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Modular Pipe-Z System for Three-Dimensional Knots

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Abstract. This paper presents a modular Pipe-Z parametric design system which comprised of a single module allows the creation of complex three-dimensional single-branch structures, for example mathematical knots. The Pipe-Z module is introduced and its parametrization explained. An algorithm for the automated creation of any PZ structure by aligning the modules along a given spatial curve is introduced and the results for a trefoil, a figure-eight knot and a pentafoil are presented.

Key Words: Pipe-Z, modular system, parametric design

MSC 2010: 51M04, 51N05

1. Introduction

At the International Conference on Geometry and Graphics (ICGG 2012) in Montreal/Canada, a climbing frame, called *Krabbelknoten* [3], was presented. The idea, in a nutshell, was to create a physical object whose shape follows the spatial curvature of a trefoil knot. The purpose was educational, and it was addressed to children. The completed work is exhibited and used as shown in Fig. 1 at the so-called *Mathematics Adventure Land* in Dresden, Germany. The main issues described in that presentation were: safety, fabrication and installation of the physical object in the exhibition space. However, the effort, labor and logistics seemed enormous, so the question is natural whether such a task could have been completed in a much simpler way.

This paper goes further and considers a more general issue: is it possible to create any spatial knot by the assembly of a single modular unit ? In other words, how to combine the spatial complexity of the geometrical concept with the practicality of its physical fabrication ? Since 1990's the custom fabrication became increasingly more efficient, and there are cases where the economic advantage of modularity becomes negligible [5]. However, it is assumed that in general modular systems still have a practical, economical and in a certain sense an intellectual advantage over customization — unfortunately often at the expense of aesthetics.

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Figure 1: Children hopefully learning about mathematical topology by climbing through the *Krabbelknoten* — © Technische Sammlungen Dresden, Erlebnisland Mathematik

However, this paper hopefully demonstrates that it is possible to combine uniformity of the components with an overall result that is not trivial and acceptable also from the designer's perspective. For further discussion on free-form vs. modularity see [6].

2. Pipe-Z

Probably already for a very long time designers and artists have been experimenting with the creation of free-forms with modular elements. In the context of this paper, particularly interesting is the sculpture at the station *Metro da Sé* in São Paulo / Brazil, shown in Fig. 2.

Although the perfect modularity may not have been the actual goal of the artist, the units which comprise the sculpture are not the same. Congruent sectors of circular tori where used to create pipe-connections where the central curve has a constant curvature [1]. This paper goes a step further and introduces *Pipe-Z* (PZ) — a system capable of approximating practically any spatial curve with a single modular unit. A congeneric system — Truss-Z, which has been created for a specific practical task, namely the improvement of pedestrian safety, was presented in [6]. PZ is based on the following assumptions:

1. The entire structure consists of congruent units — so-called *Pipe-Z modules* (PZM).
2. For a given geometrical task both, PZM and the arrangement of PZMs, can be simultaneously optimized.

These above issues are briefly described in the following subsections.

Figure 2: *"Garatuja"* (Portuguese: "Scrawl") by Marcello Nitsche (1978). Steel sheets welded and painted. Dimensions: 3.35 × 3.83 × 4.44 m; weight: 3,000 kg.
The inset on the right © Hanneorla, `http://flickr.com/photos/hanneorla/sets`.

The module

PZM is a geometrical object analog to a sector of circular torus described in [1]. It is defined by three non-negative parameters $r, \rho, \zeta \in \mathbb{R}$ which denote the radius, the corresponding radius and the central angle, respectively (see Fig. 3).

PZMs are terminated by two faces T and B, corresponding to the top and the bottom of a unit. Although they need not be congruent in principle, for practicality, however, it is desirable that PZM is symmetrical about the plane perpendicular to its axis. Such a condition implicates that T and B are congruent, and their relative position is controlled by r, ρ and ζ. The faces of T and B can have shapes of circles or of regular polygons with an arbitrary number n of sides. Polygonal faces seem easier to fabricate and assemble. In such a case the number n of sides, $n \in \mathbb{N}$, becomes an additional parameter, which, however, is set arbitrarily and therefore not subject to optimization. In the further examples n is set to 12, so T and B are regular dodecagons (12-gons).

