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Kiepert Conics in Regular CK-Geometries

Sybille Mick, Johann Lang

Institute of Geometry, Graz University of Technology
Kopernikusgasse 24, A-8010 Graz, Austria
emails: `mick@tugraz.at, johann.lang@tugraz.at`

Abstract. This paper is a contribution to the concept of *Kiepert conics* in regular CK-geometries. In such geometries a triangle ABC determines a quadruple of *first Kiepert conics* and, consequently, a quadruple of *second Kiepert conics*.

Key Words: Cayley-Klein geometries, geometry of triangle, Kiepert conics, projective geometry

MSC 2010: 51M09, 51N30

1. Introduction

Hyperbolic geometry obeys the axioms of Euclid except for the Euclidean parallel postulate which is replaced by the *hyperbolic parallel postulate*: Any line g and any point P not on g determine at least two distinct lines through P which do not intersect g. Axiomatic hyperbolic geometry $\mathcal{H}$ can be visualized by the *disk model*. It is defined by an absolute conic m (regular curve of 2^{nd} order) with real points in the real projective plane. The points of the model are the inner points of m, the lines are the open chords of m (see [1, 2, 4]). In a real projective plane the conic m also defines the hyperbolic Cayley-Klein geometry $CK_{\mathcal{H}}$ ([6, 9]). All points of the plane not on m — not only the inner points of m — are points of $CK_{\mathcal{H}}$. All lines of the real projective plane are lines of $CK_{\mathcal{H}}$. The second type of regular CK-geometry is the elliptic Cayley-Klein geometry $CK_{\mathcal{E}}$ which is determined by a real conic m without real points. In CK-geometries the measurement of distances and angles is based on cross-ratios of points and lines. The group of congruence transformations is the automorphism group of m.

In Section 2 we recall a few results on Kiepert conics in hyperbolic geometry $\mathcal{H}$ (see [8]). Section 3 is dedicated to the construction of Kiepert triangles in the hyperbolic CK-geometry $CK_{\mathcal{H}}$ which in turn is the blueprint for Section 4: the Construction of Kiepert triangles in regular CK-geometries. In Section 5 we construct a quadruple of first Kiepert conics to the given triangle. In Section 6 we determine the corresponding quadruple of second Kiepert conics. Finally, in Section 7, we add a short outline.

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2. Kiepert conics in hyperbolic geometry

In this preparatory section we recall the situation in $\mathcal{H}$ (see [8]).

2.1. Kiepert triangles in the hyperbolic geometry $\mathcal{H}$

Let ΔABC be an arbitrary hyperbolic triangle. If ΔA_1BC, ΔAB_1C, and ΔABC_1 are h-isosceles triangles with a common base angle ρ attached to the edges AB, BC, CA, respectively, the new triangle $\Delta A_1B_1C_1$ is called a *hyperbolic Kiepert triangle* to ΔABC. Varying ρ from $-\pi/2$ to $\pi/2$ we get the set $\mathcal{S}_1$ of Kiepert triangles to ΔABC (see [8]).

2.2. Kiepert conics in the hyperbolic geometry $\mathcal{H}$

The triangle ΔABC and $\Delta A_1B_1C_1$ are perspective from some centre K. All such centres K lie on a conic k called the *first Kiepert conic* (see [8]). The conic k is circumscribed to ΔABC and contains the hyperbolic centroid G and the hyperbolic orthocentre H. Due to *Desargues' Theorem* a pair of triangles perspective from a point is also perspective from a line: the Desargues axis. In [8] we proved that the set of all Desargues axes to the triangle ΔABC and triangles $\Delta A_1B_1C_1$ from $\mathcal{S}_1$ is tangent to a conic l inscribed to ΔABC. This conic l is called the *second Kiepert conic*.

