Journal for Geometry and Graphics
Volume 17 (2013), No. 2, 177–191.

Ruled Surfaces Asymptotically Normalized

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Dedicated to Nikolaus K. Stephanidis on the occasion of his 85^{th} birthday

Abstract. We consider a skew ruled surface Φ in the Euclidean space E^3 and relative normalizations of it, so that the relative normals at each point lie in the corresponding asymptotic plane of Φ. We call such relative normalizations and the resulting relative images of Φ *asymptotic*. We determine all ruled surfaces and the asymptotic normalizations of them, for which Φ is a relative sphere (proper or inproper) or the asymptotic image degenerates into a curve. Moreover we study the sequence of the ruled surfaces $\{\Psi_i\}_{i\in\mathbb{N}}$, where Ψ_1 is an asymptotic image of Φ and Ψ_i, for $i \geq 2$, is an asymptotic image of Ψ_{i-1}. We conclude the paper by the study of various properties concerning some vector fields, which are related with Φ.

Key Words: Ruled surfaces, relative normalizations

MSC 2010: 53A25, 53A05, 53A15, 53A40

1. Preliminaries

Here we sum up briefly some elementary facts concerning the relative Differential Geometry of surfaces and the Differential Geometry of ruled surfaces in the Euclidean space E^3; for notations and definitions the reader is referred to [6] and [8].

In the Euclidean space E^3 let $\Phi: \ \overline{x} = \overline{x}(u,v)$ be an injective C^r-immersion defined on a region U of $\mathbb{R}^2$, with non-vanishing Gaussian curvature. A C^s-mapping $\overline{y}: \ U \to E^3$, $r > s \geq 1$, is called a C^s-relative normalization of Φ if

$$\operatorname{rank}\left(\{\overline{x}_{/1},\, \overline{x}_{/2},\, \overline{y}\}\right) = 3, \quad \operatorname{rank}\left(\{\overline{x}_{/1},\, \overline{x}_{/2},\, \overline{y}_{/i}\}\right) = 2,\ i = 1,2,\ \forall\,(u,v) \in U, \tag{1}$$

where

$$f_{/i} := \frac{\partial f}{\partial u^i}, \quad f_{/ij} := \frac{\partial^2 f}{\partial u^i \partial u^j} \quad \text{etc.}$$

denote partial derivatives of a function (or a vector-valued function) f in the coordinates $u^1 := u$, $u^2 := v$. The covector $\overline{X}$ of the tangent plane is defined by

$$\langle \overline{X}, \overline{x}_{/i} \rangle = 0 \quad (i = 1,2) \quad \text{and} \quad \langle \overline{X}, \overline{y} \rangle = 1, \tag{2}$$

ISSN 1433-8157/$ 2.50

where $\langle\, ,\, \rangle$ denotes the standard scalar product in E^3. The relative metric G is introduced by

$$G_{ij} = \langle \overline{X}, \overline{x}_{/ij} \rangle. \tag{3}$$

The *support function of the relative normalization* $\overline{y}$ is defined by $q := \langle \overline{\xi}, \overline{y} \rangle$ (see [5, p. 196]), where $\overline{\xi}$ is the Euclidean normalization of Φ. By virtue of (1) q never vanishes on U and, because of (2), $\overline{X} = q^{-1}\overline{\xi}$. Then by (3), we also obtain

$$G_{ij} = q^{-1}\, h_{ij}, \tag{4}$$

where h_{ij} are the coefficients of the second fundamental form of Φ. Conversely, when a support function q is given, then the relative normalization $\overline{y}$ is uniquely determined by (see [5])

$$\overline{y} = -h^{(ij)}\, q_{/i}\, \overline{x}_{/j} + q\, \overline{\xi}, \tag{5}$$

where $h^{(ij)}$ are the coefficients of the inverse tensor of h_{ij}. For a function (or a vector-valued function) f we denote by $\nabla^G f$ the first Beltrami differential operator and by $\nabla^G_i f$ the covariant derivative, both with respect to the relative metric. We consider the coefficients

$$A_{ijk} := \left\langle \overline{X},\, \nabla^G_k \nabla^G_j \overline{x}_{/i} \right\rangle$$

of the Darboux tensor. Then, by using the relative metric tensor G_{ij} for "raising and lowering the indices", the Tchebychev vector $\overline{T}$ of the relative normalization $\overline{y}$ is defined by

$$\overline{T} := T^m\, \overline{x}_{/m} \quad \text{where} \quad T^m := \frac{1}{2} A_i^{im} \tag{6}$$

and the Pick invariant by

$$J := \frac{1}{2} A_{ijk} A^{ijk}. \tag{7}$$

The relative shape operator has the coefficients B_i^j defined by

$$\overline{y}_{/i} =: -B_i^j\, \overline{x}_{/j}. \tag{8}$$

Then, the relative curvature and the relative mean curvature are defined by

$$K := \det\left(B_i^j\right), \quad H := \frac{B_1^1 + B_2^2}{2}. \tag{9}$$

When we attach the vectors $\overline{y}$ of the relative normalization to the origin, the endpoints of them describe the relative image of Φ.

