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Force Driven Ruled Surfaces

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Abstract. This paper presents a mathematical strategy for shaping ruled surfaces under the mission statement: *Form Follows Force.* We motivate why we focus on the class of ruled surfaces, we show on which principles and preliminary work this paper is based on and how we exploit line geometry to directly compute optimized ruled surfaces in near real-time. Our aim is to provide a form finding tool for lightweight structures in a common design environment.

Key Words: ruled surfaces, dual sphere, dynamic relaxation, line geometry, form finding, lightweight structure

MSC 2010: 53A25, 51M30, 70G65

1. Introduction

Ruled surfaces play an important role in architecture and civil engineering. Especially their application in shell structures built of reinforced concrete benefits from the fact that ruled surfaces are generated from straight lines (see [12]). Thus reinforcement elements and formwork can be fabricated from rectilinear components. In practice formwork can be produced inexpensively with a hot-wire foam cutter instead of a time-consuming CNC milling machine. Despite their importance, the useful class of skew ruled surfaces is not yet suitably implemented in common CAD software. This is why we want to supply new intuitive methods, which correspond to the requirements of construction.

The basis for the integration of forces into a form finding method is *dynamic relaxation* [10]. Similar to the hanging chain models of Antoni Gaudí (cf. [3]), mass-spring-damper systems can be applied to the vertices of a polygon mesh. The result is a physical simulation in Euclidean Space where point-like particles are moved by linear forces (Fig. 1).

In our approach we want to ensure that the resulting surface still belongs to the class of ruled surfaces. Thus, we have to deal with oriented lines instead of points. This affects, which kind of movements we have to consider during the dynamic relaxation. A point can reach any position in space by a simple linear translation (Fig. 2, left). The path length of the corresponding trajectory is minimal compared to all possible displacements. Considering the displacement of an oriented line, the path length is given by the (infinite) area of the

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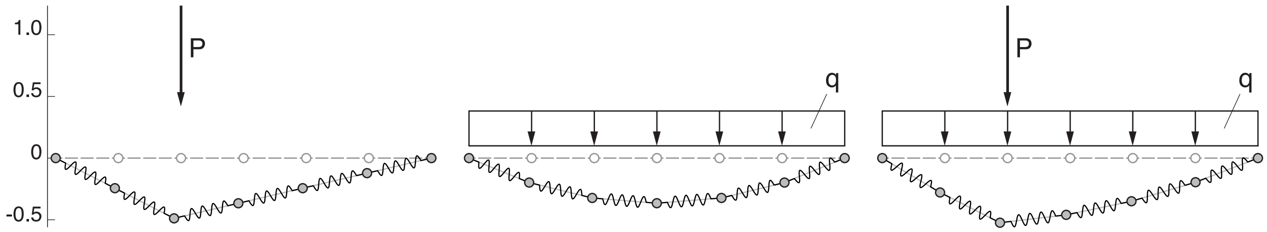

Figure 1: Application of dynamic relaxation to a planar graph in Euclidean Space

ruled surface covered by the oriented line undergoing the displacement. Therefore, we call a displacement minimal if the translation is minimal under the condition of a minimal rotation angle. Note, that oriented lines in general only need a rotation to arrive at any other position in space (Fig. 2, centre), but the latter is not neseccarily a minimal displacement. For optimization we therefore have to handle screw motions. *Chasles' theorem* states that two positions of a frame in three-dimensional space can always be connected by one continuous screw motion (Fig. 2, right) [2]. The model of dynamic relaxation, which we introduce here and which operates on frame-like particles and screw motions, is a generalization of the point model that restricts to translations.

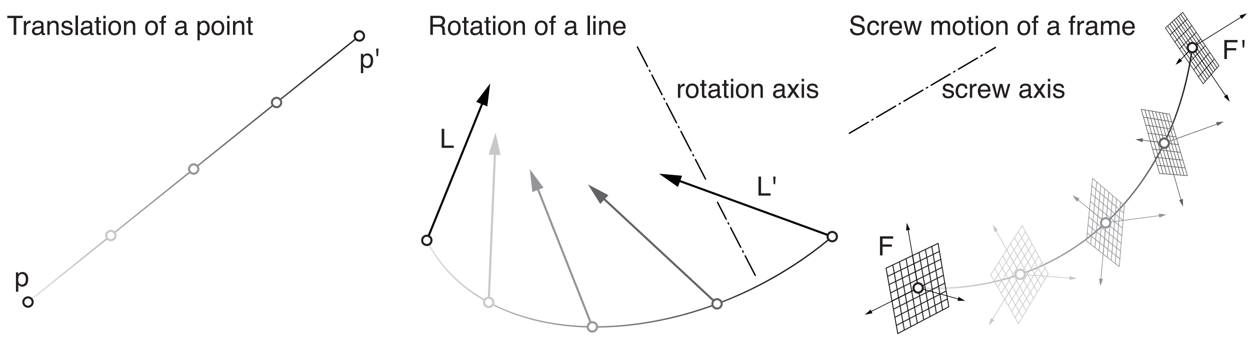

Figure 2: Particles in space and their simplest movements to connect two positions in three-dimensional space

To adequately represent the four-dimensional manifold of oriented lines in Euclidean space and in order to find the shortest screw motion between any two positions of those lines, we switch the model space to the dual unit sphere (see [15, 17]). This mathematical representation was already used by the authors on the occasion of an interpolation algorithm for discrete ruled surfaces [7]. The principles of dynamic relaxation in connection with models of line geometry have been explored to some extend during investigations on curve flows on ruled surfaces [6].