2.3. Another construction of Kiepert triangles in $\mathcal{H}$

In 2.1 we placed hyperbolic isosceles triangles on the edges of ΔABC to obtain the Kiepert triangle. Here we suggest another construction of the Kiepert triangle $\Delta A_1B_1C_1$ using a chain of h-reflections. This can be done because h-reflections do not change the angle between lines. Each line l in the plane is the axis of an h-reflection σ_l which is a homology in the automorphism group of m. The absolute pole of l is the centre of σ_l. We use the h-reflections determined by the h-perpendicular bisectors d_+ of BC, e_+ of CA, f_+ of AB and the hyperbolic angle bisectors w_A, w_B, and w_C. We apply the following chain of h-reflections to $AB_1 \in A(x)$ to generate the edge B_1C of $\Delta\ AB_1C$, the edges CA_1, A_1B of ΔCA_1B, and the edges BC_1, C_1A of ΔBC_1A:

$$\begin{array}{rcl} B_1C & = & \sigma_{s_{e_+}}(AB_1), \\ CA_1 & = & \sigma_{w_C} \circ \sigma_{s_{e_+}}(AB_1), \\ A_1B & = & \sigma_{s_{d_+}} \circ \sigma_{w_C} \circ \sigma_{s_{e_+}}(AB_1), \\ BC_1 & = & \sigma_{w_B} \circ \sigma_{s_{d_+}} \circ \sigma_{w_C} \circ \sigma_{s_{e_+}}(AB_1), \\ C_1A & = & \sigma_{s_{f_+}} \circ \sigma_{w_B} \circ \sigma_{s_{d_+}} \circ \sigma_{w_C} \circ \sigma_{s_{e_+}}(AB_1). \end{array} \tag{1}$$

Obviously, the reflection σ_{w_A} applied to C_1A delivers AB_1. For the overall composition we thus have $\sigma_{w_A} \circ \sigma_{s_{f_+}} \circ \sigma_{w_B} \circ \sigma_{s_{d_+}} \circ \sigma_{w_C} \circ \sigma_{s_{e_+}}|_{A(x)} \equiv id|_{A(x)}$. Varying the line AB_1 in the pencil $A(x)$ we get the one-parametric set $\mathcal{S}_1$ of Kiepert triangles $\Delta A_1B_1C_1$. Let $C(x)$ denote the pencil of lines x centred at C and $(w_C,\ \overline{w}_C)$ the hyperbolic angle bisectors. Both reflections in w_C and $\overline{w}_C$ operate identically on the elements of $C(x)$: $w_C|_{C(x)} \equiv \overline{w}_C|_{C(x)}$. Therefore the choice of $\sigma_{\overline{w}_C}$ instead of σ_{w_C} would not change the result of (1). The respective statement holds for $B(x)$, $(w_B,\ \overline{w}_B)$. So we can conveniently confine ourselves to the reflections in w_A, w_B, w_C. The six lines used to define the chain of reflections (1) are the triple of h-angle bisectors w_A, w_B, w_C intersecting at the incentre I of ΔABC and the triple of h-perpendicular bisectors d_+, e_+, f_+ intersecting at the circumcentre O_1 of the triangle ΔABC.

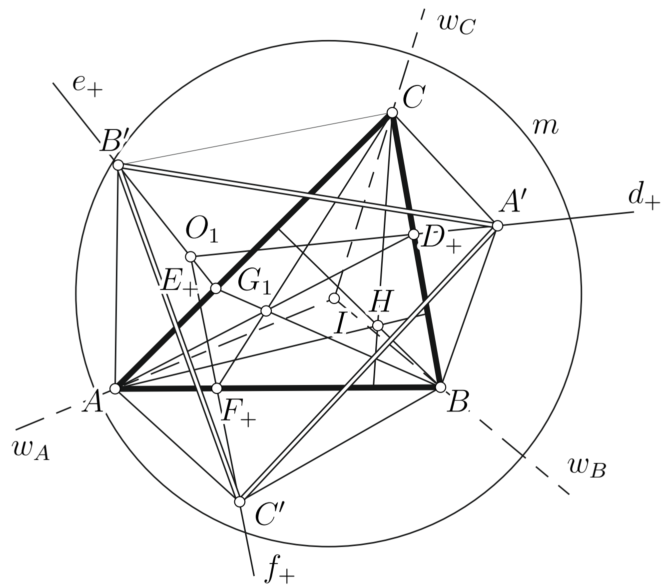

Figure 1: A triangle ΔABC in the hyperbolic plane $\mathcal{H}$, with h-angle bisectors w_A, w_B, w_C and h-perpendicular bisectors d_+, e_+, f_+ together with one of its Kiepert triangles $\Delta A'B'C'$

