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An Innovative Paradigm of Descriptive Geometry Courses

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Abstract. This paper is an introduction in a scientific and tutorial project that was created by the authors in order to change the existing paradigm of teaching Descriptive Geometry. The purpose of this project was to develop the theoretical principles of Descriptive Geometry. This improvement consists in the usage and integration of three mathematical components: the multidimensional geometry, the enumerative geometry and constructive methods. Due to this innovative paradigm it is possible to analyse the given data of a problem, to calculate the number of solutions end even to create a new research problem.

Key Words: dimension, geometric conditions, structural characteristics, innovation paradigm

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1. Introduction

There are many geometric results based on the well known Grassmann formula

$$D_n^m = (n - m) \cdot (m + 1), \tag{1}$$

giving the dimension of the m-dimensional subspaces (m-planes) in a n-dimensional space, $m < n$. Another well known formula

$$r = m + q - n \tag{2}$$

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gives the dimension r or the intersection between a m-plane and a q-plane within the n-space in the generic case.

However, classical Descriptive Geometry deals with the three-dimensional space, and here these formulas are simply unnecessary. This kind of limitation is unfounded. That is why we consider the classical Descriptive Geometry as a special case of a general course called *'Descriptive Geometry in Higher Dimensions'*.

In order to determine the basic elements of 'Descriptive Geometry in Higher Dimensions' and to avoid the numerous details connected with properties of lines, planes, 3-planes and so on, we consider the *geometric conditions* as a basis of our course. Any m-plane is in some relation with other subspaces. A relation of two subspaces is called the *geometric condition* if it reduces Grassmann's dimensions of both subspaces.

2. Geometric conditions

There are three kinds of geometric conditions,

1. the conditions of *parallelism or partial parallelism*,
2. the conditions of *orthogonality or partial orthogonality*, and
3. the general condition of *intersection or partial intersection*.

All these conditions may take analytic, constructive, descriptive and enumerative forms. The main task of our course is to show the constructive, descriptive and enumerative forms of conditions and their applications. Each condition has a dimension as its main numerical characteristic.

Taking into account the *degree of parallelism* between a m-plane and a q-plane according to

$$p_{\|} = \frac{r+1}{m}, \tag{3}$$

where r is the dimension of common subspace and $m \leq q$, we can calculate the dimension of the *condition of parallelism* by the formula

$$Q_{\|} = p_{\|} \cdot m \cdot (n - m - q + p_{\|} \cdot m), \tag{4}$$

where $p_{\|}$ is the degree of parallelism between the m-plane and the q-plane under $m \leq q$, and n is the dimension of the space, where the condition of parallelism is regarded.

If a m-plane is orthogonal or partial orthogonal to a q-plane then the *degree of orthogonality* is defined by

$$p_{\perp} = \frac{r+1}{m}, \tag{5}$$

where m is not larger than q. So we can calculate the dimension of the *condition of orthogonality* by

$$Q_{\perp} = p_{\perp} \cdot m \cdot (q - m + p_{\perp} \cdot m). \tag{6}$$

Since we want to consider the general condition of intersection or partial intersection, it is reasonable to discuss some elements of Hermann SCHUBERT's calculus (see, e.g., [1]). Any of SCHUBERT's varieties can be described by the symbol

$$e^{m,\ m-1,\ \dots,1,\ 0}_{a_m, a_{m-1}, \dots, a_1 a_0}, \tag{7}$$

where the upper indices $m, m-1, \dots, 1, 0$ mean the complete m-flag and the lower indices mean the incomplete a_m-flag. Every four indices for which

$$(m-i) + a_i \neq a_{m-i} + j \tag{8}$$

mean the condition of intersection. All such kinds of fours show us the general condition of intersection or partial intersection. The dimension of the condition may be calculated by the formula

$$Q_{ob} = \frac{(2 \cdot n - m)(m+1)}{2} - \sum_{i=0}^{m} a_i \,, \tag{9}$$

where n is the dimension of the space in which the incidence is considered, m is the dimension of the plane (element) satisfying the generalized condition of incidence, and the a_i's are the subscripts in the symbolic interpretation of the condition [3].