To gain an overview of the mathematical strategy involved in our approach we take a look at the scheme in Fig. 3. We start with a ruled surface. It is described by a number of discrete generators, the oriented lines $L_0, L_1, \ldots, L_n$, defined by ray segments from p_i to q_i $(i = 0, \ldots, n)$. Additional load cases are given and processed in connection with the generators. The input is then split up into three categories: endpoints of the (oriented) generator segments, in the following computed in homogenous coordinates, oriented lines, represented in normalized *Plücker coordinates*, and load cases handled as wrenches. In the next step a four-dimensional point model for the lines, the dual unit sphere $S^2_{\mathbb{D}}$, is used for the calculations. The ruled surface $\mathfrak{R}$ appears as a curve on the dual unit sphere. Since the

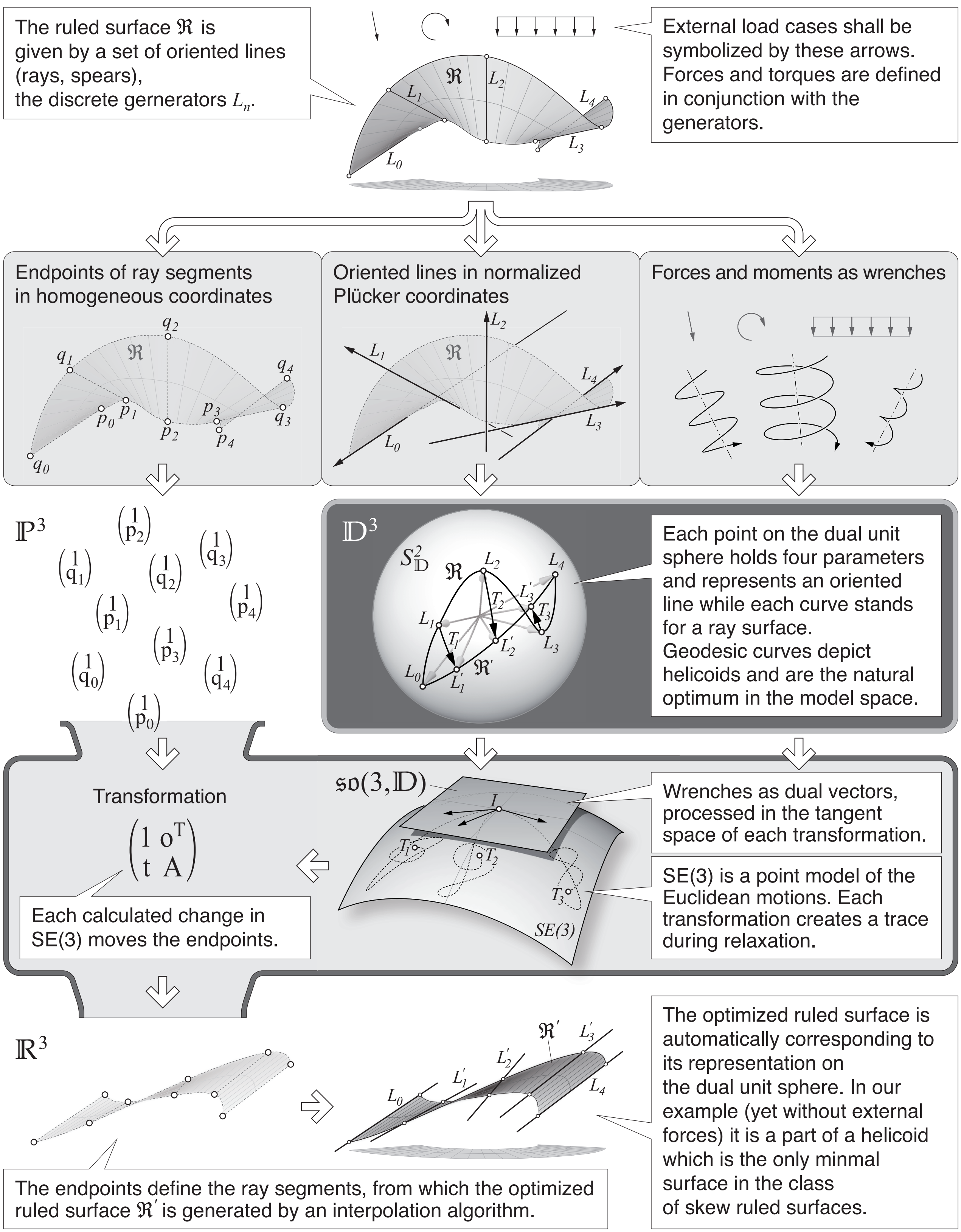

Figure 3: Computation scheme for the dynamic relaxation of ruled surfaces under internal and external forces by the means of line geometry, example of $n = 5$ lines

first and last generator shall stay in place, the other points $L_1, L_2, \dots, L_{n-1}$ will move on the dual unit sphere during optimization until they reach their end positions $L'_1, L'_2, \dots, L'_{n-1}$. The example given in the scheme is computed without external forces. This is why the optimized ruled surface $\mathfrak{R}'$ appears as a geodesic between L_0 and L_n. This is meaningful because geodesics on the dual unit sphere stand for helicoids, which are the only minimal surfaces in the class of skew ruled surfaces (proved by Catalan in 1842).

