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# Generation of Relations for Bicentric Polygons

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**Abstract.** A *bicentric polygon* is one which is simultaneously *cyclic*: all vertices lie on a circle, and *tangential*: all sides are simultaneously tangential to another circle. All triangles and regular polygons are trivially bicentric. In the late 18-th century, Leonhard Euler developed a formula which linked the radii $R$ and $r$ of the circumcircles and incircles of a triangle, and the distance $d$ between their centres: $R^2-d^2=2Rr$. Shortly after, Euler's secretary, Nicolaus Fuss, managed to develop similar formulas for bicentric polygons of orders 4 to 9; these formulas have been given in many different forms subsequently. The purpose of this paper is to demonstrate how such relations can be generated by using polynomial ideals and Gröbner bases, in a manner which can be easily implemented on any modern computer algebra system.

*Key Words:* bicentric polygon, Gröbner bases of polynomial ideals.

*MSC 2010:* 51N20, 51N35, 13P10, 68W30

## 1. Introduction

Euler's formula $R^2-d^2=2Rr$ relating the radii $R$ and $r$ of the circumcircle and incircle of a triangle, and the distance $d$ between them, is well known. Quadrilaterals present some difficulty: some are cyclic (squares, rectangles); some aren't; some are tangential; others aren't (non-square rectangles). For a bicentric quadrilateral, Fuss showed that

$$\frac{1}{(R+d)^2}+\frac{1}{(R-d)^2}=\frac{1}{r^2}\,.$$

A neat proof is given by Dörrie [4].

Bicentric polygons are of interest partly because of the remarkable result known as *Poncelet's Porism.* (For the purpose of this article, "porism" may be considered a fancy word for "Theorem"; and this result is also called *Poncelet's closure theorem.*) Suppose we have two circles $C$ and $D$, with $D$ lying entirely within $C$. Pick any point $a_0$ on the outer circle $C$. Let $a_1$ be the point at which the tangent to $D$ from $a_0$ intersects $C$. From $a_1$ we can

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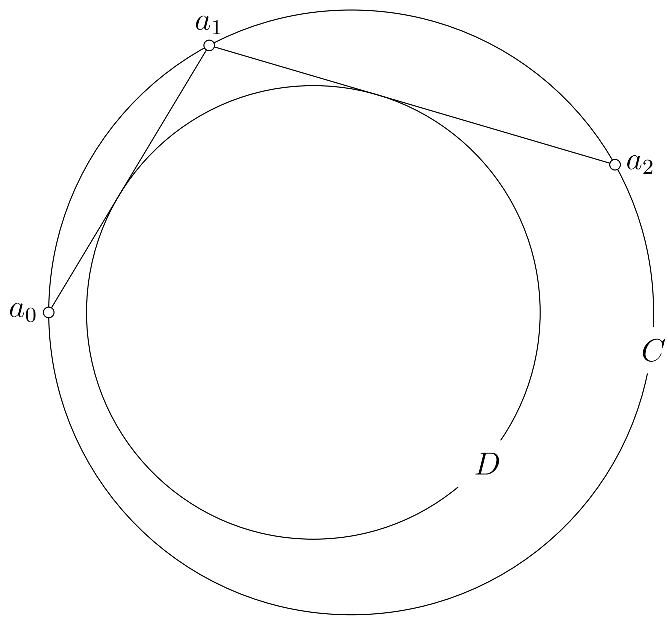


Figure 1: Tangential chords

similarly construct $a_2$ and so on as shown in Fig. 1. This sequence of points and lines is called a *Poncelet traverse*.

Poncelet's result says that if for a given point $a_0$ there is a value $a_n = a_0$ ; that is, if the lines "close up" to form a polygon, then a polygon will be formed for *any* starting value of $a_0$. Poncelet actually stated the result for general conic sections; not just for circles, and without the requirement that one lies within the other.

So for example if $D$ and $C$ are the incircle and circumcircle respectively of a triangle, then a triangle will be formed by following the above construction starting at any point on $C$. Proofs of this result can be found using elliptic functions in Dragović et al. [5]; some very elegant and accessible proofs are given by Ueno et al. [19].