Let now Φ be a skew (non-developable) ruled C^2-surface, which is defined by its striction curve $\Gamma:\ \overline{s} = \overline{s}(u)$, $u \in I$ ($I \subset \mathbb{R}$ open interval) and the unit vector $\overline{e}$ pointing along the generators. We choose the parameter u to be the arc length along the spherical curve $\overline{e} = \overline{e}(u)$ and we denote the differentiation with respect to u by a prime. Then a parametrization of the ruled surface Φ over the region $U := I \times \mathbb{R}$ is

$$\overline{x}(u, v) = \overline{s}(u) + v\, \overline{e}(u), \tag{10}$$

with

$$|\overline{e}| = |\overline{e}'| = 1, \quad \langle \overline{s}'(u), \overline{e}'(u) \rangle = 0 \quad \forall\, u \in I. \tag{11}$$

The distribution parameter $\delta(u) := (\overline{s}', \overline{e}, \overline{e}')$, the conical curvature $\kappa(u) := (\overline{e}, \overline{e}', \overline{e}'')$ and the function $\lambda := \cot\sigma$, where $\sigma := \sphericalangle(\overline{e}, \overline{s}')$ is the striction of Φ ($-\frac{\pi}{2} < \sigma \leq \frac{\pi}{2}$, $\text{sign}\sigma = \text{sign}\delta$), are the fundamental invariants of Φ and determine uniquely, up to Euclidean rigid motions, the ruled surface Φ. The moving frame of Φ is the orthonormal frame which is attached to the striction point $\overline{s}(u)$, and consists of the vector $\overline{e}(u)$, the central normal vector $\overline{n}(u) := \overline{e}'(u)$ and the central tangent vector $\overline{z}(u) := \overline{e}(u) \times \overline{n}(u)$. It fulfils the equations [6, p. 280]

$$\overline{e}' = \overline{n}, \quad \overline{n}' = -\overline{e} + \kappa\,\overline{z}, \quad \overline{z}' = -\kappa\,\overline{n}. \tag{12}$$

Then, we have

$$\overline{s}' = \delta\lambda\overline{e} + \delta\overline{z}. \tag{13}$$

By (10) and (13) we also obtain

$$\overline{x}_{/1} = \delta\lambda\overline{e} + v\overline{n} + \delta\overline{z}, \quad \overline{x}_{/2} = \overline{e}, \tag{14}$$

and thus

$$\overline{\xi} = \frac{\delta\overline{n} - v\overline{z}}{w}, \quad \text{where} \quad w := \sqrt{v^2 + \delta^2}. \tag{15}$$

The coefficients g_{ij} and h_{ij} of the first and the second fundamental form of Φ take the form

$$g_{11} = w^2 + \delta^2\lambda^2, \quad g_{12} = \delta\lambda, \quad g_{22} = 1, \tag{16}$$

$$h_{11} = -\frac{\kappa w^2 + \delta' v - \delta^2\lambda}{w}, \quad h_{12} = \frac{\delta}{w}, \quad h_{22} = 0. \tag{17}$$

The Gaussian curvature $\widetilde{K}$ of Φ is given by (E. Larmarle's formula [6])

$$\widetilde{K} = -\frac{\delta^2}{w^4}. \tag{18}$$

In this paper only skew ruled surfaces of the space E^3 are considered with parametrization like in (10) and (11).