It is also convenient to introduce a new parameter $s = \rho/r$ with $s \in (0, \infty)$. Therefore, r is a global parameter relating the size of PZM to the size of the geometrical environment, and s is the relative parameter defining the "slenderness" of PZMs (see Fig. 4).

A *PZ structure* is constructed by assembling PZMs in a sequence such that the top face of the previous unit becomes the base for the next one. The successive unit i is rotated by a twist angle κ_i, which can have real or discrete values. In the latter case such rotations are denoted by k_i. In all further examples PZMs are based on dodecagons; therefore the subsequent unit can be added at twelve rational (dihedral) angles, so that the facets of adjacent units coincide — as shown in Fig. 3.

An entire PZ structure is encoded as $PZ = \{\{n, r, s, \zeta\}, V_s, L\}$, where n, r, s, and ζ are the PZM parameters; V_s is the initial vector which positions the first unit in space, and L is the sequence of dihedral twist angles k_i, where i is the index of the i^{th} unit (Fig. 4).

In order to allow a more systematical experimentation for larger problems, a simple algorithm as described below was implemented which aligns PZMs along a given spatial curve.

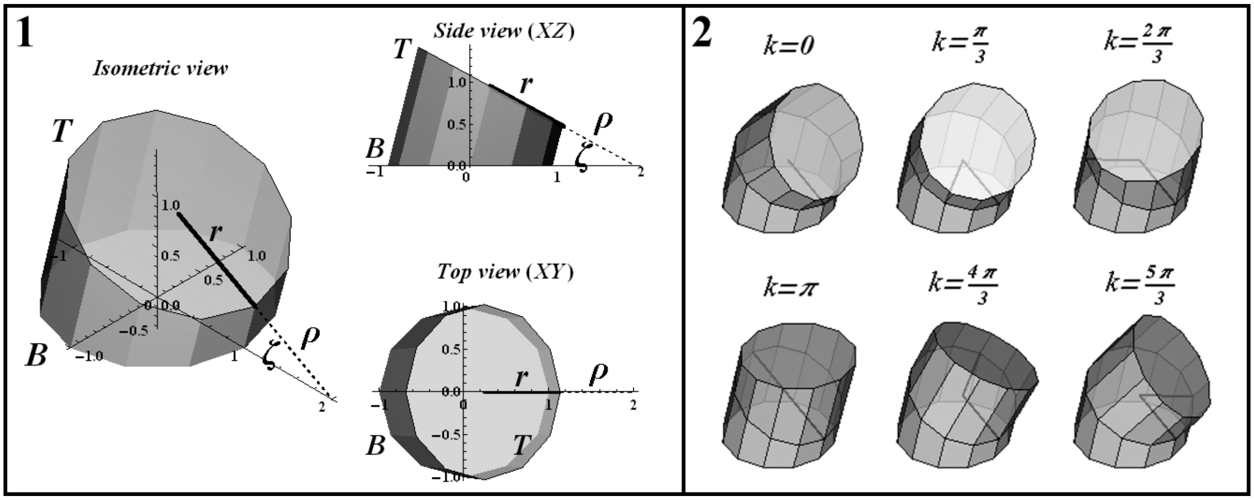

Figure 3: 1. A visualization of PZM which is defined by the parameters r, ρ, ζ, and n. In the depicted case $n=12$ the top and bottom faces are dodecagons. 2. Two PZMs can be connected at six out of twelve possible dihedral rotations of the twist angle k.