Finding relations between $R$, $r$ and $d$ has generated much interest; original papers by Fuss, Steiner and Richelot (see references to Weisstein [22]); in the first half of the 20th century first Chaundy [2, 3], and later Gulasekharam [7] and Kerawala [8]; more recently Radić et al. [13, 14, 15, 16, 17]. The purpose of this article is to show how with a modern computer algebra system, relations similar to those of Euler and Fuss providing the requirements between $R$, $r$ and $d$ for a bicentric polygon to exist, can be automatically generated. Although a similar sounding paper has been published [11], the approach given here is both more generic, more strongly connected to modern ring theory, and more easily transferable to any other computer algebra system.

## 2. Algebraic background

### The geometry

We start by investigating the requirements for a chord of one circle to be tangent to another. For this article, we suppose that the outer circle (for which the polygon is cyclic) is given by $x^2+y^2 = R^2$, and the inner circle (to which the polygon is tangential) is given by $(x-d)^2+y^2 = r^2$, with $R > r + d$. Following Kerawala [8] we consider a chord between two points as shown in Fig. 2.

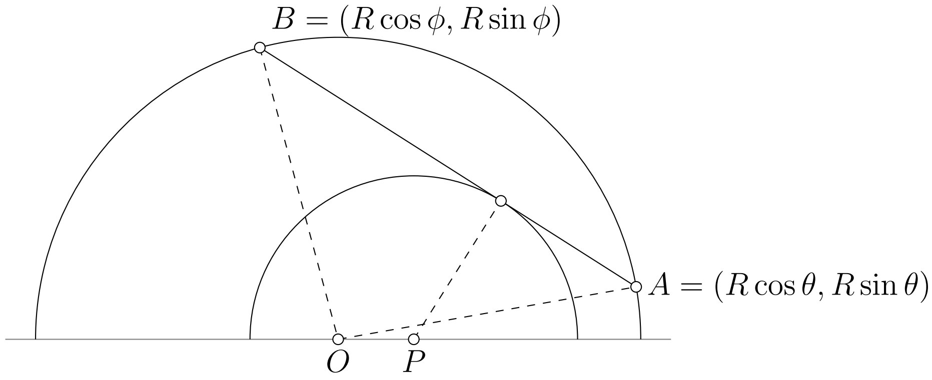


Figure 2: A chord tangential to an inner circle

Consider the perpendicular bisector of the chord $AB$, as shown in Fig. 3. Since the chord subtends the angle $\phi - \theta$, the length $\overline{OM}$ will be $R\cos\dfrac{\phi-\theta}{2}$.

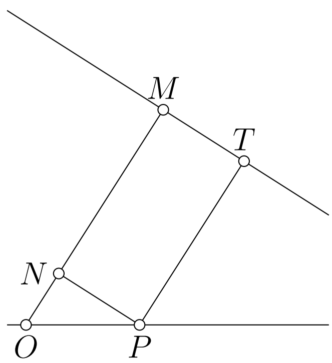


Figure 3: Determining a relation between angles on a tangential chord

Since $\overline{OP}$ has length $d$, and the angle $\angle NOP$ is $\dfrac{\theta+\phi}{2}$, we have $\overline{ON} = d\cos\dfrac{\theta+\phi}{2}$. Since $\overline{NM} = r$:

$$R\cos\left(\frac{\phi}{2}-\frac{\theta}{2}\right) - d\cos\left(\frac{\theta}{2}+\frac{\phi}{2}\right) = r. \tag{1}$$

Following Kerawala, we expand the above using standard trigonometric addition and subtraction formulas, and introduce the tangent half-angle substitutions

$$s = \tan\frac{\theta}{4}, \quad t = \tan\frac{\phi}{4}$$

(note that from equation 1 we are taking half angles of $\dfrac{\theta}{2}$ and $\dfrac{\phi}{2}$ for which of course

$$\cos\frac{\theta}{2} = \frac{1-s^2}{1+s^2}, \quad \sin\frac{\theta}{2} = \frac{2s}{1+s^2}$$

and similarly for $t$, $\phi$. Substituting into the expansion of equation (1) produces

$$R\big((1-t^2)(1-s^2)+4st\big) - d\big((1-s^2)(1-t^2)-4st\big) = r(1+s^2)(1+t^2). \tag{2}$$