2. Ruled surfaces relatively normalized

Let $\overline{y}$ be a relative normalization of a given ruled C^2-surface Φ ($\delta \neq 0$) and let q be the corresponding support function. Then, on account of (4) and (17) the coefficients of the inverse relative metric tensor are computed by

$$G^{(11)} = 0, \quad G^{(12)} = \frac{wq}{\delta}, \quad G^{(22)} = \frac{wq\,(\kappa w^2 + \delta' v - \delta^2\lambda)}{\delta^2}. \tag{19}$$

The relative normalization $\overline{y}$ of Φ can be expressed with respect to the moving frame $\{\overline{e}, \overline{n}, \overline{z}\}$, by using (5), (14), (15) and (17), as follows:

$$\overline{y} = -w\frac{\delta q_{/1} + q_{/2}(\kappa w^2 + \delta' v)}{\delta^2}\overline{e} + \frac{\delta^2 q - w^2 v q_{/2}}{\delta w}\overline{n} - \frac{vq + w^2 q_{/2}}{w}\overline{z}. \tag{20}$$

It is well known [5, p. 199], that the components of the Tchebychev vector $\overline{T}$ of $\overline{y}$ are given by

$$T^i = \left[\ln\left(\frac{|q|}{q_{AFF}}\right)\right]_{/j} G^{(ij)}, \tag{21}$$

where, by virtue of (18),

$$q_{AFF} = |\widetilde{K}|^{1/4} = \frac{|\delta|^{1/2}}{w} \tag{22}$$

denotes the support function of the equiaffine normalization $\overline{y}_{AFF}$. From the relations (18) and (19) we have

$$T^1 = \frac{w^2 q_{/2} + vq}{\delta w}, \quad T^2 = \frac{2\delta w^2 q_{/1} + \delta' q\,(\delta^2 - v^2)}{2\delta^2 w} + \frac{T^1\,(\kappa w^2 + \delta' v - \delta^2\lambda)}{\delta}. \tag{23}$$

Thus, by using (6) and (14), we obtain

$$\overline{T} = w\frac{q\,(2\kappa v + \delta') + 2\delta q_{/1} + 2q_{/2}(\kappa w^2 + \delta' v)}{2\delta^2}\,\overline{e} + \frac{vq + w^2 q_{/2}}{\delta w}\,(v\overline{n} + \delta\overline{z})\,. \tag{24}$$

Especially, the Tchebychev vector $\overline{T}_{EUK}$ of the Euclidean normalization ($q = 1$) reads

$$\overline{T}_{EUK} = w\frac{2\kappa v + \delta'}{2\delta^2}\,\overline{e} + \frac{v}{\delta w}\,(v\overline{n} + \delta\overline{z})\,. \tag{25}$$

We introduce now the tangential vector

$$\overline{Q} := \frac{1}{4}\,\nabla^G\left(\frac{1}{q}, \overline{x}\right) \tag{26}$$

of Φ. On account of (5) and (19) we have

$$\overline{y} - q\,\overline{\xi} = 4\,q\,\overline{Q}.$$

Thus, by (26), the vector $\overline{Q}$ *is in the direction of the tangential component of* $\overline{y}$.

Definition 1. *We call* $\overline{Q}$ *the support vector of* $\overline{y}$.

Its components with respect to the local basis $\{\overline{x}_{/1}, \overline{x}_{/2}\}$, because of (19) and (26), are

$$Q^1 = \frac{-w\,q_{/2}}{4\delta q}, \quad Q^2 = -w\frac{(\kappa w^2 + \delta' v - \delta^2\lambda)\,q_{/2} + \delta q_{/1}}{4\delta^2 q}. \tag{27}$$

By using (14) we find

$$\overline{Q} = -w\frac{\delta\,q_{/1} + q_{/2}(\kappa w^2 + \delta' v)}{4\delta^2 q}\,\overline{e} - \frac{w q_{/2}}{4\delta q}\,(v\overline{n} + \delta\overline{z})\,. \tag{28}$$

Denoting by $\overline{Q}_{AFF}$ the support vector of the equiaffine normalization $\overline{y}_{AFF}$ and using (22), (24), (25) and (28), we get the relations

$$\overline{T}_{EUK} = 4\,\overline{Q}_{AFF}, \quad \overline{T} = q\,\overline{T}_{EUK} - 4q\,\overline{Q}.$$

3. Asymptotic normalizations of ruled surfaces

First in this section we find all relative normalizations $\overline{y}$, so that the relative normals at each point P of Φ lie in the corresponding asymptotic plane, i.e., in the plane $\{P; \overline{e}, \overline{n}\}$. On account of (20), this is valid iff $vq + w^2 q_{/2} = 0$, or, equivalently, iff the support function q of $\overline{y}$ is of the form $q = f\, w^{-1}$, where $f = f(u)$ is an arbitrary non-vanishing C^1-function. By virtue of (24) we have

Proposition 2. *The following statements are equivalent:*

(a) The relative normals at each point P of Φ lie on the corresponding asymptotic plane.

