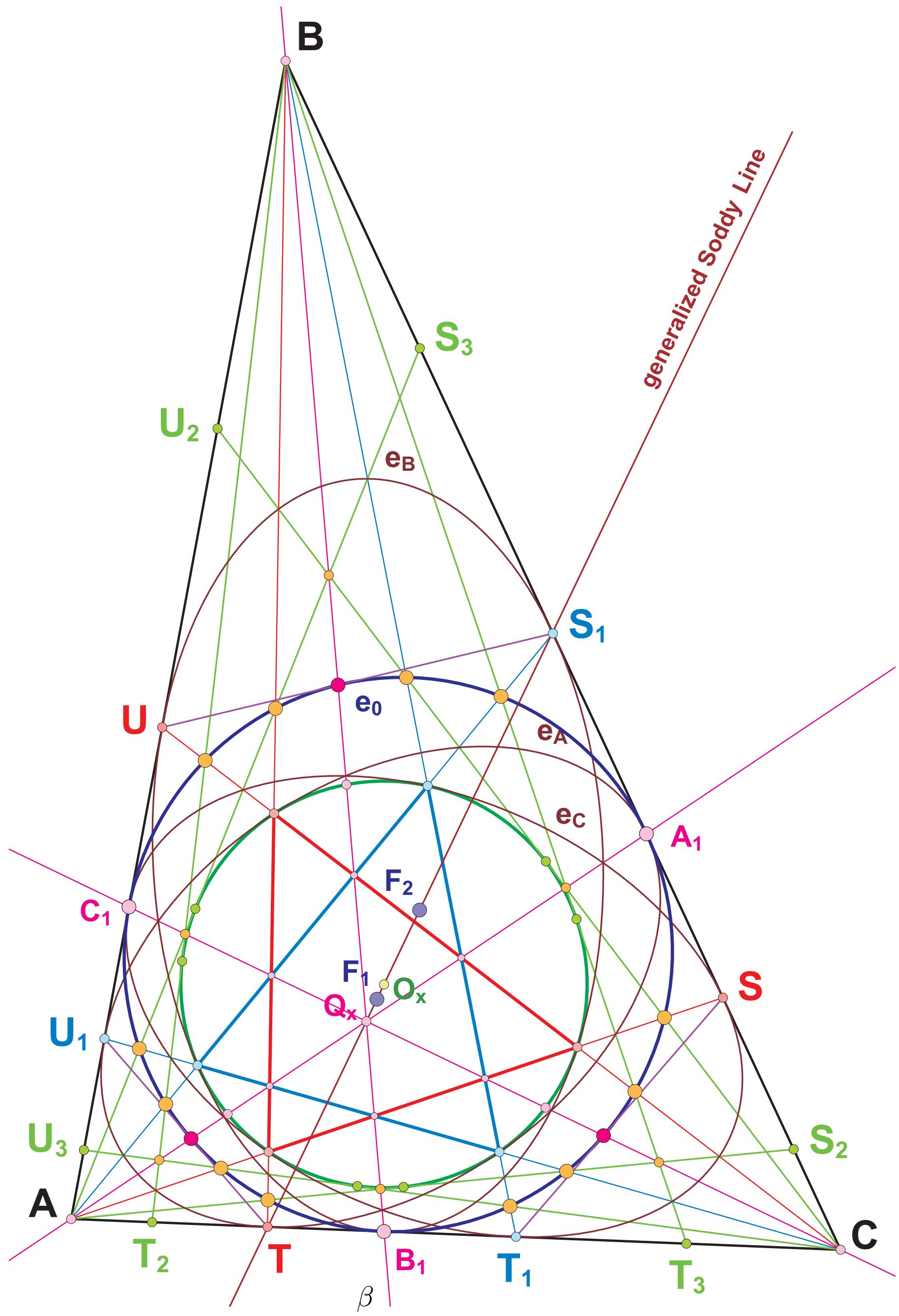

Figure 7: Special case of amicable triangles and their properties

the both foci F_1 and F_2 of this ellipse lie on the generalized Soddy Line and we get the following equation which holds for any perfect circle): $|O_xF_1|/|O_xF_2| = |QF_1|/|QF_2|$ (Fig. 7).