This last equation can be written equivalently as

$$R\big((1-st)^2-(s-t)^2\big)-d\big((1-st)^2-(s+t)^2\big)=r\big((1+st)^2+(s+t)^2\big) \tag{3}$$

or as

$$4(R+d)st+(R-d)(1-s^2)(1-t^2)=r(1+s^2)(1+t^2). \tag{4}$$

This relation between the five variables is called by Kerawala the "quadratic involution" which connects the points $A$ and $B$, although given by him in a form equivalent to

$$2R(1+st)^2-(R+d+r)(1-st)^2-(R-d+r)(s+t)^2. \tag{5}$$

**Algebra**

Much of our work will be in eliminating variables from a system of equations. As we will show below, the equations connecting the values of $R$, $r$ and $d$ for a triangle can be written as

$$\begin{aligned}(R-d)(1-t^2)&=r(1+t^2),\\ R(1-6t^2+t^4)&=(d-r)(1+t^2)^2.\end{aligned}$$

The problem is to eliminate $t$ from these equations, producing a result in $R$, $r$ and $d$ only.

Suppose $K$ is a field, and $R=K[x_1,x_2,\dots,x_n]$ is the ring consisting of all polynomials in the variables $x_1,x_2,\dots,x_n$ with coefficients in $K$. Let $p_i(x_1,x_2,\dots,x_n)$ for $i=1,2,\dots m$ be elements of $K[x]$. The the *ideal generated by* $p_i$ is the set $I$ of all linear combinations of the $p_i$:

$$I=\{\alpha_1p_1+\alpha_2p_2+\cdots+\alpha_mp_m,\ \alpha_i\in R\}.$$

One problem in ring theory is the *ideal membership problem*: given $q\in R$, can we determine if $q\in I$?

A *Gröbner basis* is an alternative generating set for $I$ which satisfies some very useful properties. To define a Gröbner basis, we first need a polynomial ordering. One such is *lexicographic ordering.* Suppose we order the variables

$$x_1<x_2<\cdots<x_n$$

Then monomials can be ordered:

$$x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}<x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$$

if the first variable (from the left) with different exponents has $a_k<b_k$. This is equivalent to "dictionary order" of words. Using this order, any polynomial can be written with terms in a prescribed order. Given an ordering, any polynomial in $R$ can be expressed as terms in increasing order:

$$p=c_1m_1+c_2m_2+\cdots+c_nm_n$$

where the $m_i$ are ordered monomials: $m_1<m_2<\cdots<m_n$. We define $\mathrm{LM}(p)=m_1$ to be the leading monomial, and for an ideal $I$ we define $\mathrm{LM}(I)$ to be the ideal generated by the leading monomials of all elements of $I$. Note that if $I$ is generated by $p_1,p_2,\dots,p_k$, it is not generally true that $\mathrm{LM}(I)$ is generated by $\mathrm{LM}(p_i)$.

If $I\subseteq R$ is an ideal, then a generating set $G=\{g_1,g_2,\dots,g_k\}$ for $I$ is a *Gröbner basis* if the monomials $\mathrm{LM}(g_i)$ generate $\mathrm{LM}(I)$.

Alternatively, $G$ is a Gröbner basis for $I$ if for every $p \in I$, $\mathrm{LM}(p)$ is a multiple of $\mathrm{LM}(g_i)$ for some $g_i \in G$.

For the purposes of this article, a most important property of a Gröbner basis is the *elimination property.* Suppose we have an ordering $x_1 < x_2 < \cdots < x_n$, and a Gröbner basis $G$ for an ideal $I$. Then the intersection of $G$ with $K[x_1, x_2, \dots, x_k]$ is a Gröbner basis for the intersection of $I$ with $K[x_1, x_2, \dots, x_k]$. Using the notation $G(I)$ for the Gröbner basis, we can write

$$G(I \cap K[x_1, x_2, \dots, x_k]) = G(I) \cap K[x_1, x_2, \dots, x_k].$$

What this means is that if we have a set of polynomial equations $p_i = 0$, we can consider the ideal $I$ generated by the $p_i$. By choosing an appropriate ordering of the variables, and then by finding a Gröbner basis $G$, we can find equivalent generators for $I$ which eliminate some variables.

The use of Gröbner bases for geometry is well established, and it fact goes back to Buchberger [1] who presented the first modern algorithm for computing Gröbner bases; more modern examples can be found in Pech [12]. There are also older uses of elimination theory in general for investigating problems in geometry, for example van der Waerden [20] and his more recent discussion [21].

Note that much of our computations could be achieved by other means; for example polynomial resultants. However, we will in general be eliminating multiple variables from multiple polynomials, which would require the generalized Dixon resultant as revised by Kapur et al. [9] or possibly Kronecker resolvents [6]. These are not implemented in many computer algebra systems, whereas Gröbner bases are readily available.