(b) The Tchebychev vector $\overline{T}$ of $\overline{y}$ at each point P of Φ is parallel to the corresponding generator.

(c) The support function is of the form

$$q = \frac{f(u)}{w}, \quad f(u) \in C^1(I), \quad f(u) \neq 0. \tag{29}$$

Definition 3. *We call a support function of the form (29), as well as the corresponding relative normalization*

$$\overline{y} = \left[-\left(\frac{f}{\delta}\right)' + \frac{\kappa f}{\delta^2} v \right] \overline{e} + \frac{f}{\delta} \overline{n}, \tag{30}$$

and the resulting relative image of Φ asymptotic.

It is apparent from (22) and (29), that the *equiaffine normalization $\overline{y}_{AFF}$ is contained in the set of the asymptotic ones.* Support functions of ruled surfaces of the form (29) were introduced by the first author in [9].

We consider an asymptotically normalized by (30) ruled surface Φ. The Pick invariant of Φ is computed from (7), by using the well known equation [5, p. 196]

$$A_{ijk} = \frac{1}{q} \langle \overline{\xi}, \overline{x}_{/ijk} \rangle - \frac{1}{2} \left(G_{ij/k} + G_{jk/i} + G_{ki/j} \right) \tag{31}$$

and the relations (4), (14), (15) and (17). We easily find $A_{222} = 0$. Then, since the Darboux tensor is fully symmetric, we have

$$J = \frac{3}{2} \left(A_{112} A^{112} + A_{122} A^{122} \right). \tag{32}$$

On account of (31), by straightforward calculations, we get

$$A_{112} = \frac{2\delta f' - \delta' f}{2 f^2}, \quad A^{112} = A_{122} = 0, \quad A^{122} = f \frac{2\delta f' - \delta' f}{2\delta^3}.$$

Substitution in (32) gives $J = 0$. This generalizes a result on equiaffinely normalized ruled surfaces (see [1, p. 217]).

The relative curvature and the relative mean curvature of Φ are computed on account of (9). By using (8), (14) and (30), we find the coefficients of the relative shape operator

$$B_1^1 = \frac{-\kappa f}{\delta^2}, \quad B_2^1 = 0, \quad B_2^2 = \frac{-\kappa f}{\delta^2}, \tag{33}$$

$$B_1^2 = \frac{2\delta' f\,(\kappa v + \delta') - \delta\,[\kappa f' v + 2\delta' f' + f\,(\kappa' v + \delta'')] + \delta^2\,[f\,(1 + \kappa\lambda) + f'']}{\delta^3}, \tag{34}$$

so that

$$K = \frac{\kappa^2 f^2}{\delta^4}, \quad H = \frac{-\kappa f}{\delta^2}. \tag{35}$$

It is obvious that

- *The relative curvature and the relative mean curvature are constant along each generator of Φ. Moreover they are both constant iff the function f is of the form $f = c\,\delta^2\,\kappa^{-1}$, $c \in \mathbb{R}^*$.*
- *The only asymptotically normalized ruled surfaces, which are relative minimal surfaces (or of vanishing relative curvature) are the conoidal ones.*

The scalar curvature S of the relative metric G, which is defined formally and is the curvature of the pseudo-Riemannian manifold (Φ, G), is obtained by direct computation to be $S = H$. Substituting J, H and S in the *Theorema Egregium of the relative Differential Geometry* (see [5, p. 197]), which states that

$$H - S + J = 2\,T_i\,T^i,$$

it turns out that *the norm $\|T\|_G$ with respect to the relative metric of the Tchebychev vector $\overline{T}$ of any asymptotic normalization $\overline{y}$ of Φ vanishes identically.*

Let the ruled surface Φ be non-conoidal. We consider the covariant coefficients $B_{ij} = B_i^k\,G_{kj}$ of the relative shape operator and we denote by $\widetilde{B}$ the scalar curvature of the metric $B_{ij}\,du^i\,du^j$, which is defined formally just as the curvature S. Then, on account of (4), (17), (29), (33) and (34), it turns out that $\widetilde{B}$ equals 1.