3. Construction of Kiepert triangles in the CK-geometry $CK_{\mathcal{H}}$

The CK-geometry $CK_{\mathcal{H}}$ is closely related to the hyperbolic geometry $\mathcal{H}$. In order to clearly see the essential differences we start with the midpoint configuration of a triangle ΔABC.

3.1. The configuration of midpoints of a triangle in the CK-geometry $CK_{\mathcal{H}}$

Let ΔABC be a triangle in $CK_{\mathcal{H}}$ such that all points A, B, C are either inner points of m or outer points of m and no edge of ΔABC is tangent to m. We further demand that either all three edges of the triangle intersect the conic m in three pairs of real points or that all three edges intersect in pairs of conjugate complex points. If these conditions are fulfilled the triangle will be referred to as *admissible*.

In the following considerations we use that an admissible triangle ΔABC determines a complete midpoint-configuration: Each side of ΔABC has two midpoints. We label them D_+, D_- on BC, E_+, E_- on CA, and F_+, F_- on AB. These points are the six corners of a quadrilateral. Each side of the quadrilateral carries a triple of collinear points: (D_+, E_+, F_-), $(D_-, E_+, F_+,)$, (D_+, E_-, F_+), (D_-, E_-, F_-). Applying the absolute polarity we obtain a new configuration. The six h-midpoints are mapped onto the six h-perpendicular bisectors d_-, d_+, e_-, e_+, f_- and f_+. The four sides of the quadrilateral are mapped onto the four h-circumcentres O_1, O_2, O_3, O_4 of ΔABC. Therefore each h-circumcentre O_i coincides with three h-perpendicular bisectors

$$\begin{aligned} O_1 &= d_+ \cap e_+ \cap f_+, \\ O_2 &= d_- \cap e_+ \cap f_-, \\ O_3 &= d_+ \cap e_- \cap f_-, \\ O_4 &= d_- \cap e_- \cap f_+. \end{aligned} \tag{2}$$

Each triple of collinear midpoints determines three remaining midpoints that form a triangle. We get the triangles $\Delta_1 D_+, E_+, F_+$, $\Delta_2 D_-, E_+, F_-$, $\Delta_3 D_+, E_-, F_-$, $\Delta_4 D_-, E_-, F_+$ and the