The transformations $T_1, T_2, \dots, T_{n-1}$ on the dual unit sphere are represented in another point model, the group $SE(3)$ of direct Euclidean motions in $\mathbb{R}^3$, which is a *Lie group*. To be exact, each step of the dynamic relaxation is computed in the tangent space of the actual transformation, which is a local *Lie algebra*. In the beginning there is no movement and thus the transformations of all generators are depicted by the identity I. Then the forces between the generators and all external forces are brought in as vectors into the Lie algebra of the identity I and the resulting vectors induce the first movements. Step by step the points $T_1, T_2, \dots, T_{n-1}$ will generate traces in the Lie group, until all forces are in equilibrium. Now the transformations, translated into transformations on the dual sphere, result in new positions of oriented generators as a whole, not considering the endpoints of the generator segments. To provide the new positions of the endpoints we use the transformation matrices computed in $SE(3)$ to directly affect the point coordinates. Finally the optimized ruled surface $\mathfrak{R}'$ is obtained by an interpolation algorithm.

We will describe the mathematical background of our approach, compare the new results with standard dynamic relaxation and give an example for the implementation of external forces.

2. Mathematical background

2.1. Dual numbers

In order to describe oriented lines, we first want to introduce the calculus of dual numbers (cf. [16]). The set of dual numbers is denoted by $\mathbb{D}$ and given by

$$\mathbb{D} := \left\{ z = a + \epsilon b \,|\, a, b \in \mathbb{R}, \epsilon^2 = 0 \right\}.$$

This is in analogy to the complex numbers $\mathbb{C}$, where $i^2 = -1$. Here, ϵ is called dual unit. However, with these assumptions we are able to define addition and multiplication as follows

$$\begin{aligned} z_1 + z_2 &= (a_1 + a_2) + \epsilon(b_1 + b_2), \\ z_1 z_2 &= (a_1 a_2) + \epsilon(a_1 b_2 + b_1 a_2). \end{aligned}$$

It is easy to verify that a dual number with vanishing real part is a zero divisor, *i.e.*, it has no inverse. Therefore, the dual numbers $\mathbb{D}$ only form a commutative ring with 1. Furthermore, the dual number calculus can be extended to differentiable functions $f : \mathbb{D} \to \mathbb{D}$ using the Taylor expansion with respect to the dual unit ϵ (see [1])

$$f(a + \epsilon b) = f(a) + \epsilon b f'(a).$$

This allows the adaption of arbitrary functions. Especially trigonometric functions play an import role in order to determine dual angles.

2.2. Representation of lines

A unique representation of an oriented line L or *"spear"* [15] or *"ray"* [12] in the Euclidean space is given by its normalized direction vector $l \in \mathbb{R}^3$ and its moment vector $m \in \mathbb{R}^3$ which can be determined by

$$m = p \times l,$$

where p is an arbitrary point on the line (see Fig. 4 left). The pair (l, m) is known as Plücker coordinates and in the following we will write (l, m) in dual form as $L = l + \epsilon m$. In this notation L is an element of the 3-dimensional modul $\mathbb{D}^3 = \mathbb{R}^3 + \epsilon\mathbb{R}^3$ and therefore called a dual vector (cf. [13]). In addition to this, we define the bilinear form

$$\langle L_1, L_2 \rangle_{\mathbb{D}} := \langle l_1, l_2 \rangle + \epsilon(\langle l_1, m_2 \rangle + \langle m_1, l_2 \rangle). \tag{1}$$

For a normalized line $L := l + \epsilon m$ we have the additional conditions $\|l\| = 1$ and $l \perp m$. Hence, the dual scalar product of L with itself results in

$$\langle L, L \rangle_{\mathbb{D}} = \underbrace{\langle l, l \rangle}_{=1} + 2\epsilon \underbrace{\langle l, m \rangle}_{=0} = 1.$$

Therefore, the set of all lines in $\mathbb{R}^3$ is given by the dual unit sphere

$$S^2_{\mathbb{D}} := \left\{ L \in \mathbb{D}^3 \mid \|L\| = 1 \right\},$$

where $\|L\|$ denotes the dual square root of $\langle L, L \rangle_{\mathbb{D}}$. This dual unit vector representation for oriented lines is of advantage for the computation of Euclidean invariants between two spears L_1, L_2, e.g., their included angle θ and their shortest distance δ (see Fig. 4 right). Note, that there exist two opposite spears that are contained by one line. These spears correspond to antipodal points on the dual unit sphere. The scalar product by Eq. (1) yields

$$\langle L_1, L_2 \rangle_{\mathbb{D}} = \cos(\theta) - \epsilon\delta \sin(\theta) = \cos(\theta + \epsilon\delta) \tag{2}$$

where $\phi := \theta + \epsilon\delta$ is called dual angle between L_1 and L_2. Thus, Eq. (2) gives a simple method to determine distances and angles between lines using the arc cosine function for dual arguments. In addition to this, the oriented common normal N of L_1 and L_2 can be calculated from the normalized dual cross-product

$$L_1 \times_{\mathbb{D}} L_2 := (l_1 \times l_2) + \epsilon(l_1 \times m_2 + m_1 \times l_2)$$
$$N := \frac{L_1 \times_{\mathbb{D}} L_2}{\|L_1 \times_{\mathbb{D}} L_2\|}$$