From (30) it is obvious, that the asymptotic image of Φ degenerates into a point or into a curve iff Φ is conoidal. In this case we have

$$\overline{y} = -\left(\frac{f}{\delta}\right)'\overline{e} + \frac{f}{\delta}\,\overline{n}.$$

Furthermore, computing the derivative of $\overline{y}$ and using (12), it follows immediately that the asymptotic image of Φ degenerates

a) into a curve Γ_1, iff $f \neq \delta(c_1 \cos u + c_2 \sin u)$, $c_1, c_2 \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$, or

b) into a point, iff $f = \delta(c_1 \cos u + c_2 \sin u)$, $c_1, c_2 \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$.

In case (a) one readily verifies, that Γ_1 is a planar curve, whose radius of curvature equals $r = |(\frac{f}{\delta})'' + \frac{f}{\delta}|$. In case (b) the asymptotic normalization of Φ is constant. Consequently the ruled surface Φ is an improper relative sphere [3]. Hence we have

Proposition 4. *Let Φ be an asymptotically normalized ruled surface. The asymptotic image of Φ degenerates*

(a) into a curve, which is planar, iff Φ is conoidal and $f \neq \delta(c_1 \cos u + c_2 \sin u)$, $c_1, c_2 \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$,

(b) into a point, whereupon Φ is an improper relative sphere, iff Φ is conoidal and $f = \delta(c_1 \cos u + c_2 \sin u)$, $c_1, c_2 \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$.

Let now Φ be a proper relative sphere, i.e., its relative normals pass through a fixed point [4]. It is well known, that this is valid iff there exists a constant $c \in \mathbb{R}^*$ and a constant vector

$\overline{a}$, such that $\overline{x} = c\,\overline{y} + \overline{a}$. Taking partial derivatives of this last equation on account of (10), (12), (13), (30) and (35), we obtain

$$f = \frac{\delta^2}{c\kappa}, \quad (\kappa \neq 0), \tag{36}$$

$$\left(\frac{\delta}{\kappa}\right)'' + \frac{\delta}{\kappa}(1 + \kappa\lambda) = 0 \tag{37}$$

and

$$c\overline{y} = \left[-\left(\frac{\delta}{\kappa}\right)' + v\right]\overline{e} + \frac{\delta}{\kappa}\overline{n}. \tag{38}$$

We notice, that the relative curvature and the relative mean curvature of a proper relative sphere are constant.
Conversely, let us suppose, that the equations (36) and (37) are valid, where $c \in \mathbb{R}^*$. Then, because of (30), the equation (38) is valid as well. Moreover, from (13) and (37) we obtain

$$\left[-\left(\frac{\delta}{\kappa}\right)'\overline{e} + \frac{\delta}{\kappa}\overline{n}\right]' = \overline{s}'.$$

Therefore the striction curve Γ of Φ is parametrized by

$$\overline{s} = -\left(\frac{\delta}{\kappa}\right)'\overline{e} + \frac{\delta}{\kappa}\overline{n} + \overline{a}, \quad \overline{a} = \text{const.} \tag{39}$$

By combining this last relation with (10) and (38) we get $\overline{x} = c\,\overline{y} + \overline{a}$, which means that Φ is a proper relative sphere. Thus, we arrive at

Proposition 5. *An asymptotically normalized ruled surface Φ is a proper relative sphere iff the function f is given by (36) and its fundamental invariants are related as in the equation (37).*

We now assume, that the relative normals of Φ are parallel to a fixed plane E. Let $\overline{c}$ be a constant normal unit vector on E. Then $\langle \overline{y}, \overline{c}\rangle = 0$, whence, on account of (30), we find

$$\frac{\kappa f}{\delta^2}\langle \overline{e}, \overline{c}\rangle v + \left[-\left(\frac{f}{\delta}\right)'\langle \overline{e}, \overline{c}\rangle + \frac{f}{\delta}\langle \overline{n}, \overline{c}\rangle\right] = 0. \tag{40}$$

Differentiation of (40) relative to v yields $\kappa\langle \overline{e}, \overline{c}\rangle = 0$. Then, again from (40), we derive the system

$$\kappa\langle \overline{e}, \overline{c}\rangle = 0, \quad \left(\frac{f}{\delta}\right)'\langle \overline{e}, \overline{c}\rangle - \frac{f}{\delta}\langle \overline{n}, \overline{c}\rangle = 0.$$

In case of $\langle \overline{e}, \overline{c}\rangle \neq 0$, we obtain

$$\kappa = 0, \quad \left(\frac{f}{\delta}\right)'' + \frac{f}{\delta} = 0.$$

In this case $\overline{y}$ is constant, i.e., Φ is an improper relative sphere. In case of $\langle \overline{e}, \overline{c}\rangle = 0$, we have $\kappa = 0$ and (40) is identically fulfilled. So we have proved