## 3. Euler's and Fuss's formulas

In this section we shall show how the machinery described in the previous section can be used to determine Euler's formula for a triangle, and Fuss's formula for a bicentric quadrilateral. By Poncelet's porism, we may assume that the first point $A$ is at $(R, 0)$ so that $\theta = s = 0$.

If our polygon is to be a triangle, then by symmetry the side opposite $A$ must be vertical, as shown in Fig. 4.

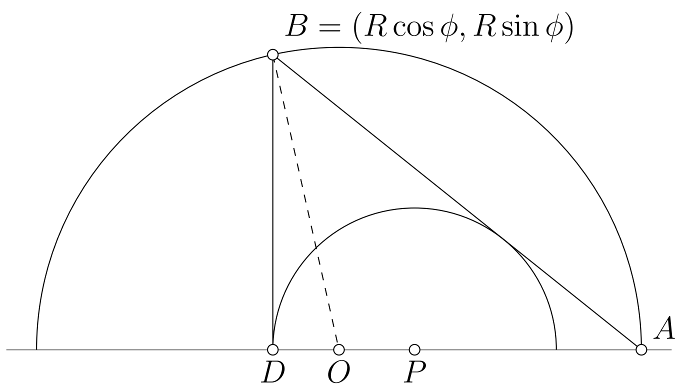


Figure 4: A vertical tangent

From the diagram, if $\angle BOA = \phi$ then $\angle DBO = \phi - \frac{\pi}{2}$, and so

$$R \sin\left(\phi - \frac{\pi}{2}\right) = r - d$$

or

$$\cos\phi = \frac{d-r}{R}.$$

Since $\tan\dfrac{\phi}{4} = t$ we can write this equation as

$$\frac{1-6t^2+t^4}{(1+t^2)^2} = \frac{d-r}{R}. \tag{6}$$

This equation was given in the previous section. From Fig. 4 there are two equations; the other being the chordal-tangent equation (4), with $s = 0$:

$$(1+t^2)(R-d)(1-t^2) = r(1+t^2). \tag{7}$$

To eliminate $t$ from equations (6) and (7) we need to set up a polynomial ring with an ideal generated by these equations, and use a Gröbner basis to eliminate $t$. This can be done very easily with the open source computer algebra system Sage [18]. The commands are self explanatory:

```
sage: PR.<R,r,d,t> = PolynomialRing(QQ)
sage: I = PR.ideal([(R-d)*(1-t^2)-r*(1+t^2),\
....: R*(1-6*t^2+t^4)-(d-r)*(1+t^2)^2])
sage: ie = I.elimination_ideal([t])
sage: ie.gens()

   [R^2 - 2*R*r - d^2]
```

The result can be interpreted as

$$R^2 - 2Rr - d^2 = 0$$

which is Euler's relation.

For a quadrilateral, there will be three points to consider: $A$ at $(R, 0)$, $B$ on the circle, and $C$ also on the $x$-axis. Since the angle at $C$ is $\pi$, we will use $u = \tan\frac{\pi}{4} = 1$. In order to ease the input, we start with a function which describes the chordal-tangent relation:

```
sage: t1(s,t) = 4*(R+d)*s*t+(R-d)*(1-s^2)*(1-t^2)-\
....: r*(1+s^2)*(1+t^2)
```

Then the Sage commands are similar to above:

```
sage: PR.<R,r,d,t> = PolynomialRing(QQ)
sage: I = PR.ideal([t1(0,t),t1(t,1)])
sage: ie = I.elimination_ideal([t])
sage: ie.gens()

   [R^4 - 2*R^2*r^2 - 2*R^2*d^2 - 2*r^2*d^2 + d^4]
```

This last means that

$$R^4 - 2R^2r^2 - 2R^2d^2 - 2r^2d^2 + d^4 = 0$$

or that

$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2)$$

which is one of the forms of Fuss's relation.