3. First principles of the course

We now want to discuss the following problem: Let a m-plane or a set of m-planes or a m-surface or a set of m-surfaces and so on be given in the n-space. These entities will be called *originals*. We want to find their correct images in a 2-plane or in subspaces of various dimensions under a given projection. The image is a set of points, lines, planes, curves, surfaces, and so on.

Now we formulate three basic statements.

Statement 1. The image of a given original is correct if and only if the set of images and the set of originals are of equal dimension.

Statement 2. The image of given original is correct if and only if the set of images and the set of originals are of equal structural characteristics.

Statement 1 is obvious. Let us consider Statement 2: The structure of the n-dimensional space is a set of subspaces together with a set of binary relations, for example, incidence relations. This structure is *linear* if the intersection of two subspaces is a subspace of the same set. For example, the structure of line spaces is nonlinear. They have a quadratic structure or a structure of higher degree.

The structure of the 2-dimensional image is a set of plane figures together with a set of binary relations. The structure of images is linear if the relations are expressed by linear functions.

If we are fully confident that Statements 1 and 2 are true we may formulate Statement 3.

Statement 3. We have the possibility to solve in the image space all problems which are formulated in the original space.

4. Examples

Taking a point as the primary object of the three-dimensional space, we can see that the dimension of the image set has to be three. The image set is a pencil of lines with two points on each line.

The pencil of lines is represented by the condition $e^{1,0}_{2,0}$, and the condition e^1_1 means that a point is on a line.

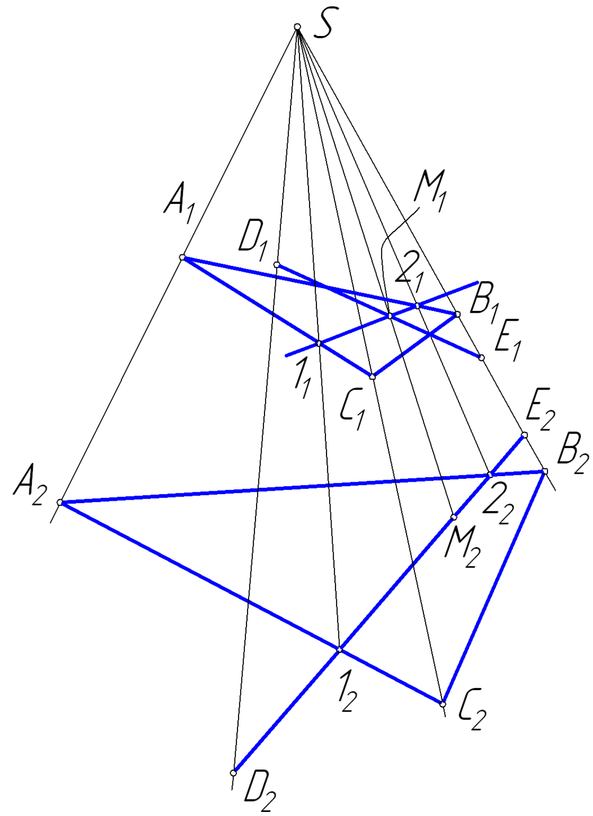

Figure 1: The intersection of the line DE with the plane ABC

Taking into account that $\dim\, e^{1,0}_{2,0} = \dim\, e^0_1 = 1$, we calculate the summary dimension

$$\dim\, e^{1,0}_{2,0} + \dim\, e^0_1 + \dim\, e^0_1 = 3.$$

Since the space is linear the image set has to be linear, too. We can prove it by means of the multiplication of conditions:

$$e^{1,0}_{2,0} \cdot e^{1,0}_{2,0} = e^{1,0}_{1,0};\; e^0_1 \cdot e^0_1 = e^0_0, \quad e^{1,0}_{2,0} \cdot e^0_1 = e^{1,0}_{1,0} + e^0_0 \tag{10}$$

where all factors are equal to one.

A basic problem in the three-dimensional space is to find a point M which lies in a given plane ABC and on a given line DE (see Fig. 1).