The Nobbs Points O_{A_1}, O_{B_1} and O_{C_1} are the points of convergence of three groups of the lines related to the incircle of the triangle ABC triangle and especially to three main cevians AA_1, BB_1 and CC_1. In case of incircle all three Nobbs Points are collinear and lie on the Gergonne Line. The first Nobbs Point O_{A_1} lies on the extension of the side a, the second O_{B_1} on the extension of the side b, and the last one O_{C_1} on the extension of the side c of the reference triangle ABC. Let's draw the midpoints of the segments ML_1, M_1L and KK_1

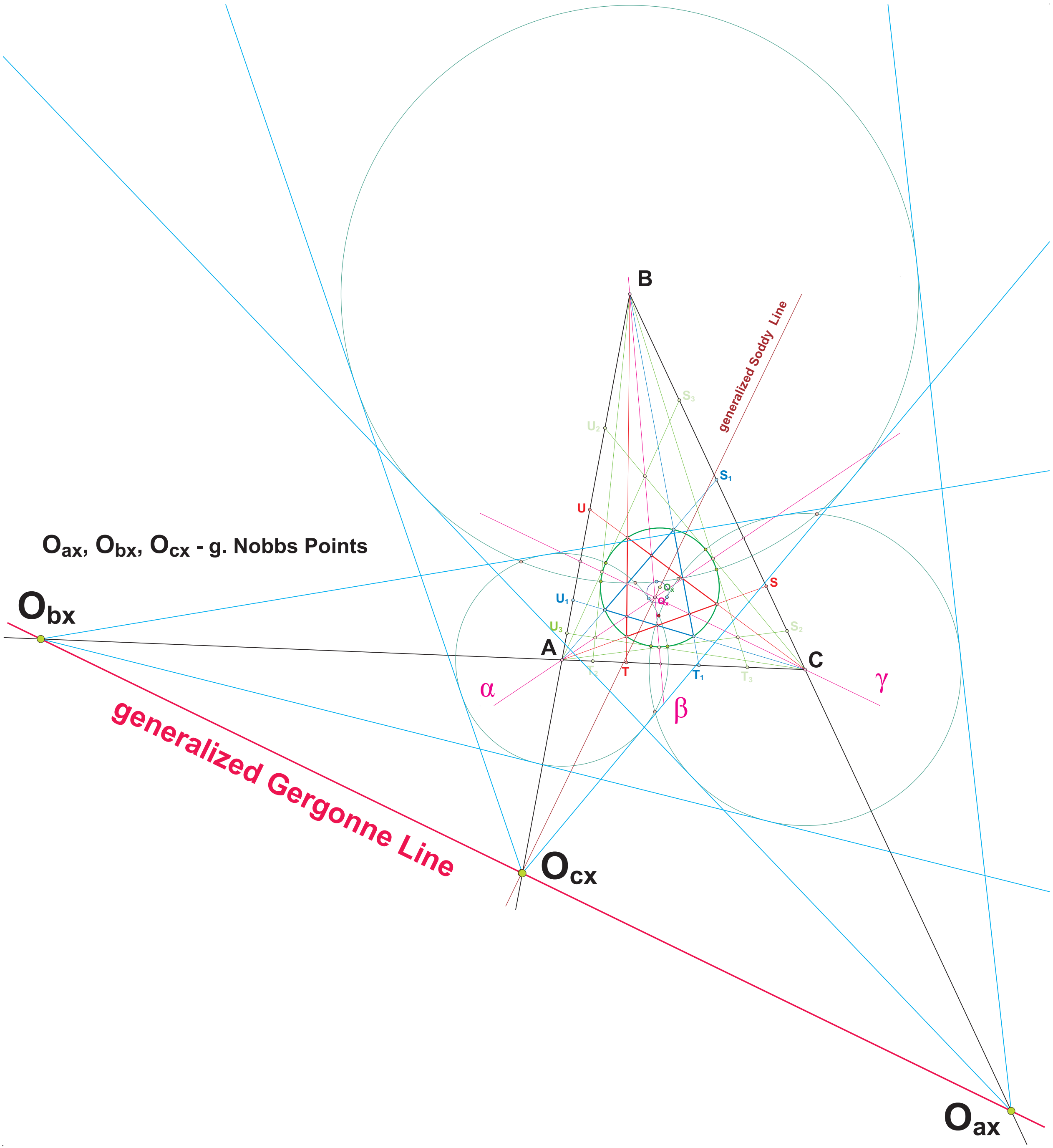

Figure 8: The vertical circles of perfect circles and common tangents

and call them V_{ML_1}, V_{M_1L} and V_{KK_1}. Now, let's draw the circle $c(V_{ML_1}, V_{M_1L}, V_{KK_1})$ passing through these points. It turns out, that this circle passes also through the incenter O and through the points A_1, A_2 and the Nobbs Point O_{A_1}, as well. These properties hold for each perfect circle. We get here the *generalized Nobbs Points.*