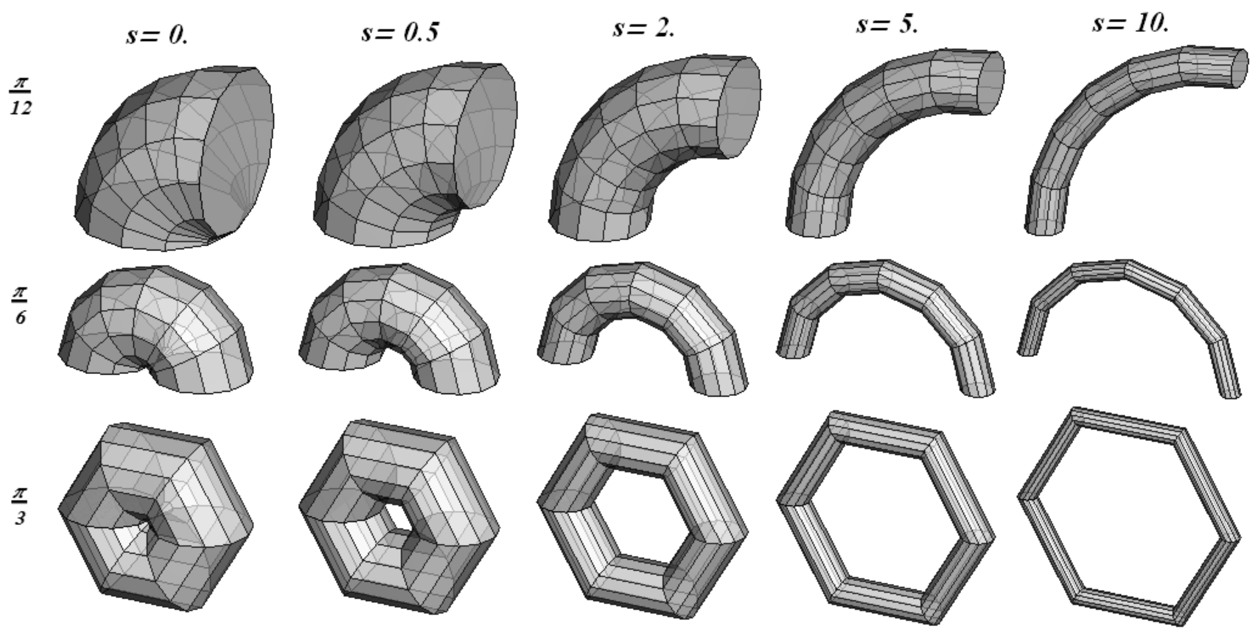

Figure 4: Variety of PZ shapes resulting from the same sequence of six units (initial unit + five subsequent units) with $k_1 = k_2 = \cdots = k_5 = 0$ at the same value of $r = 1$ and by increasing values of the parameters s and ζ. The images are zoomed-to-fit.

3. Alignment of units along a guide path

The units of a PZ structure are added piece by piece. Before the i^{th} PZM is added, k_i can be optimized such that the distance δ_i between the center of the top face of the new unit $C(T_i)$ to the given spatial curve, called the *guide path* (GP) is minimal, as shown in Fig. 5.

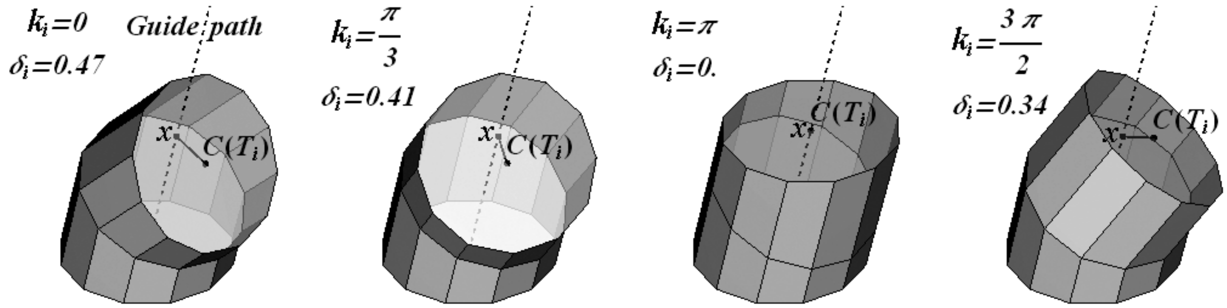

Figure 5: Selected values of k_i and resulting δ_i. The closest points on GP to $C(T_i)$ is indicated by x. In these examples the optimal value of k_i is π, since it gives the minimal value of δ_i.

3.1. The algorithm

Since the search space is discrete due to dihedral twist angles, a simple "greedy" algorithm based on breadth-first search (BFS) [2] was implemented. Thus when an i^{th} unit is added, all twelve values of twist angle are provided ($k_i^1 = 0,\ k_i^2 = \pi/6,\ \ldots,\ k_i^{12} = 2\pi$), and the one that returns the minimal δ_i is selected. The algorithm takes the PZM parameters and the GP and returns the entire PZ structure as shown in Fig. 6.