## 4. Substitutions

Many substitutions are provided by Kerawala [8], which allow for the complex expressions for higher $n$ to be written more simply. We shall consider only two:

$$a = \frac{1}{R+d}, \quad b = \frac{1}{R-d}, \quad c = \frac{1}{r}$$

and

$$p = \frac{R+d}{r}, \quad q = \frac{R-d}{r}.$$

Using these substitutions, equation (4) can be written as

$$4bcst + ac(1-s^2)(1-t^2) = ab(1+s^2)(1+t^2) \tag{8}$$

and as

$$4pst + q(1-s^2)(1-t^2) = (1+s^2)(1+t^2). \tag{9}$$

These can also be written in "Kerawala's form", as given in equation (5):

$$\begin{aligned}(p+q)(st+1)^2 - (p+1)(st-1)^2 - (q+1)(s+t)^2 = 0\\ (b+c)(st+1)^2 - (a+c)(st-1)^2 - (b+c)(s+t)^2 = 0.\end{aligned}$$

To use these equations we can write them as functions, as well as the vertical tangent function:

```
sage: t1abc(s,t) = 4*b*c*s*t+a*c*(1-s^2)*(1-t^2)-a*b*(1+s^2)*(1+t^2)
sage: t1pq(s,t) = 4*p*s*t+q*(1-s^2)*(1-t^2)-(1+s^2)*(1+t^2)
```

Similarly, equation (6) can be written as

$$\frac{1-6t^2+t^4}{(1+t^2)^2} = \frac{bc-ac-2ab}{(a+b)c} \tag{10}$$

and also as

$$\frac{1-6t^2+t^4}{(1+t^2)^2} = \frac{p-q-2}{p+q}, \tag{11}$$

and the generators of the ideals can be entered as

```
sage: vabc(t) = (a+b)*c*(1-6*t^2+t^4)-(b*c-a*c-2*a*b)*(1+t^2)^2
sage: vpq(t) = (p+q)*(1-6*t^2+t^4)-(p-q-2)*(1+t^2)^2
```

Then Euler's formula can be produced, first with $a$, $b$ and $c$:

```
PP.<t,a,b,c> = PolynomialRing(QQ)
I = PP.ideal([t1abc(0,t),vabc(t)])
ie = I.elimination_ideal([t,R,r,d])
factor(ie.gen(0))

   c * b * a * (a + b - c)
```

This means that $abc(a+b-c) = 0$, and since we assume each of $a$, $b$, $c$ is non-zero, we have $a+b=c$ or that

$$\frac{1}{R+d} + \frac{1}{R-d} = \frac{1}{r}$$

which is a more symmetric version that the originally cited form. Using $p$ and $q$ is similar:

```
PP.<t,p,q> = PolynomialRing(QQ)
I = PP.ideal([t1pq(0,t),vpq(t)])
ie = I.elimination_ideal([t])
ie.gen(0)

   p*q - p - q
```

This can be read as

$$pq = p + q$$

or alternatively,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Applying the same substitutions to FUSS's quadrilateral relation leads to

$$a^2 + b^2 = c^2$$

and

$$p^2q^2 = p^2 + q^2$$

or alternatively,

$$\frac{1}{p^2} + \frac{1}{q^2} = 1\,.$$

Note that if a relation is found for $p$ and $q$, the relation for $a$, $b$ and $c$ can be obtained with $p = c/a$, $q = c/b$; these being immediate consequences of their definitions.

## 5. Pentagon and hexagon

The relations now can be obtained automatically using the methods of the previous section: first enter the chord-tangent relations, and then produce the elimination ideal. For a pentagon, starting at $(R, 0)$ with $s = 0$, there will be two points on the upper semicircle, with tangent values $t$ and $u$, say. The point corresponding to $u$ will be on a vertical tangent from the inner circle, so that using the $p$, $q$ substitution the ideal can be created with

```
sage: I = PP.ideal([t1pq(0,t),t1pq(t,u),vpq(u)])
```

Eliminating the variables $t$ and $u$ produces a single expression which can be factored as

$$(pq - p - q)(p^3q^3 + p^3q^2 + p^2q^3 - p^3q - 2p^2q^2 - pq^3 - p^3 + p^2q + pq^2 - q^3).$$

Only the second term is of interest: the first is just EULER's formula, which we would expect if $t = u$. Calling the second term $g$ we notice that

$$g - (pq - p - q)^3 = 4pq(p-1)(q-1)(p+q).$$

Alternatively

$$g - (p^3q^3 - p^3 - q^3) = -q(p-1)(q-1)(p+q).$$

Comparing these last two equations we can write

$$(pq - p - q)^3 = 4\left(p^3q^3 - p^3 - q^3\right). \tag{12}$$

Using the $a$, $b$, $c$ substitution produces

$$(a+b-c)^3 = 4(a^3+b^3-c^3). \tag{13}$$

Without any substitution:

$$(R^2-d^2-2Rr)^3 = 4((R^2-d^2)^3 - 2Rr^3(R^2+3d^2). \tag{14}$$

For a hexagon, and using the $p$, $q$ substitution:

```
sage: I = PP.ideal([t1pq(0,t),t1pq(t,u),t1pq(u,1)])
```