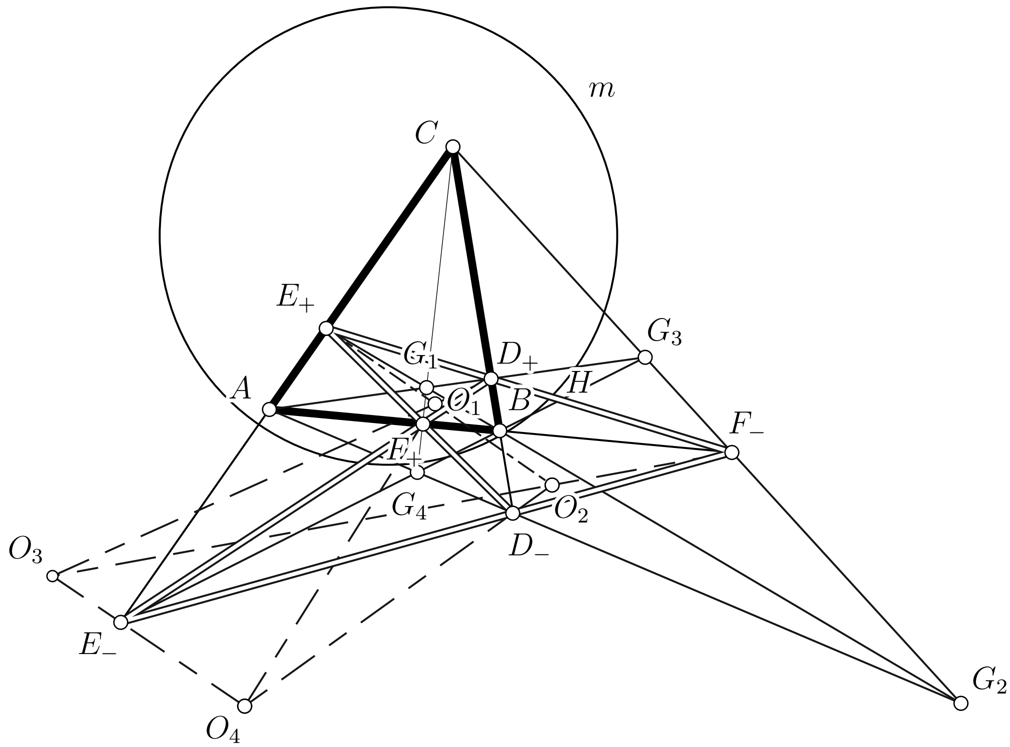

Figure 2: A triangle ΔABC in the hyperbolic plane $\mathcal{H}$, its h-midpoints of edges and h-circumcentres and h-centroids

four h-centroids G_1, G_2, G_3, G_4 of ΔABC:

$$\begin{aligned} G_1 &= AD_+ \cap BE_+ \cap CF_+, \\ G_2 &= AD_- \cap BE_+ \cap CF_-, \\ G_3 &= AD_+ \cap BE_- \cap CF_-, \\ G_4 &= AD_- \cap BE_- \cap CF_+. \end{aligned} \tag{3}$$

We visualize the configuration in Fig. 2 where A, B and C are inner points of m. Then three midpoints are inner points, denoted by D_+, E_+, F_+, the h-circumcentres O_2, O_3, O_4 are outer points of m. In Fig. 2 O_1 is an inner point, but O_1 could either be an inner or an outer point of m, or possibly even a point on m. Mind that there can also be admissible triangles where all six midpoints are outer points and four h-circumcentres are inner points of m.

3.2. Construction of Kiepert triangles in CK-geometry $CK_{\mathcal{H}}$

Changing from the hyperbolic geometry $\mathcal{H}$ to the Cayley-Klein geometry $CK_{\mathcal{H}}$ we construct Kiepert triangles. Due to Subsection 3.1 we can generate a set of Kiepert triangles as in Subsection 2.3 with four different sets of reflection axes. In order to select one set of reflection axes we just have to pick out one h-circumcentre O_i. We call the set of Kiepert triangles $\mathcal{S}_i$ *associated with the h-circumcentre* O_i, $i \in \{1, \dots, 4\}$.

In Fig. 3 we selected the circumcenter O_2, i.e. the triple of h-perpendicular bisectors d_-, e_+, f_- to construct a Kiepert triangle $\Delta A'B'C'$ of the set $\mathcal{S}_2$. The triangle $\Delta_2 D_- E_+ F_-$ determines the centroid G_2.

4. Construction of Kiepert triangles in regular CK-geometries

The considerations in $CK_{\mathcal{H}}$ enable us to also complete the elliptic case $CK_{\mathcal{E}}$ and to record the overall result for regular CK-geometries.

4.1. Construction of Kiepert triangles in the elliptic CK-geometry $CK_{\mathcal{E}}$

In $CK_{\mathcal{E}}$ each line intersects m in a pair of conjugate complex points and each segment has two real midpoints in $CK_{\mathcal{E}}$ (e-midpoints). This is why every triangle ΔABC in $CK_{\mathcal{E}}$ is admissible. The construction of Kiepert triangles in $CK_{\mathcal{E}}$ is the same as in $CK_{\mathcal{H}}$.