The next term of the proposed course is the algorithmization of the objects' construction. For example, a general algorithm for the construction of surfaces can be proposed as follows [5]:

1. We calculate the dimension of the general set of m-planes in the n-dimensional space;
2. We choose some of the geometric conditions:
   (a) conditions of incidence;
   (b) conditions of partial parallelity;
   (c) conditions of partial orthogonality.
3. We choose those conditions which can determine the constructed surfaces.

4. We calculate the structural characteristics of the constructed surfaces.
5. We consider all constructive methods of bulding the surfaces.

Let us illustrate the proposed algorithm by an example of constructing the hypersurface of lines in the four-dimensional space.

1. The dimension of the set of lines in a 4-plane is $D_4^1 = (4-1)\cdot(1+1) = 6$.
2. All conditions of incidence and their dimensions are

$$\begin{array}{lll} e_{4,2}^{1,0} - 1; & e_{4,1}^{1,0} - 2; & e_{4,0}^{1,0} - 3; \\ e_{3,2}^{1,0} - 2; & e_{3,1}^{1,0} - 3; & e_{3,0}^{1,0} - 4; \\ e_{2,1}^{1,0} - 4; & e_{2,0}^{1,0} - 5; & e_{1,0}^{1,0} - 6. \end{array}$$

   All conditions of partial parallelity are:
   - a 1-plane is parallel to a hyperplane,
   - a 1-plane is parallel to a 2-plane,
   - a 1-plane is parallel to 1-plane.
3. Since the hypersurface has a two-dimensional set of 1-planes, the total dimension of the conditions must be 4 .
4. We write some of the constructed hypersurfaces as products:

$$1.\ \left(e_{4,2}^{1,0}\right)^4; \quad 2.\ \left(e_{4,2,}^{1,0}\right)^2 \cdot e_{4,1}^{1,0}; \quad 3.\ \left(e_{4,2}^{1,0}\right)^2 e_{3,2}^{1,0}; \quad 4.\ \left(e_{4,2}^{1,0}\right)^2 e_{4,1}^{1,0}.$$

5. We define the structural characteristics of the hypersurface which are represented by $\left(e_{4,2}^{1,0}\right)^2 \cdot e_{4,1}^{1,0}$.

Theorem 1. *The order of the hypersurface* $\left(e_{4,2}^{1,0}\right)^2 \cdot \left(e_{4,1}^{1,0}\right)$ *is two and the class of it is one.*

em Proof: We have

$$\left(e_{4,2}^{1,0}\right)^2 \cdot e_{4,1}^{1,0} = \left(e_{4,1}^{1,0} + e_{3,2}^{1,0}\right) \cdot e_{4,1}^{1,0} = 2e_{3,0}^{1,0} + e_{2,1}^{1,0}.$$

The order is calculated by

$$\left(2e_{3,0}^{1,0} + e_{2,1}^{1,0}\right) \cdot e_{4,1}^{1,0} = 2e_{1,0}^{1,0} + 0 = 2e_{1,0}^{1,0}.$$

The class follows from

$$\left(2e_{3,0}^{1,0} + e_{2,1}^{1,0}\right) \cdot e_{3,2}^{1,0} = 0 + e_{1,0}^{1,0} = e_{1,0}^{1,0}.$$ □

5. Analysis of conditions

Let's consider the following problem: Find a plane which is

1. passing through a given point A;
2. orthogonal to a given plane $(a \bigcap b)$;
3. at a given distance R to a given point B.

The graphical form of these conditions is shown in Fig. 2. Find a plane which passes through the given point A and which is orthogonal to the given plane.