Elimination of $t$ and $u$ produces:

$$3p^4q^4 - 2p^4q^2 - 2p^2q^4 - p^4 + 2p^2q^2 - q^4$$

as the generator of the elimination ideal; this may be written as

$$4p^2q^2(p^2-1)(q^2-1) - (p^2+q^2-p^2q^2)^2$$

and so the relation is

$$(p^2+q^2-p^2q^2)^2 = 4p^2q^2(p^2-1)(q^2-1). \tag{15}$$

Alternatively, the generator may be written as

$$(p^2q^2-p^2-q^2)^2 + 2(p^4q^4-p^4-q^4)$$

resulting in the relation

$$(p^2q^2-p^2-q^2)^2 = 2(p^4+q^4-p^4q^4).$$

Using $a$, $b$ and $c$ the relation is

$$(a^2c^2+a^2b^2-b^2c^2)^2 = 4a^2b^2c^2(c^2-a^2)(b^2-a^2) \tag{16}$$

or

$$(a^2c^2+a^2b^2-b^2c^2)^2 = 2(a^4c^4+a^4b^4-b^4c^4).$$

## 6. Higher order polygons

It is clear now how expressions for higher orders can be easily created. Suppose we wish to determine the relations for a $2n$-sided polygon. By symmetry, we will have points $A_1$ at $(R,0)$ and again $A_{n+1}$ on the $x$-axis. There will be $2(n-1)$ points on the circle $C$ above the $x$-axis, and similarly below. If the points $A_k$ is at angle $\theta_k$ to the positive $x$-axis, with $s_k = \tan(\theta_k/4)$, then the ideal is generated by

$$t_1(0,s_2),\ t_1(s_2,s_3),\ \ldots,\ t_1(s_{n-1},s_n),\ t_1(s_n,1).$$

We can use Gröbner bases to eliminate $s_2, s_3, \ldots, s_n$ and so obtain the relation we require.

For a polygon with $2n+1$ sides, we will have a vertical side (as with the triangle and pentagon), so generate an ideal with

$$t_1(0, s_2),\ t_1(s_2, s_3),\ \ldots,\ t_1(s_{n-1}, s_n),\ v(s_n)$$

and then eliminate the $s_i$ values as above.

An immediate problem is the computational difficulty of computing a Gröbner base for an ideal with a large generating set. For this reason we establish relations between points on the outer circle $C$ which are not consecutive. We start by considering the relation between $A_k$ and $A_{k+2}$. Suppose that we have $s = \tan\dfrac{\theta_k}{4}$, $t = \tan\dfrac{\theta_{k+1}}{4}$ and $u = \tan\dfrac{\theta_{k+2}}{4}$. To establish a relation between $s$ and $u$ we can set up an ideal with

```
sage: I = PP.ideal([t1pq(s,t),t1pq(t,u)])
```

and eliminate $t$. The final relation can be written as

$$p^2q^2(su+1)^2 - p^2(su-1)^2 - q^2(s+u)^2 = 0 \tag{17}$$

or alternatively as

$$(p^2q^2 - p^2)(s^2u^2+1) + 2(p^2q^2 + p^2 - q^2)su - q^2(s^2+u^2) = 0 \tag{18}$$

and we can include this in future computations with

```
sage: t2pq = p^2*q^2*(s*u+1)^2-p^2*(s*u-1)^2-q^2*(s+u)^2
```

Note that if $(s, u) = (0, 1)$ then equation (17) produces Fuss's relation for the quadrilateral, as we would expect.

Note that if we substitute $c/a$ and $c/b$ for $p$ and $q$ respectively in equation (17), and multiply out by $a^2b^2/c^2$, we obtain

$$c^2(su+1)^2 - b^2(su-1)^2 - a^2(s+u)^2 = 0$$

which is equivalent to an expression given by Kerawala [8].