4.2. Kiepert triangles in regular CK-geometries

Summing up the results we can maintain for both geometries:

Theorem 1. *In regular CK-geometries $CK_{\mathcal{H}}$ and $CK_{\mathcal{E}}$ each triangle ΔABC determines four different one-parametric sets $\mathcal{S}_i$ of Kiepert triangles $\Delta A'_i B'_i C'_i$, $i = 1, 2, 3, 4$.*

The concept of Kiepert conics in hyperbolic geometry $\mathcal{H}$ and Theorem 1 enable us to formulate the general results on Kiepert conics for regular CK-geometries.

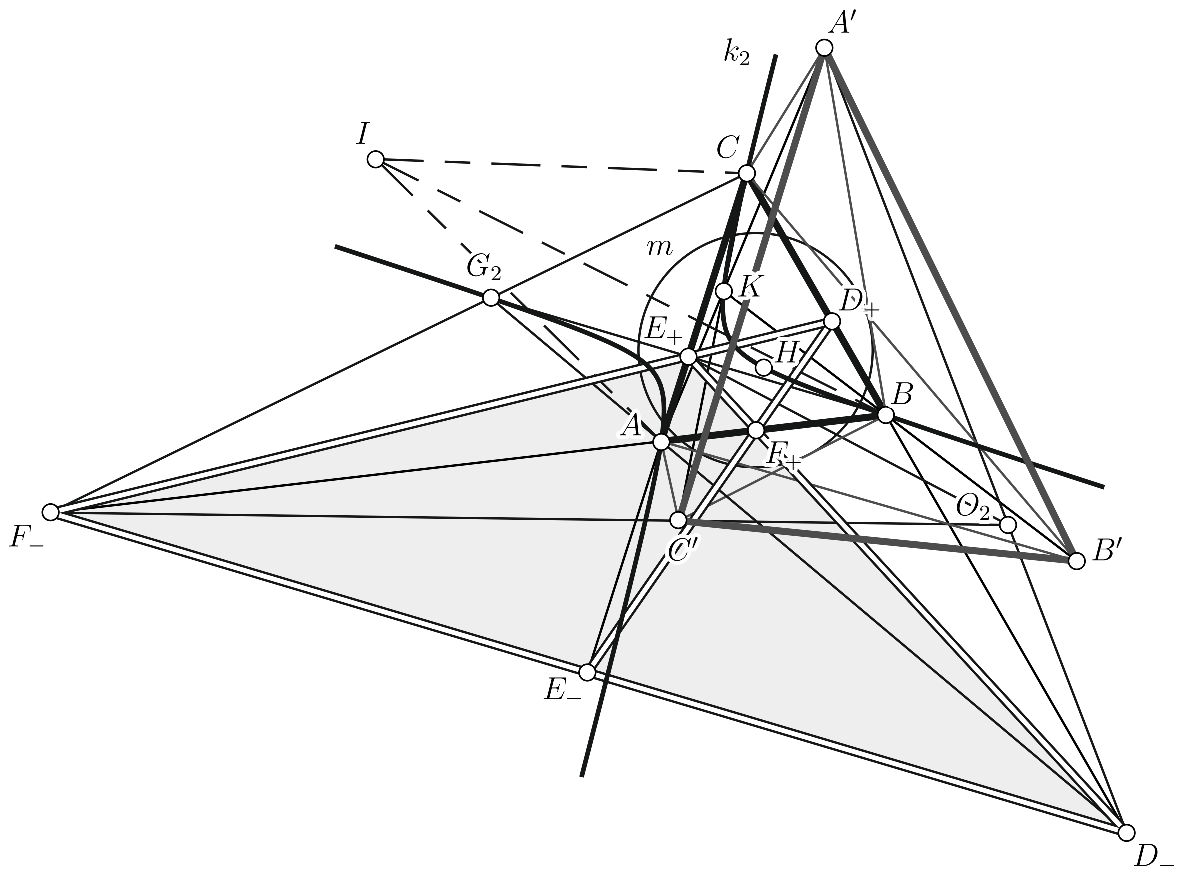

Figure 3: A triangle ΔABC in the CK-geometry $CK_{\mathcal{H}}$, h-perpendicular bisectors through the h-circumcentre O_2, the Kiepert triangle $\Delta A'B'C'$ and the Kiepert conic k_2