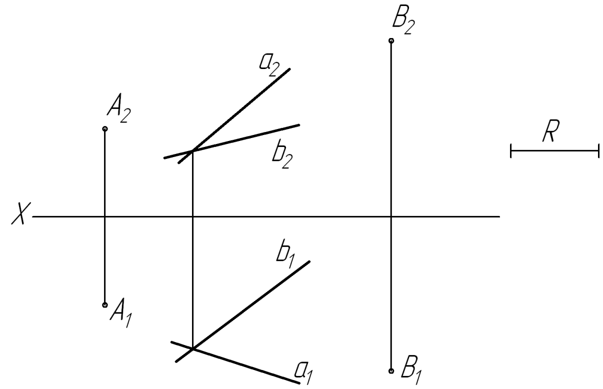

Figure 2: A graphical part of the condition of the problem

Analyzing the conditions of the problem

1. Calculation of the dimension of the required objects:
   The object of the problem is the plane. The general set of planes is three-dimensional. So, in order to get a solution it is necessary to impose conditions, where the total dimension is three.
2. Calculation of dimensions of the given geometric conditions:
   There are three conditions given:
   Condition U_1 is the condition of incidence of planes and a given point: $e^{2,1,0}_{3,2,0}$ is the set of planes passing through a point. Its dimension is $Q_{ob}\left(e^{2,1,0}_{3,2,0}\right) = 1$.
   Condition U_2 is the orthogonality of one given plane and the set. The dimension of this condition is 1. This condition can again be written as the condition of the set of planes passing through a given point: $\bar{e}^{2,1,0}_{3,2,0}$.
   Condition U_3 is condition of given distance between the plane and a given point: $2 \cdot e^{2,1,0}_{3,2,0}$ is the set of planes contacting the sphere with radius equal to the given distance. The dimension of this condition is $Q_{ob}\left(e^{2,1,0}_{3,2,0}\right) = 1$.
3. Testing the adequacy of the initial data or the accuracy of the specified conditions:
   On this stage it is necessary to compare the dimension of the requested element, and in our problem it is three (see item 1). On the other hand, the sum of dimensions of the specified conditions is again $1 + 1 + 1 = 3$. Hence, these dimensions are equal, and therefore all given conditions are sufficient for determining the required set.
4. Testing the conditions with the criterion of computability:
   If we consider the product of the given conditions and carry out their reduction, we'll get
   $$e^{2,1,0}_{3,2,0} \cdot \bar{e}^{2,1,0}_{3,2,0} \cdot 2 \cdot e^{2,1,0}_{3,2,0} = 2 \cdot \left(e^{2,1,0}_{3,2,0}\right)^3 = 2 \cdot e^{2,1,0}_{2,1,0}.$$
   It means that the conditions are compatible.
5. Calculation of the number of solutions or the dimension and algebraic properties of the required object:
   As a result, the process of reduction is $2 \cdot e^{2,1,0}_{2,1,0}$. The factor 2 before the symbolic designation means that the problem has two solutions, namely, two planes.
6. Implementation of the construction graph and mathematical algorithms for solving the problem, based on the analysis of the given conditions:
   This step allows us to find the optimal algorithm for solving the problem [4].

6. Algorithms for solving the problem

The first algorithm is $(U_1 \cap U_2) \cap U_3$

The composition of the conditions $e^{2,1,0}_{3,2,0} \cdot \bar{e}^{2,1,0}_{3,2,0} = \bar{e}^{2,1,0}_{3,1,0}$ means the set of planes passing through a given point A and being orthogonal to the given plane $a \bigcap b$. This set is the pencil of planes passing through a perpendicular to a given plane, and the perpendicular passes through the given point A.