We ask now, where lies the center O_A of the circle $c(V_{ML_1}, V_{M_1L}, V_{KK_1})$? The center O_A is the midpoint of the line segment $O_{AS_1}O$. All three circles $c(V_{ML_1}, V_{M_1L}, V_{KK_1})$, $c(V_{KM_1}, V_{K_1M}, V_{LL_1})$ and $c(V_{LK_1}, V_{L_1K}, V_{MM_1})$ intersect at two points: at the incenter O and at the Fletcher Point Fl – the point of intersection of the Soddy Line and the Gergonne Line.

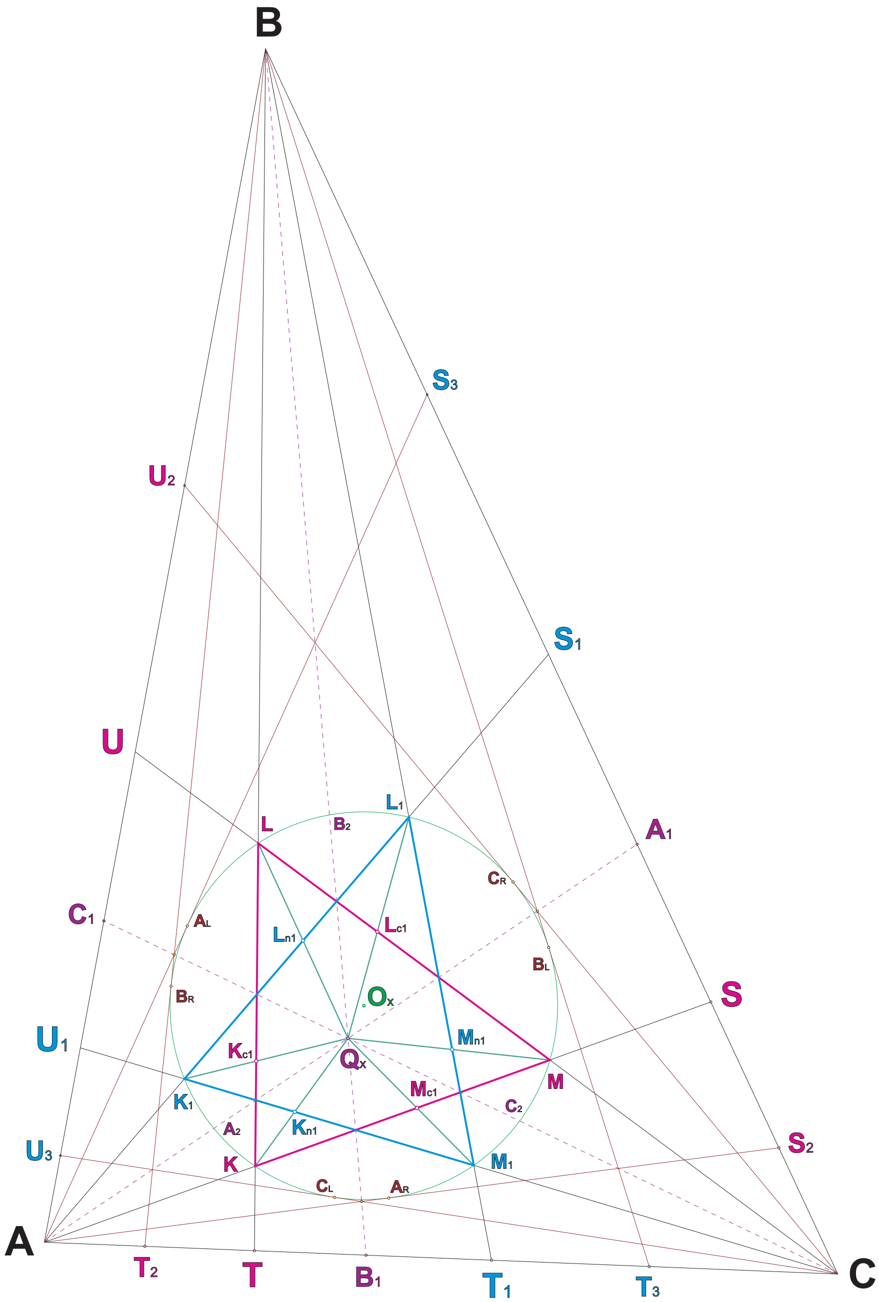

Figure 9: One of the various properties of perfect circles

All described above holds also for any perfect circle (Fig. 8), but with the points A_3, B_3 and C_3 instead of A_1, B_1 and C_1, respectively, which in case of the incircle cover mutually.

The lines tangent to each perfect circle at the points N_2 and N_3 intersect at the generalized Nobbs points O_{N_1} lying on the side n (or its extension) of the triangle ABC. For each (not necessarily perfect) circle we get the very important equation:

$$[O_{A_1}, C, A_1, B]_c = [O_{B_1}, A, B_1, C]_c = [O_{C_1}, B, C_1, B]_c = 2. \tag{19}$$

Both amicable triangles in each perfect circle have common Brocard Points, but the first Brocard Point for one of the triangles is the second Brocard Point for the other triangle.

Let's take the pair of amicable triangles (always based on any perfect circle) and let's connect the generalized Gergonne Point Q_x with all vertices of these triangles (Fig. 9). We get six line segments: QK, QL, QM, QK_1, QL_1 and QM_1. Each of them intersects one of the sides l_1, m_1, k_1, m, k and l at the respective points K_{n_1}, L_{n_1}, M_{n_1}, K_{c_1}, L_{c_1} and M_{c_1}. We obtain the following equation:

$$\begin{aligned}\frac{|QK_{n_1}|}{|K_{n_1}K|}+\frac{|QL_{n_1}|}{|L_{n_1}L|}+\frac{|QM_{n_1}|}{|M_{n_1}M|}&=\frac{|QK_{c_1}|}{|K_{c_1}K|}+\frac{|QL_{c_1}|}{|L_{c_1}L|}+\frac{|QM_{c_1}|}{|M_{c_1}M|}\\ =\sigma_Q(r_x)=\frac{s+s_1}{\sqrt{s+s_1}}&=\frac{t+t_1}{\sqrt{t+t_1}}=\frac{u+u_1}{\sqrt{u+u_1}}.\end{aligned} \tag{20}$$