5. First Kiepert conics in regular CK-geometries

Let ΔABC be an admissible triangle in $CK_{\mathcal{H}}$. We select one of the one-parameter sets $\mathcal{S}_i$, $i \in \{1, 2, 3, 4\}$ of Kiepert triangles as explained in Subsection 3.2. According to Subsection 2.2 each triangle in $\mathcal{S}_i$ is perspective to ΔABC. From Subsection 2.2 and [8] we get that the centres K_i lie on the first Kiepert conic k_i associated with the circumcentre O_i. We additionally have that k_i is the conic through the points A, B, C, the h-orthocentre H and the respective h-centroid G_i.

If we replace the absolute conic m governing $CK_{\mathcal{H}}$ by a conic m without real points we arrive at $CK_{\mathcal{E}}$. In that case the set of outer points of m is empty. The projective construction of $\mathcal{S}_i$ and the construction of the first Kiepert conic k_i remains the same. So we can describe the whole process in a uniform way for these CK-geometries:

Theorem 2. *Let ΔABC be an admissible triangle in a regular CK-geometry. ΔABC and each Kiepert triangle $\Delta A'_iB'_iC'_i \in \mathcal{S}_i$ are perspective and the centre K_i is the intersection point of AA'_i, BB'_i and CC'_i. The locus of all such points K_i is a conic k_i. The triangle ΔABC defines a quadruple of* first Kiepert conics k_i, $i \in \{1, 2, 3, 4\}$ *(Fig. 4).*

Theorem 3. *Let ΔABC be an admissible triangle in a regular CK-geometry. A first Kiepert conic k_i is determined by the points A, B, C, the orthocentre H and a centroid G_i, $i \in \{1, 2, 3, 4\}$.*

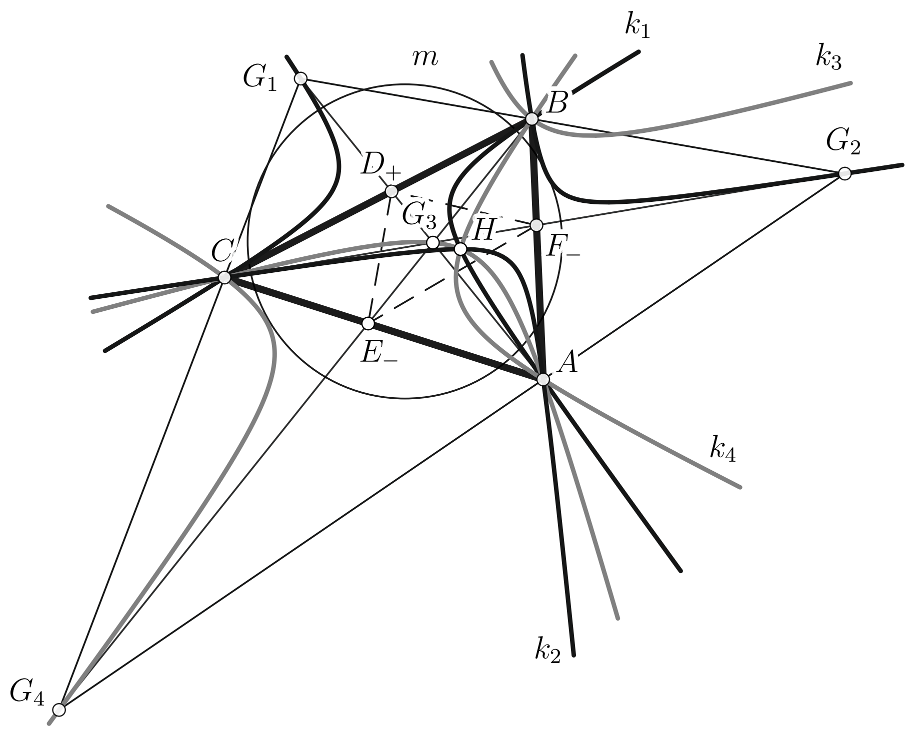

Figure 4: CK-geometry $CK_{\mathcal{H}}$: A triangle ΔABC and its quadruple of first Kiepert conics