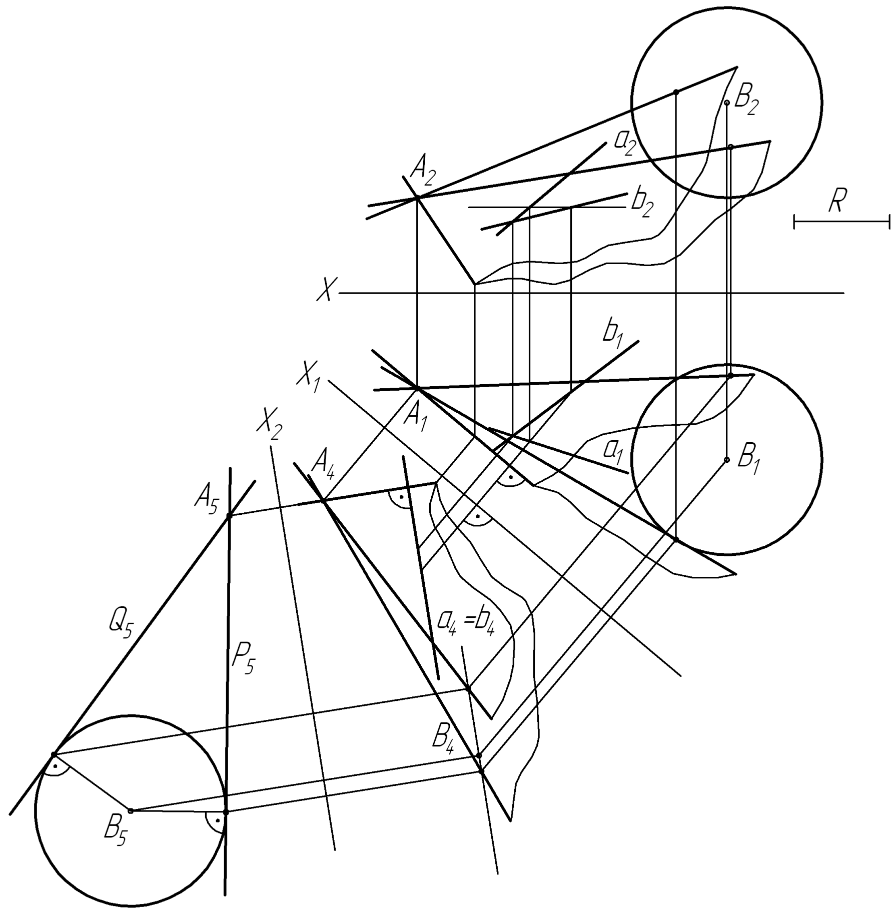

Figure 3: The solution of the problem according to the first algorithm

In the complex drawing displayed in Fig. 3 this stage is realized by building the perpendicular to the plane $a \bigcap b$ passing through the point A. The constructed straight line — a perpendicular — will be one of the lines that defines the required set. So it will be a common line of the two planes — the line of their intersection.

The second action $\bar{e}^{2,1,0}_{3,2,0} \cdot 2 \cdot^{2,1,0}_{3,2,0} = 2 \cdot e^{2,1,0}_{2,1,0}$ means the selection of two planes which are tangent to the sphere centered at point B, and the radius of the sphere equals the given distance R.

In order to fulfill the second action it is necessary to perform a replacement of the projection planes for the conversion of the previously constructed perpendicular into a projecting line. Two planes q and p which are tangent to the sphere (center B and radius R) are the requested ones.

The second algorithm $(U_1 \cap U_3) \cap U_2$

The first step is the simultaneous fulfillment of the conditions U_1 and U_3: $e^{2,1,0}_{3,2,0} \cdot e^{2,1,0}_{3,2,0} = 2 \cdot e^{2,1,0}_{3,1,0}$. The result of this action is the pencil of planes of second order with its own center, passing through the lines which are tangent to the sphere. It is sufficient to construct a complex of lines forming a conical surface. Moreover, each of these lines passes through point A and is tangent to the sphere (Fig. 4).

The second action assumes the composition of the conditions: $2 \cdot e^{2,1,0}_{3,1,0} \cdot \bar{e}^{2,1,0}_{3,2,0} = 2 \cdot e^{2,1,0}_{2,1,0}$. There are two planes of the beam passing through a perpendicular to the given plane $a \bigcap b$. In other words, it is necessary to draw the perpendicular to the plane $a \bigcap b$ through point A. Then we draw the auxiliary plane through the perpendicular which contacts the conical surface. The perpendicular itself and the lines of tangency on the auxiliary conical surface — two generators — will span the required two-planes.