The norm of the cross-product itself is equal to $\sin(\phi)$.

2.3. Group of direct Euclidean motions $SE(3)$

The basic principle of this paper is working in a transformation group. For our purpose, the convenient transformation group is the group of Euclidean motions in $\mathbb{R}^3$, denoted by $SE(3)$. In addition to the group structure, $SE(3)$ is also a smooth manifold whose smooth structure is compatible with group operations and therefore a *Lie group*. For a matrix group based introduction to Lie groups we refer to [9].

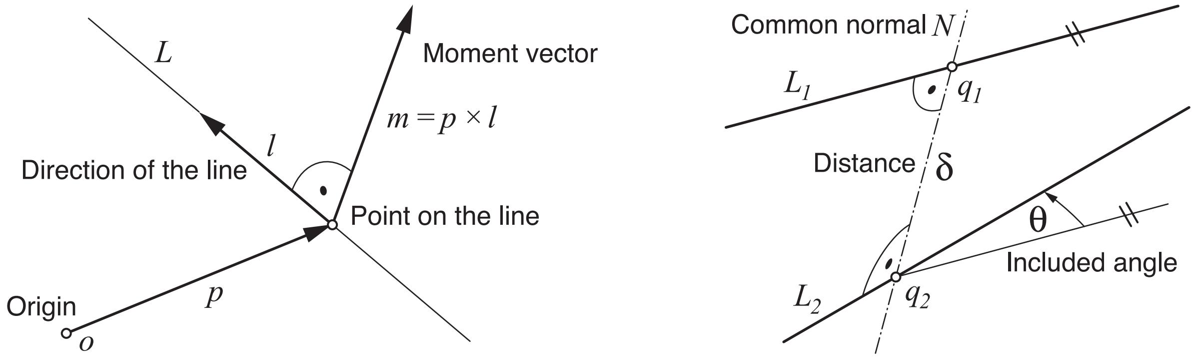

Figure 4: Direction and moment vector (left), Euclidean invariants (right)

In the following, we are interested in the action of $SE(3)$ on a special manifold M. To get a feeling for this, we give a short example and consider $M = \mathbb{R}^3$. The Lie group action $\Lambda\colon\ SE(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ then corresponds to the motion of points $p \in \mathbb{R}^3$ under the displacement $\alpha \in SE(3)$ and is usually given by a matrix-vector product of a 4×4 transformation matrix $M_{\mathbb{R}^3}$ with p in homogeneous coordinates

$$\Lambda(\alpha, p) = M_{\mathbb{R}^3} \cdot \begin{pmatrix} 1 \\ p \end{pmatrix} = \begin{pmatrix} 1 & o^T \\ t & A \end{pmatrix} \begin{pmatrix} 1 \\ p \end{pmatrix} = \begin{pmatrix} 1 \\ Ap + t \end{pmatrix}. \tag{3}$$

$M_{\mathbb{R}^3}$ consists of a 3×3 rotation matrix A and a translation vector $t \in \mathbb{R}^3$. This mapping works fine for points, but actually we are interested in the action on lines. For this reason, we use $M = S^2_{\mathbb{D}}$ as an appropriate manifold, where each point can be interpreted as a spear (ray). Using dual quaternions H_d, it can be shown that the Lie group action Λ of $SE(3)$ on the dual unit sphere is also given by a matrix-vector product (see [8]). More precisely, the transformation is represented by a special orthogonal 3×3 matrix $M_{S^2_{\mathbb{D}}}$, whose entries are dual numbers. These matrices form a group that is denoted by $SO(3, D)$. For an arbitrary point L on $S^2_{\mathbb{D}}$, the action Λ is given by

$$\Lambda(\alpha, L) = M_{S^2_{\mathbb{D}}} \cdot L = (A + \epsilon B)(l + \epsilon m) = Al + \epsilon(Am + Bl)$$

The special orthogonality of $M_{S^2_{\mathbb{D}}}$ implies

$$M^T_{S^2_{\mathbb{D}}} M_{S^2_{\mathbb{D}}} = (A^T + \epsilon B^T)(A + \epsilon B) = A^T A + \epsilon(A^T B + B^T A) = I + \epsilon 0$$

and therefore A is a rotation matrix and B is an arbitrary matrix with $C := B^T A$ is skew-symmetric. For an arbitrary motion A includes the rotational and C includes the translational part of the motion. Note, that there is a one-to-one correspondence between matrices acting on $\mathbb{R}^3$ (see Eq. (3)), and the group of special orthogonal matrices acting on $S^2_{\mathbb{D}}$. Therefore, $SE(3)$ is isomorphic to $SO(3, \mathbb{D})$. The action of $SE(3)$ on $\mathbb{R}^3$ and $S^2_{\mathbb{D}}$ is presented in Fig. 5.