6. Second Kiepert conics in regular CK-geometries

Let ΔABC be an admissible triangle and $\mathcal{S}_i$ be one of the sets of Kiepert triangles determined by ΔABC. According to Theorem 2 the triangle ΔABC and any of the Kiepert triangles

$\Delta A_i'B_i'C_i' \in \mathcal{S}_i$ are perspective from a centre K_i on the first Kiepert conic k_i , $i \in \{1,2,3,4\}$. In order to describe the set of Desargues axes we use the arguments from [8] which are genuinely projective; thus, the result can be transferred to $CK_{\mathcal{H}}$ and $CK_{\mathcal{E}}$. This way we arrive at a general description of the second Kiepert conics in CK-geometries:

Theorem 4. *Let ΔABC be an admissible triangle in a regular CK-geometry. Let $\mathcal{S}_i$, $i \in \{1,2,3,4\}$ be a one-parametric subset of Kiepert triangles. Each triangle $\Delta A_i'B_i'C_i' \in \mathcal{S}_i$ is perspective to ΔABC and the set of Desargues axes envelops a conic l_i inscribed to ΔABC. We overall get four conics l_1, l_2, l_3, l_4, the quadruple of second Kiepert conics.*

7. Conclusions

In this article we have been developing results on Kiepert triangles and Kiepert conics for both types of regular CK-geometries. In [8] we had been dealing with Kiepert triangles and Kiepert conics in the hyperbolic plane $\mathcal{H}$. The considerations and computations in that former paper enabled us to prove results on Kiepert triangles and Kiepert conics in regular CK-geometries. In doing so we extensively used projective geometry.

In the isotropic plane — a singular CK-geometry — the geometry of triangles has been addressed in [10]. Some further contributions to projective triangle geometry can be found in [11]. Recently [7] presented new results on Feuerbach hyperbolae in the affine plane which can directly be transferred to elliptic CK-geometry.

References

[1] J. Coolidge: *The Elements of Non-Euclidean Geometry.* 1st ed., Clarendon Press, Oxford 1909, reprinted 1927.

[2] H.M.S. Coxeter: *Non-Euclidean Geometry.* 5th ed., University of Toronto Press 1965, reprinted 1968.

[3] R.H. Eddy, R. Fritsch: *The Conics of Ludwig Kiepert – A Comprehensive Lesson in the Geometry of the Triangle.* Mathematics Magazine **67**, no. 3, 188–205 (1997).

[4] M.J. Greenberg: *Euclidean and Non-Euclidean Geometries.* 3rd ed., W.H. Freeman, New York 1993.

[5] L. Kiepert: *Solution de question 864.* Nouvelles Annales de Mathématiques **8**, 40–42 (1869).

[6] F. Klein: *Vorlesungen über Nicht-Euklidische Geometrie.* Springer, Berlin 1928, reprinted 1968.

[7] T. Kouřilová, O. Röschel: *A remark on Feuerbach Hyperbolas.* J. Geom. **104**, 317–328 (2013).

[8] S. Mick, J. Lang: *On Kiepert Conics in the Hyperbolic Plane.* J. Geometry Graphics **16**, 1–11 (2012).

[9] J. Richter-Gebert: *Perspectives on Projective Geometry.* Springer, Heidelberg 2011.

[10] V. Volenec, Z. Kolar-Begović, R. Kolar-Šuper: *Kiepert Triangles in an Isotropic Plane.* Sarajevo Journal of Mathematics **7**, no. 19, 81–90 (2011).

[11] N.J. Wildberger: *Universal Hyperbolic Geometry II.* KoG **14**, 3–24 (2010).

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