Similarly we can create a function $t_4pq$ which gives the relation between $A_k$ and $A_{k+4}$; assuming that the quarter tangent values are $s$ and $u$, we establish the ideal generated by $t_2pq(s,t)$ and $t_2pq(t,u)$ and eliminate $t$. Considering equations (5) and (18), we might expect that the result can be written in the form

$$X(su+1)^2 + Y(su-1)^2 + Z(s+u)^2 = 0. \tag{19}$$

And in fact when the coefficients of equation (19) are compared with the result of the elimination, we find that the expression is

$$A^2B^2(su+1)^2 - B^2C^2(su-1)^2 - A^2C^2(s+u)^2 = 0 \tag{20}$$

where

$$\begin{aligned} A &= p^2q^2 - p^2 + q^2, \\ B &= p^2q^2 + p^2 - q^2, \\ C &= p^2q^2 - p^2 - q^2. \end{aligned}$$

Equation (20) is a neater formulation than KERAWALA's equation 1.8, obtained by him "after wading through a mass of algebra".

The version for $a$, $b$ and $c$ differs only in the equations for $A$, $B$ and $C$:

$$A = c^2 - b^2 + a^2, \quad B = c^2 + b^2 - a^2, \quad C = c^2 - a^2 - b^2.$$

Comparing equations (5), (17) and (20) we see that if

$$t_n(s,t) = x(st+1)^2 - y(st-1)^2 - z(s+t)^2$$

then

$$t_{2n}(s,t) = X^2Y^2(st+1)^2 - X^2Z^2(st-1)^2 - Y^2Z^2(s+t)^2$$

where

$$\begin{aligned} X &= x - y + z \\ Y &= x + y - z \\ Z &= x - y - z. \end{aligned}$$

In particular this allows us to create relations whose order is a power of 2 by a simple recursive algorithm: compute the expression for $t_k(s,t)$, with $k = 2^n$, and put $(s,t) = (0,1)$.

Similarly, we can develop an expression for $t_3(s,v)$, when we eliminate the variables $t$ and $u$ from the ideal generated by $t_1(s,t)$, $t_1(t,u)$, $t_1(u,v)$. The result has the same form as previously:

$$t_3(s,v) = X(sv+1)^2 - Y(sv-1)^2 - Z(s+v)^2 \tag{21}$$

where

$$\begin{aligned} X &= (p+q)(pq-p+q)^2(pq+p-q)^2, \\ Y &= (p+1)(pq-p-q)^2(pq+p-q)^2, \\ Z &= (q+1)(pq-p-q)^2(pq-p+q)^2. \end{aligned}$$

Since the $X$, $Y$ and $Z$ are functions of $p$ and $q$ only, this can be extended recursively to any power of 3. Note, as above, that if $(s,v) = (0,1)$, then equation (21) reduces to $X - Y - Z = 0$, and after making the substitutions with $p$ and $q$ becomes the equations for the bicentric hexagon shown at the end of Section 5.

Clearly this approach can be used to develop relations for polygons of any order, but as the number of sides $n$ gets larger, the relations get more unwieldy, and especially so for when $n$ is prime.

## 7. Other computer algebra systems

Previous computations have been done with Sage, which is convenient, powerful, and free. We show briefly here how similar computations can be done with another system, in this case Maxima [10], which has the advantage over Sage in running natively on multiple systems. We shall just show how FUSS's quadrilateral relation may be obtained. In Maxima only three steps are required: load the `grobner` package, define the polynomial, perform an elimination:

```
(%i1) load(grobner);
```

```
(%i2) tpq(s,t) := (p+q)*(s*t+1)^2-(p+1)*(s*t-1)^2-(q+1)*(s+t)^2;
(%i3) poly_elimination_ideal([tpq(0,s),tpq(s,1)],1,[s,p,q]);
```

(%o3) $[p^2q^2 - p^2 - q^2]$

Clearly similar computations can be performed in any system which supports Gröbner basis computations.

## 8. Conclusion

We have seen that the use of Gröbner bases within a modern computer algebra system allows for the easy creation of relations between the radii $R$ and $r$ of two circles, and the distance $d$ between them, so that bicentric polygons of $n$ sides exist. The CAS not only can produce a relation by elimination of variables in an ideal of a polynomial ring, but allows for experimentation with the resulting relation. The algebra involved has been noted by previous authors to be difficult ("heartbreaking" [3]); our approach completely removes the need for exhaustive pencil and paper computations, and brings this problem into the realm of modern experimental mathematics and algebra.

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