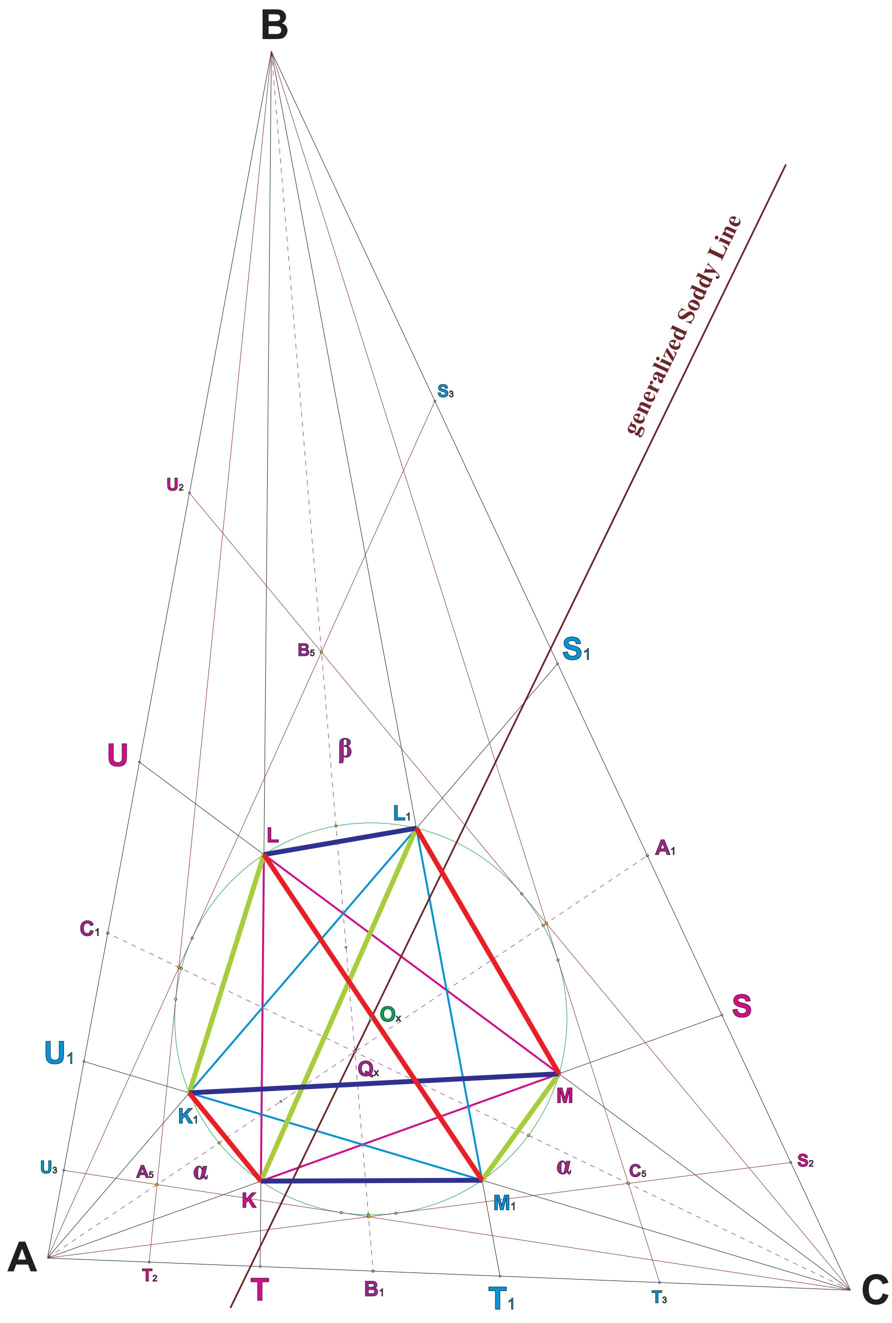

Figure 10: The cevianic triangle KLM and its 'amigo' $K_1L_1M_1$

The following three properties I'd like to present in this paper are based on the cevianic triangle KLM and its 'amigo' $K_1L_1M_1$ (see Fig. 10). The first property holds for each circle:

$$\frac{|AK|}{|AK_1|} \cdot \frac{|BL|}{|BL_1|} \cdot \frac{|CM|}{|CM_1|} = 1. \tag{21}$$

However, the second and third property are fulfilled only for perfect circles:

$$|KK_1| \cdot |M_1L| \cdot |ML_1| = |LL_1| \cdot |K_1M| \cdot |KM_1| = |MM_1| \cdot |L_1K| \cdot |LK_1|. \tag{22}$$

$$\frac{|KM_1|}{|KK_1|} \cdot \frac{|LK_1|}{|LL_1|} \cdot \frac{|ML_1|}{|MM_1|} = stu. \tag{23}$$

The perfect circles allow to generalize many points and lines related to the incircle. Under the generalizations almost all properties of these objects are preserved from the special case – also the modes of construction.

We conclude with two theorems:

Theorem 2. *The amicable triangles KLM and $K_1L_1M_1$ have a common Symmedian Point, which coincides with the generalized Gergonne Point Q_x of the reference triangle ABC.*

Theorem 3. *The amicable triangles KLM and $K_1L_1M_1$ share also other triangle centers (according to Clark Kimberling's Encyclopedia* [1]*, in particular the first isodynamic point X_{15}, the second isodynamic point X_{16}, the third power point X_{32}, the Brocard midpoint X_{39}, the first isogonal conjugate point X_{61}, the second isogonal conjugate point X_{62}, the midpoint X_{182} of the Brocard diameter X_3X_6, and the Schoute Center X_{187}.*

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[3] E.W. Weisstein: *CRC Concise Encyclopedia of Mathematics.* Chapman & Hall, 2003.

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Received August 3, 2012; final form March 18, 2013