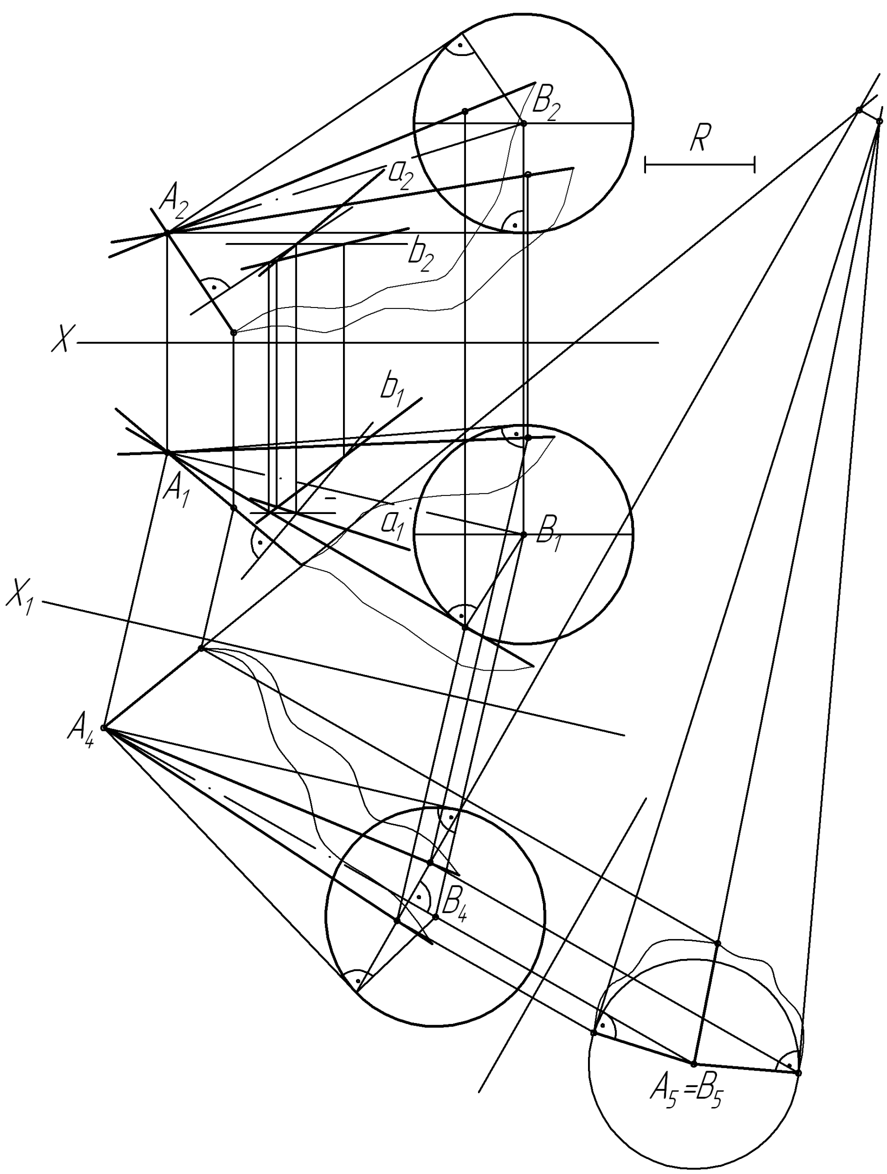

Figure 4: The solution of the problem according to the second algorithm

The third algorithm $(U_2 \cap U_3) \cap U_1$

The first step of the algorithm is $\bar{e}^{2,1,0}_{3,2,0} \cdot 2 \cdot e^{2,1,0}_{3,2,0} = 2 \cdot e^{2,1,0}_{3,1,0}$. This means a pencil of planes of second order with infinite center, passing through tangent lines of a sphere, thus forming a cylindrical surface. At this stage the construction (Fig. 5) of the beam is difficult. So it is enough to construct a cylindrical surface which is tangent to the sphere (center B and radius R) and in orthogonal position to the plane $a \bigcap b$.

The second step is $2 \cdot e^{2,1,0}_{3,1,0} \cdot e^{2,1,0}_{3,2,0} = 2 \cdot e^{2,1,0}_{2,1,0}$. It means the existence of two planes, selected from the constructed set by the condition of passing through the given point A. There are two planes passing through point A and being tangent to the cylindrical surface.

The analysis of the graphical solutions of the problem shows that the first algorithm $(U_1 \cap U_2) \cap U_3$ is optimal with respect to the simplicity of implementation in the graphic construction.