Proposition 6. *If the relative normals of an asymptotically normalized ruled surface Φ are parallel to a fixed plane E, then Φ is conoidal. Furthermore Φ is either an improper relative sphere or its generators are parallel to E.*

We consider now a non-conoidal ruled surface which is asymptotically normalized by (30). In view of (35) we observe that *all points of Φ are relative umbilics* ($H^2 - K = 0$), a result which generalizes a result on equiaffinely normalized ruled surfaces (see [1, p. 218]) Thus, the relative principal curvatures k_1 and k_2 equal H. The parametrization of the unique relative focal surface of Φ, which initially reads

$$\overline{x}^* = \overline{s} + v\overline{e} + \frac{1}{H}\,\overline{y},$$

becomes

$$\overline{x}^* = \overline{s} - \frac{\delta}{\kappa}\,\overline{n} + \frac{\delta f' - \delta' f}{\kappa f}\,\overline{e},$$

i.e., *the focal surface degenerates into a curve Γ^* and all relative normals along each generator form a pencil of straight lines.* This generalizes a result on equiaffinely normalized ruled surfaces (see [8, p. 204]).

Let $P(u_0)$ be a point of the striction curve Γ of Φ and $R(u_0)$ the corresponding point on the focal curve Γ^*. If we consider all asymptotic normalizations of Φ, then the locus of the points $R(u_0)$ is a straight line parallel to the vector $\overline{e}\,(u_0)$. In this way we obtain a ruled surface Φ^*, whose generators are parallel to the vectors $\overline{e}(u)$, a parametrization of which reads

$$\Phi^*:\ x^* = \overline{s} - \frac{\delta}{\kappa}\,\overline{n} + v^*\,\overline{e},$$

which is *the asymptotic developable of* Φ (see [2, p. 51]). One easily verifies, that

$$\overline{s}^* = \overline{s} - \frac{\delta}{\kappa}\,\overline{n} + \left(\frac{\delta}{\kappa}\right)'\overline{e}$$

is a parametrization of the striction curve of Φ^*.

4. The relative image of an asymptotically normalized ruled surface

In this paragraph we consider a non-conoidal ruled surface Φ, which is asymptotically normalized by $\overline{y}$ via the support function $q = fw^{-1}$. The parametrization (30) of $\overline{y}$ shows, that the asymptotic image Ψ_1 of Φ is also a ruled surface, whose generators are parallel to those of Φ. Then, by a straightforward computation we can find the following parametrization of its striction curve

$$\Gamma_1:\ \overline{s}_1 = -\left(\frac{f}{\delta}\right)'\overline{e} + \frac{f}{\delta}\,\overline{n}. \tag{41}$$

Thus, if we put for convenience $\overline{y} = \overline{y}_1$, we can rewrite the parametrization (30) as

$$\Psi_1:\ \overline{y}_1 = \overline{s}_1 + v_1\,\overline{e}, \quad v_1 := -H\,v,$$

where H denotes the relative mean curvature of Φ (see (35)). Obviously Ψ_1 is parametrized like in (10) and (11). We use $\{\overline{e}, \overline{n}, \overline{z}\}$ as moving frame of Ψ_1. The fundamental invariants of Ψ_1 are given by

$$\delta_1 = -\delta H, \quad \kappa_1 = \kappa, \quad \lambda_1 = -\frac{\left(\frac{f}{\delta}\right)'' + \frac{f}{\delta}}{\kappa\frac{f}{\delta}}. \tag{42}$$

From the above the following results, which can be checked fairly easily are listed:

- *If* Φ *and its asymptotic image* Ψ_1 *are congruent* ($\delta=\delta_1, \kappa=\kappa_1, \lambda=\lambda_1$), *then*

$$f=\frac{\delta^2}{\kappa} \quad \text{and} \quad \left(\frac{\delta}{\kappa}\right)'' + \frac{\delta}{\kappa}(1+\kappa\lambda)=0,$$

  and thus Φ *is a proper relative sphere* (see Proposition (5)).

- Ψ_1 *is orthoid* ($\lambda_1=0$) *iff* $f=\delta\,(c_1\cos u + c_2 \sin u)\,,\ c_1,c_2\in\mathbb{R},\ c_1^2+c_2^2\neq 0$.