Let t be a real parameter, then $M(t) \in SO(3, \mathbb{D})$ represents a one-parameter motion with $M(t = 0) = I$. The image of a line L_0 is then given by

$$L(t) = M(t) L_0 \tag{4}$$

For further investigations we are interested in tangential elements to $SO(3, \mathbb{D})$. Differentiating Eq. (4) leads to

$$\dot{L}(t) = \dot{M}(t) L_0 = \dot{M}(t) \underbrace{M^T(t) M(t)}_{=I} L_0 = \underbrace{\dot{M}(t) M^T(t)}_{=:\Omega(t)} L(t) \tag{5}$$

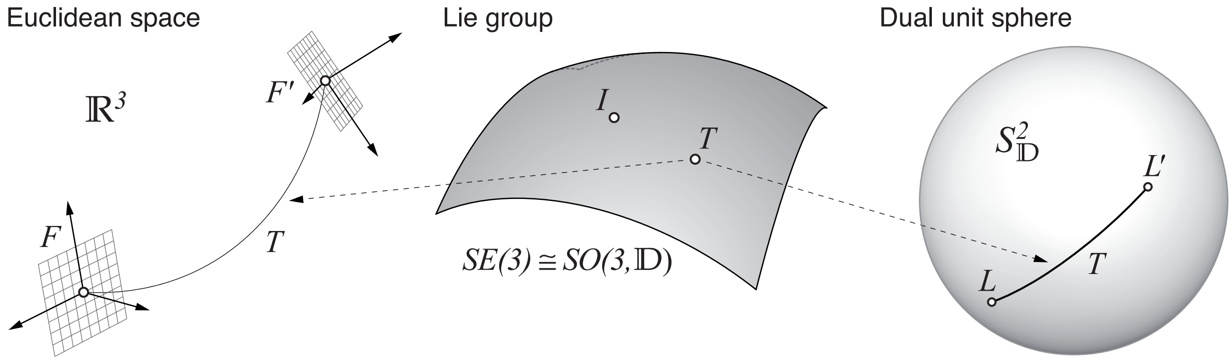

Figure 5: Lie group action $SE(3)$ on $\mathbb{R}^3$ ans $S^2_{\mathbb{D}}$

where I denotes the 3×3 identity matrix.

Due to the special orthogonality of M, the dual matrix $\Omega(t)$ is skew-symmetric and can be interpreted as tangential element at $M(t) \in SO(3, \mathbb{D})$. Generally, the tangent space at the identity of a Lie group is given by its corresponding Lie algebra. In our case, this equals the set of dual skew-symmetric matrices, denoted by $\mathfrak{so}(3, \mathbb{D})$. The specific structure allows an interpretation as matrix representation of a dual cross-product, *i.e.*, $\Omega x = [\omega]_\times x = \omega \times x$, $x \in \mathbb{D}^3$. Therefore, a mapping from $\mathfrak{so}(3, \mathbb{D})$ to $\mathbb{D}^3$ can be defined. A geometric interpretation of ω is given by its normalization $N := \|\omega\|^{-1}\omega$ and its norm $\phi := \|\omega\|$. While N is an element of the dual unit sphere and thus represents a line which corresponds to the instantaneous screw axis, ϕ is a (dual) velocity consisting of an angular velocity around the axis and an ordinary velocity along the axis.

The solution of the differential Eq. (5) for constant ω is given by the matrix exponential

$$\exp\colon\ \mathfrak{so}(3, \mathbb{D}) \to SO(3, \mathbb{D}).$$

More precisely, each one-dimensional subspace of $\mathfrak{so}(3, \mathbb{D})$ defines a screw group, *i.e.*, rotations around and translations along an axis, where the translation magnitude is a linear function of the rotation angle. One-parameter screw motions can be parameterized by the dual version of the Rodrigues-Formula (see [14])

$$\exp(\Omega t) = I + \sin(\phi t)N + (1 - \cos(\phi t))N^2$$

In addition to this, the inverse map is given by the logarithm

$$\log\colon\ SO(3, \mathbb{D}) \to \mathfrak{so}(3, \mathbb{D}).$$

A schematic presentation of the exponential and logarithm map is illustrated in Fig. 6.