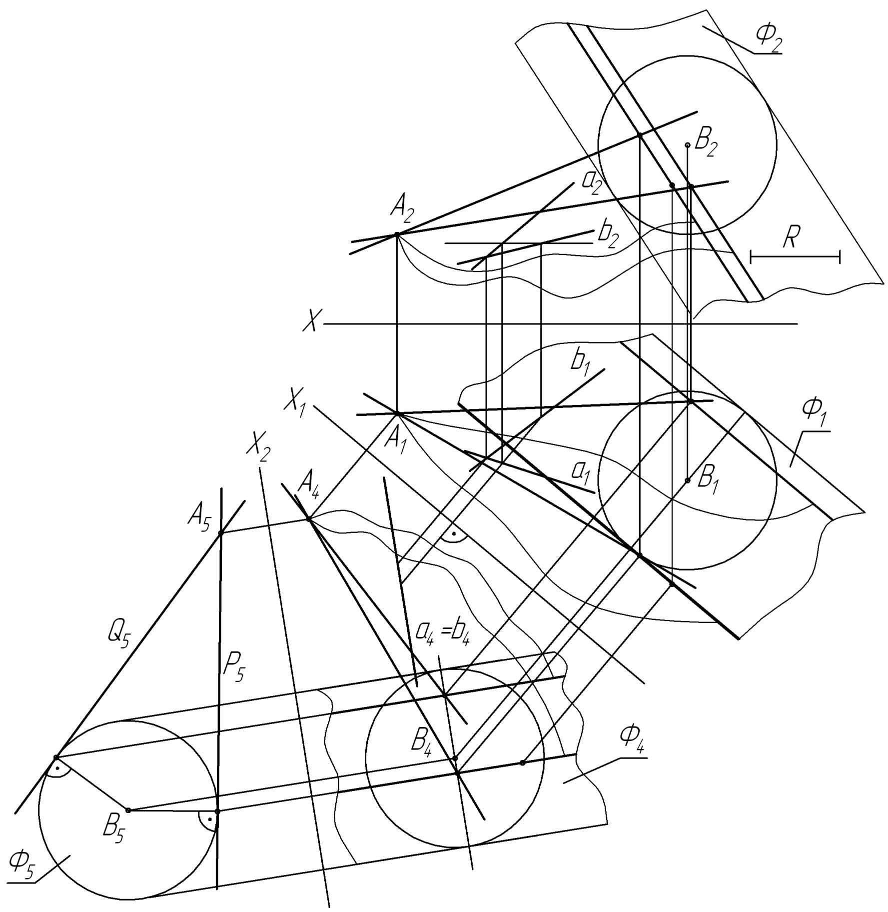

Figure 5: The solution of the problem according to the third algorithm

7. Conclusions

In this paper we presented some general aspects of how to solve geometric problems. We showed the determination of the problem's correctness and the test of conditions' compatibility.

Besides, each condition allows us to provide a subset of the set of required objects. Considering the intersection of these subsets, we can not only choose the optimal algorithm of solution, but we can also foster the students' competence in developing new properly formulated problems with necessary and sufficient conditions.

References

[1] H. Schubert: *Kalkül der abzählenden Geometrie*. B.G. Teubner, 1879; reprint Springer, Berlin, Heidelberg, New York 1979.

[2] V.Ya. Volkov, M.A. Chizhik: *Graphic optimization models of multivariate processes: a monograph* [in Russian]. Omsk Univ OmGIS, 2009, 101 p.

[3] V.Ya. Volkov, V.Yu. Yurkov: *Multivariate enumerative geometry: a monograph* [in Russian]. Omsk Univ OmPU, 2008, 244 p.

[4] V.Ya. Volkov: *The theory of the parameterization and modeling of geometric objects of multidimensional spaces and its applications*. Abstract of PhD theses for Dr. of technical. Science, Moscow Aviation Institute, 1983, 27 p.

[5] V.Ya. Volkov, V.Yu. Yurkov, K.L. Panchuk, N.V. Kaygorodtseva: *The course of descriptive geometry on the base of geometric modeling: a textbook* [in Russian]. Omsk Univ SibADI, 2010, 253 p.

[6] V.Ya. Volkov, V.Yu. Yurkov, K.L. Panchuk, N.V. Kaygorodtseva: *Collection of problems and exercises on descriptive geometry (to the textbook "The course of descriptive geometry on the base of geometric modeling")* [in Russian]. Omsk Univ SibADI, 2010, 73 p.

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