- *The striction curve of* Ψ_1 *is an asymptotic line of it* ($\kappa_1=\lambda_1$) *iff*

$$\left(\frac{f}{\delta}\right)'' + \frac{f}{\delta}\left(1+\kappa^2\right)=0,$$

  and *it is an Euclidean line of curvature of it* ($1+\kappa_1\lambda_1=0$) *iff* $f=\delta\,(c_1 u + c_2)$, $c_1,c_2\in\mathbb{R},\ c_1^2+c_2^2\neq 0$.

- Ψ_1 *is an Edlinger surface*[1] ($\delta_1'=1+\kappa_1\lambda_1=0$ [2, p. 36]) *iff*

$$f=\frac{c\delta}{\kappa} \quad \text{and} \quad \kappa=\frac{1}{c_1 u + c_2}, \quad c,c_1,c_2\in\mathbb{R}, \quad c\neq 0,\ c_1^2+c_2^2\neq 0.$$

For $f=|\delta|^{1/2}$, i.e., for the equiaffine normalization, some of the above results were obtained in [10, § 4].

We now assume that Φ has a "precedent" ruled surface, i.e., that there exists another skew ruled surface, say Ψ^*, with parallel generators, an asymptotic image of which is Φ. We consider a parametrization of Ψ^* like in (10)–(11) and let $\delta^*,\kappa^*,\lambda^*$ be its fundamental invariants. We denote likewise all magnitudes of Ψ^* by the usual symbols supplied with a star (*). We normalize Ψ^* asymptotically via the support function $q^*=f^*{w^*}^{-1}$, and suppose that the resulting normalization of it, say Ψ^{**}, is the given ruled surface Φ. Then, on account of (42), clearly $\kappa^*=\kappa$ and

$$\delta=-\delta^* H^*, \quad \lambda=-\frac{\left(\frac{f^*}{\delta^*}\right)''+\frac{f^*}{\delta^*}}{\kappa\frac{f^*}{\delta^*}}, \tag{43}$$

where, in view of (35), $H^*=-{\delta^*}^{-2}\,\kappa\, f^*$ is the relative mean curvature of Φ^*. Thus the system (43) becomes

$$\frac{f^*}{\delta^*}=\frac{\delta}{\kappa}, \quad \left(\frac{\delta}{\kappa}\right)''+\frac{\delta}{\kappa}(1+\kappa\lambda)=0. \tag{44}$$

Let, conversely, the relations (44) be valid. We consider an arbitrary skew ruled surface Ψ^*, whose generators are parallel to those of Φ, and let δ^* be its distribution parameter. The conical curvature of Ψ^* equals κ. We normalize asymptotically Ψ^* via the support function $q^*=f^*\,{w^*}^{-1}$, where $f^*=\delta\,\delta^*\,\kappa^{-1}$. We can easily verify, by using (42) and (44), that the fundamental invariants of the asymptotic image Ψ^{**} of Ψ^* coincide with the corresponding fundamental invariants of Φ. Hence Ψ^{**} and Φ are congruent. So we arrive at

Proposition 7. *The ruled surface* Φ *is the asymptotic image of a ruled surface* Ψ^* *iff the second of the conditions (44) is valid.*

[1] i.e., its osculating quadrics are rotational hyperboloids [2]

We suppose now that Φ is not a proper relative sphere ($\Phi \neq \Psi_1$) and we normalize asymptotically its asymptotic image Ψ_1. Let $q_1 = f_1 w_1^{-1}$ be the support function of $\overline{y}_1$. Analogously to the computations above we get the following parametrization of the asymptotic image Ψ_2 of Ψ_1:

$$\Psi_2\colon\ \overline{y}_2 = \overline{s}_2 + v_2\,\overline{e}, \quad v_2 := -H_1\,v_1, \quad H_1 = \frac{f_1}{fH},$$

where

$$\Gamma_2\colon\ \overline{s}_2 = -\left(\frac{f_1}{\delta_1}\right)'\overline{e} + \frac{f_1}{\delta_1}\,\overline{n}$$

is its striction curve and H_1 is the relative mean curvature of Ψ_1. Thus Ψ_2 is parametrized like in (10) and (11). Obviously, the Tchebychev vector $\overline{T}_1$ of $\overline{y}_1$ is parallel to $\overline{e}$. The fundamental invariants of Ψ_2 are computed by (see (42))

$$\delta_2 = -\delta_1 H_1, \quad \kappa_2 = \kappa, \quad \lambda_2 = -\frac{\left(\frac{f_1}{\delta_1}\right)'' + \frac{f_1}{\delta_1}}{\kappa\frac{f_1}{\delta_1}}.$$