3. Mass-spring-damper systems

A mass-spring-damper system is a method to study the deformation behaviour of objects, which are generally represented by polygonal meshes. To each knot $k_i \in \mathbb{R}^3$ of the mesh a mass m_i is assigned. The sum of all masses m_i results the total mass m assigned to the object. Edges e_{ij} between two knots are coupled by springs following Hooke's law (linear elasticity, stiffness c_{ij}) and an optional parallel damper element with constant damping d_{ij}. We can

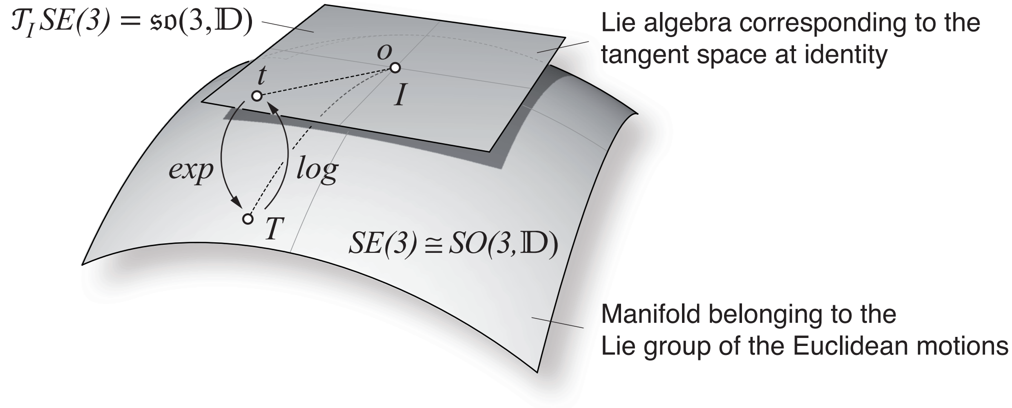

Figure 6: $SE(3)$ and the corresponding Lie algebra as tangent space at identity

determine the internal forces resulting from the springs for $k_i' = k_i + x_i$ with $J_i := J \setminus \{i\}$ by

$$F_{i,int} = \underbrace{\sum_{j \in J_i} c_{ij}(x_j + x_i)}_{=:F_{i,var}} + \underbrace{\sum_{j \in J_i} c_{ij}(\|k_j - k_i\| - l_{ij}^{Rest}) \frac{k_j - k_i}{\|k_j - k_i\|}}_{=:F_{i,const}}$$

where x_i denotes the displacement for each knot k_i and J determines the index set $J := \{1, \ldots, n\}$. Keeping Newton's second law in mind, we can formulate a second order differential equation (ODE) with respect to the time

$$F_i = F_{i,var} + F_{i,const} + F_{i,ext} = m_i \ddot{x}_i \,.$$

For our purpose, we use an additional global damping D to reduce the total mechanical energy of the problem. This is necessary because an ideal system without damping does not loose energy and therefore it will not result in a stable status. Considering the equations for all knots and displacements $x = (x_0, \ldots, x_n)^T$ we get

$$M\ddot{x} - D\dot{x} - Cx = F_{ext} - F_{const}. \tag{6}$$

Using the substitution $v = \dot{x}$ and rewriting Eq. (6) as system of first order ordinary differential equation leads to

$$\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ M^{-1}(F_{ext} - F_{const} - Cx - Dv) \end{pmatrix} = AX + B. \tag{7}$$

This system can easily be solved by arbitrary methods for initial value problems.

3.1. Relation to transformation groups

The preceding explanation of a mass-spring-damper system in $\mathbb{R}^3$ shows that a displacement x_i and a velocity v_i for each knot k_i has to be found. Taking into account the concept of transformation groups, then x_i is both a vector of $\mathbb{R}^3$ and an element of the translation group $T(3)$. Due to the additive structure of $T(3)$, the application of x_i to k_i is computed by $k_i' = k_i + x_i$. This is the action of the Lie group $T(3)$ on $\mathbb{R}^3$ thought as manifold. Admittedly, this is a very trivial case, especially the corresponding Lie algebra $\mathfrak{t}(3)$ which consists of the derivatives of x_i is also equivalent to $\mathbb{R}^3$. For better understanding of the principle, we want to consider a mesh on the unit sphere $S^2 \subset \mathbb{R}^3$. To guarantee that the mesh does not leave the sphere during the discrete solving process, we have to restrict the computation to rotations instead of translations. Therefore, we are searching for elements of the matrix group of rotations, denoted by $SO(3)$. Knots k_i of the mesh are now coupled by "spherical" instead of "linear" spring elements. Tangential elements to $SO(3)$ are skew-symmetric matrices $\mathfrak{so}(3) \cong \mathbb{R}^3$, representing a cross-product. The associated vector defines the instantaneous rotational axis of velocity or acceleration through the origin and its length corresponds to the angular velocity or angular acceleration, respectively.

In order to describe the dynamics of ruled surfaces, we go a little further and transfer the idea of springs onto the dual unit sphere $S^2_{\mathbb{D}}$. In this case, knots represent lines and edges comply with dual great circle arcs, which represent helicoids. Due to the fact that helicoids are minimal surfaces, they adapt very well to the concept of "shortest" curves connecting two knots, such as line segments in $\mathbb{R}^3$ or great circle arcs on S^2 do (see Fig. 7).

Since this paper is restricted to ray segment surfaces we only consider one-dimensional meshes, also known as chords. Because of that, knots of the chord can be thought as a discrete set of generators of a ruled surface. Additionally, the calculus of dual numbers allows dual system parameters, e.g., dual stiffness or dual damping.