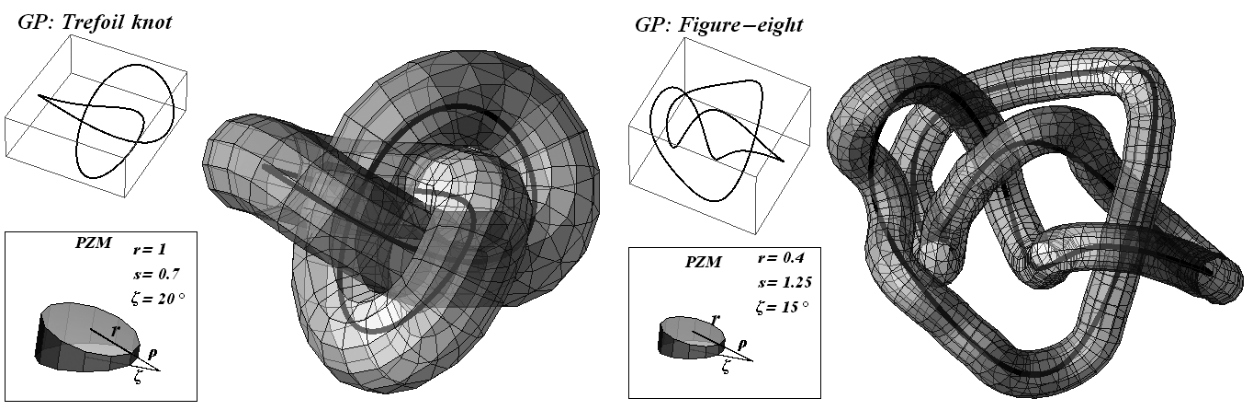

Figure 6: Trefoil and Figure-eight knot constructed with 54 and 190 PZMs respectively. Parameters r, s and ζ were adjusted manually for each knot; $n = 12$.

3.2. Self-intersections

This simple algorithm does not prevent PZ from self-intersecting, which is a well known and very difficult problem in surface modeling [4]. However, in this particular case, if GP is free of self-intersections, the self-intersections of the PZ can be avoided simply by reducing r. It seems, however, that adjusting the geometry of GP will often give better results, as demonstrated in Fig. 7.

3.3. Reducing the diversity of twist angles k

In many cases, the number of twist angles k can be naturally reduced due to symmetries in the PZ structure. For instance, the trivial cases shown in Fig. 4 use only one value:

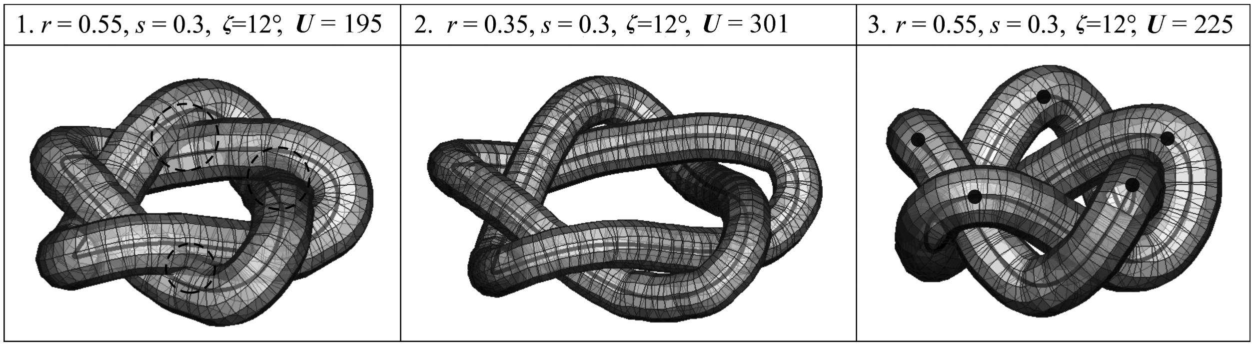

Figure 7: Pentafoil constructed with PZ: 1. Self-intersections are indicated by dashed circles. 2. A reduction of r, which is the width of the PZMs, solves the problem. 3. Alternatively, self-intersections can be avoided by elevating selected nodes of GP (indicated by black dots). U stands for the number of PZMs.