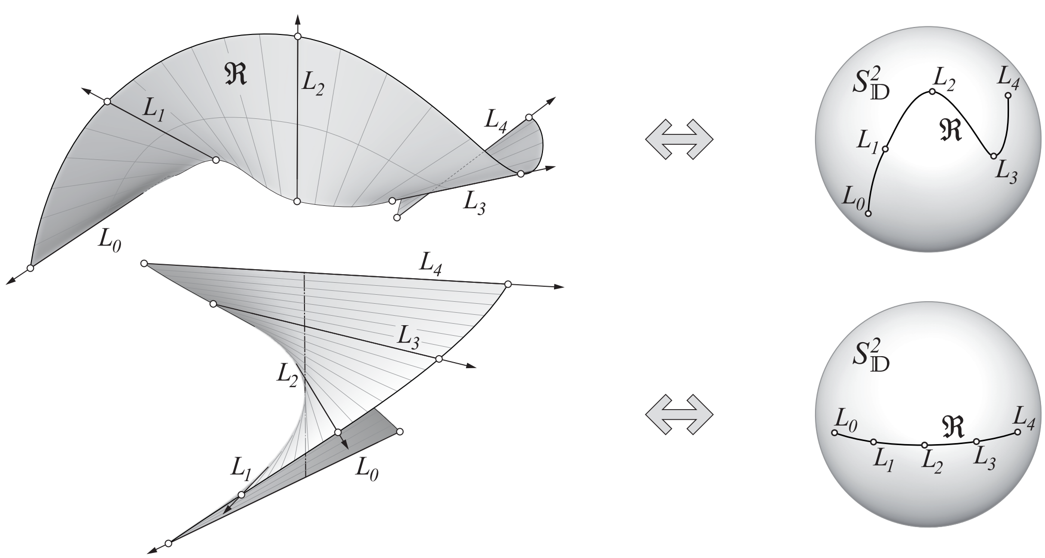

Figure 7: Ruled surfaces as curves on the dual unit sphere

4. Implementation

Differential equations on manifolds involve difficulties in solving. That is because common numerical methods for initial value problems, neither linear one-step nor multistep methods, are based on additive transformation groups. An introduction to integrators on Lie groups can be found in [4]. In our case, we use a manifold-adapted explicit one-step method (Runge-Kutta-Munthe-Kaas, abbr. RKMK) in order to handle changing tangent spaces for different time steps (see [11]). In more detail, this method calculates tangent elements at different points on the manifold and uses these to determine a best-approximating tangent element.

As far as we know, a solver for ODE's which deals with dual numbers does not exist. Fortunately, such a solver is not necessary for our problem, because we can treat dual numbers by extending the system of ODE's. Therefore we have to apply Eq. (7) to dual vectors and dual matrices. Thus, Eq. (7) turns into

$$\begin{aligned}\dot{X}_{real} + \epsilon \dot{X}_{dual} &= (A_{real} + \epsilon A_{dual})(X_{real} + \epsilon X_{dual}) + (B_{real} + \epsilon B_{dual}) \\ &= (A_{real} X_{real} + B_{real}) + \epsilon (A_{real} X_{dual} + A_{dual} X_{real} + B_{dual})\end{aligned}$$

Furthermore, we get

$$\frac{d}{dt}\begin{pmatrix} X_{real} \\ X_{dual} \end{pmatrix} = \begin{pmatrix} A_{real} & 0 \\ A_{dual} & A_{real} \end{pmatrix}\begin{pmatrix} X_{real} \\ X_{dual} \end{pmatrix} + \begin{pmatrix} B_{real} \\ B_{dual} \end{pmatrix}$$

After each time step we have to determine the new displacement (Lie algebra element) of the initial points by using the *Campbell-Baker-Hausdorff formula* [9]

$$X_{new} = \log\left(\exp\left(\Delta t \left[\frac{d}{dt} X_{old}\right]_\times\right) \exp\left([X_{old}]_\times\right)\right)$$

5. Simulation

In $SE(3)$ a load case is generally described by an acceleration screw, a wrench, consisting of a screw axis and a dual acceleration. In addition to the basic cases "force" and "moment", arbitrary load cases can be generated by combining both. Moreover, we are able to consider multiple forces and/or moments. This is due to a mechanical equilibrium which yields a resulting acceleration screw.

First of all we study the unloaded case. As shown in Fig. 8 the resulting ruled surface is a helicoid. This is because of a constant dual stiffness and an undefined dual rest length of the helicoidal springs. With these assumptions, any initial ruled surface will have this behaviour. This is the reason why helicoids are of special interest for us. Furthermore, they are also well-suited for light weight structures since the minimal surface property guarantees an optimization of the surface area.

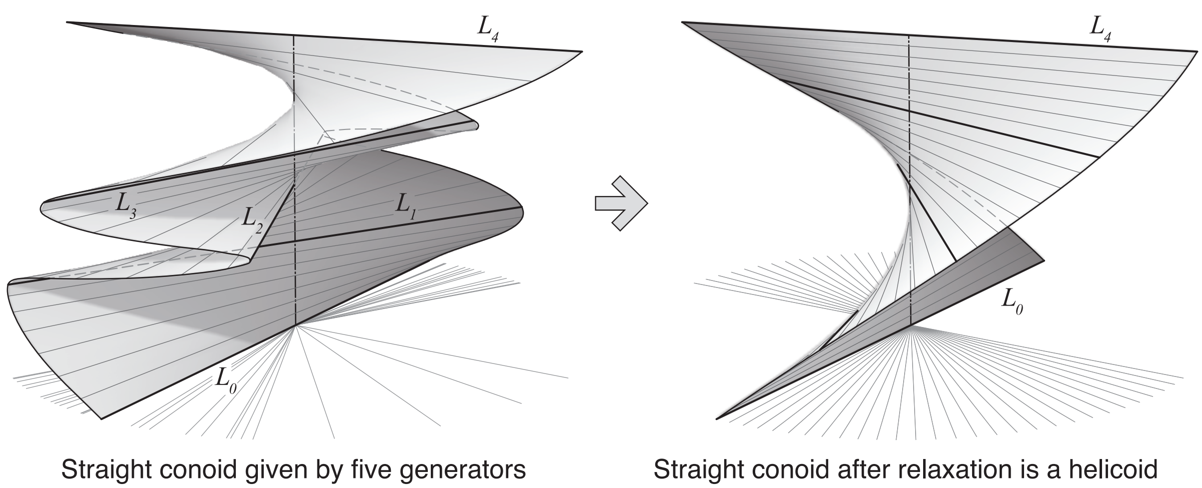

Figure 8: Simulation for a straight conoid

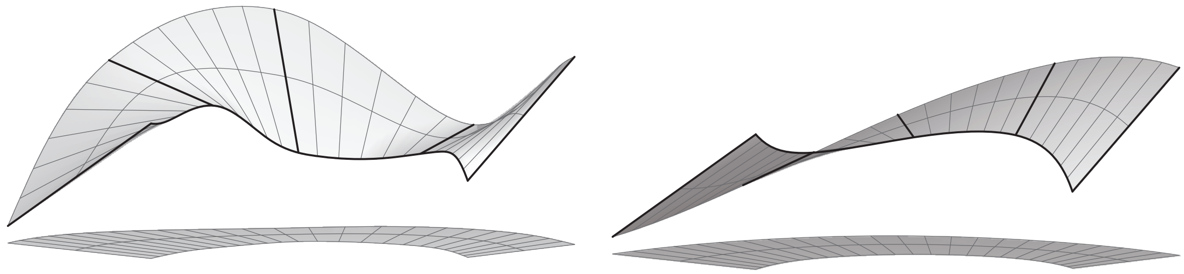

Figure 9: Relaxed ruled surface automatically adapts to the footprint of the initial surface

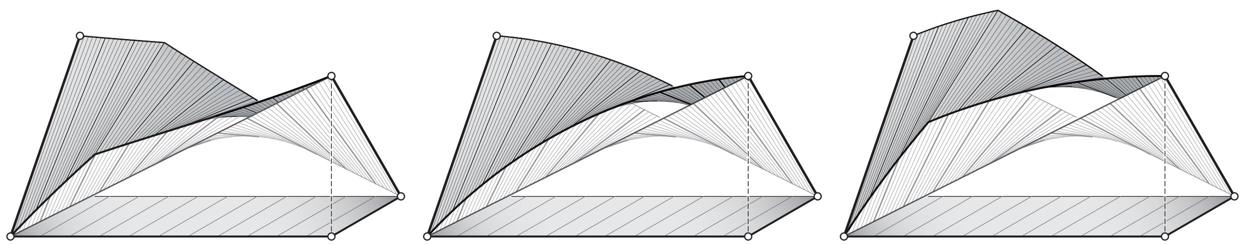

Figure 10: Reactions of a hyperbolic paraboloid to three exemplary sets of forces comparable with Fig. 1

6. Comparision to the point model

Applying the principle of mesh relaxation to a discrete point mesh of a ruled surface would generally not preserve the characteristic property of ruled surfaces, *i.e.*, lines would not stay lines under the evolution in time. Of course, we can represent a ruled surface by the end points of the line segments instead. It is intuitive to couple these endpoints with linear springs, which is like a rope ladder model. This approach guarantees the preservation of rulings, but the disadvantage is the dependency on the coupling of the springs. For example, longer lines would affect the result. This raises further questions, like how to set up the particular springs. Generally, the rope ladder approach has one degree of freedom concerning the relative position of two rulings. To avoid this, additional springs have to be included. The presented method handles these difficulties at once by considering dual spring elements. In addition to this the relative positions of the lines are preserved.

7. Conclusions

A new method for dynamic relaxation of ruled surfaces under the premise that their characteristic properties are preserved is presented in this paper. External forces and moments were integrated in the modelling process to model influences of for example wind, gravitation and snow. The resulting ruled surfaces are helicoid-like and therefore optimal for light-weight purposes.

Ruled surfaces do not have the attention in computer aided design they deserve. Skew ruled surfaces deliver a vast diversity of possibilities for CAD. Furthermore, arbitrary surfaces with non-positive Gaussian curvature can be approximated by ruled surfaces.

In this paper we did not treat structure preserving dynamic relaxation processes. This is still under construction and will be published later. The presented method can be extended to a discrete version of line congruences, *i.e.*, 2-dimensional sets of lines.

Acknowledgements

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