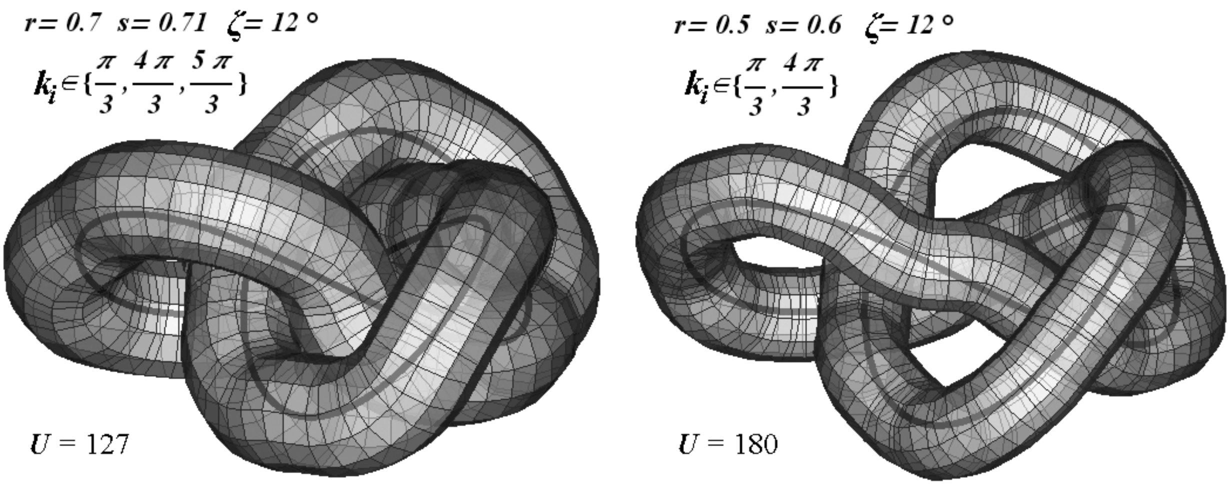

Figure 8: Two examples of a trefoil knot approximated by a PZ with only three and two different dihedral twist angles are shown on the left and right, respectively.

$k_1 = k_2 = \cdots = k_5 = 0$. In the example of a trefoil shown in Fig. 6, the k-values $2\pi/3$ and $5\pi/6$ do not occur. The figure-eight knot (Fig. 6) and the pentafoil (Fig. 7) use all twelve dihedral twist angles. In some practical applications it might be desirable to have as few different connection configurations as possible. Thus, the minimization of diversity of k is another optimization problem worth addressing. Obviously, such a reduction in many cases may also impair the smoothness of the PZ structure, which may be objectionable. Two examples of a trefoil knot constructed with only three and two different k values are shown in Fig. 8.

4. Conclusions

- As demonstrated above, complex free-form paths can be approximated well by PZs, that are composed of relatively simple congruent units. The accuracy of such an approximation depends on three independent parameters, on r, ρ and ζ. A secondary parameter n influences the geometry of a PZ, but in a very specific way is not related to the aforementioned accuracy.

- In certain cases, as in the examples of a figure-eight knot (Fig. 6) and pentafoil (Fig. 7), it is also rational to adjust the guide path — for example by translating certain nodes or by constraining the minimal allowable curvature, et cetera.
- The concept of PZ can be practically applied for designing complex three-dimensional linkages in any scale, from ventilation ducts to structural tubings.
- In the examples shown above, although PZs approximate the shape of GPs rather satisfactorily, they do not form exactly closed structures, as required by the strict mathematical definition of a knot. Thus, future research will include the construction of PZs such that the top face of the last unit T_U covers exactly the bottom face of the initial unit B_0.
- Also, the optimization of a PZ structure which performs a given geometrical task without a given GP, for example linking a given sequence of points in space with a minimal number of PZMs and with maximal "smoothness" of the overall structure, is one of the major challenges for future research.

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