According to Proposition (5) we have: *The asymptotic image* Ψ_1 *of* Φ *is a proper relative sphere iff there exists a constant* $c \neq 0$*, such that* $cf_1 = fH$ (the condition (37) is identically fulfilled). Thus, we obtain the following results:

- Φ *and* Ψ_2 *are congruent iff*

$$f_1 = f \quad \text{and} \quad \left(\frac{\delta}{\kappa}\right)'' + \frac{\delta}{\kappa}(1+\kappa\lambda) = 0.$$

- Ψ_1 *and* Ψ_2 *are congruent iff* $\delta^2 f_1 = \kappa f^2$.
- Ψ_2 *is orthoid iff* $f_1 = \frac{\kappa f}{\delta}\,(c_1\cos u + c_2\sin u)$, $c_1, c_2 \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$.
- *The striction curve of* Ψ_2 *is an asymptotic line of it iff*

$$\left(\frac{\delta f_1}{\kappa f}\right)'' + \frac{\delta f_1}{\kappa f}\left(\kappa^2 + 1\right) = 0,$$

  and *it is an Euclidean line of curvature of it iff*

$$f_1 = \frac{\kappa f}{\delta}\left(c_1 u + c_2\right), \quad c_1, c_2 \in \mathbb{R}, \quad c_1^2 + c_2^2 \neq 0.$$

- Ψ_2 *is an Edlinger surface iff*

$$f_1 = \frac{cf}{\delta} \quad \text{and} \quad \kappa = \frac{1}{c_1 u + c_2}, \quad c, c_1, c_2 \in \mathbb{R}, \quad c \neq 0, \quad c_1^2 + c_2^2 \neq 0.$$

Continuing in the same way we obtain a sequence $\{\Psi_i\}_{i\in\mathbb{N}}$ of ruled surfaces, such that Ψ_i is the asymptotic image of Ψ_{i-1}. Moreover, if $q_{i-1} = f_{i-1}\,w_{i-1}^{-1}$ is the asymptotic support function of Ψ_{i-1}, we can easily check that the parametrization of Ψ_i reads

$$\Psi_i\colon\ \overline{y}_i = \overline{s}_i + v_i\,\overline{e}, \quad v_i := -H_{i-1}\,v_{i-1},$$

where

$$\Gamma_i\colon\ \overline{s}_i = -\left(\frac{f_{i-1}}{\delta_{i-1}}\right)'\overline{e} + \frac{f_{i-1}}{\delta_{i-1}}\,\overline{n}$$

is its striction curve and H_{i-1} is the relative mean curvature of Ψ_{i-1}. Ψ_i is parametrized like in (10) and (11) and its fundamental invariants are computed by

$$\delta_i = -\delta_{i-1}H_{i-1}, \quad \kappa_i = \kappa, \quad \lambda_i = -\frac{\left(\frac{f_{i-1}}{\delta_{i-1}}\right)'' + \frac{f_{i-1}}{\delta_{i-1}}}{\kappa\frac{f_{i-1}}{\delta_{i-1}}}.$$

The relative magnitudes of Ψ_{i-1} are recursively computed by

$$J_{i-1} = 0, \quad H_{i-1} = S_{i-1} = \frac{f_{i-1}}{f_{i-2}H_{i-2}}, \quad K_{i-1} = H_{i-1}^2.$$

Finally, we notice that the Tchebychev vectors of all asymptotic normalizations of the sequence $\{\Psi_i\}_{i\in\mathbb{N}}$ are parallel to $\overline{e}$ and that their asymptotic developables coincide with the director cone of Φ [6, p. 263].

5. Some results on the Tchebychev and the support vector fields

We consider a ruled surface Φ, which is asymptotically normalized by $\overline{y}$ via the support function $q = fw^{-1}$. The Tchebychev vector of $\overline{y}$ can be computed by using (24) and (29). We find

$$\overline{T} = \frac{2\delta f' - \delta' f}{2\delta^2}\,\overline{e}.$$

The divergence $\operatorname{div}^I \overline{T}$ and the rotation $\operatorname{curl}^I \overline{T}$ of $\overline{T}$ with respect to the first fundamental form I of Φ, which initially read [10, p. 304, 305]

$$\operatorname{div}^I \overline{T} = \frac{(wT^i)_{/i}}{w}, \quad \operatorname{curl}^I \overline{T} = \frac{(g_{12}T^1 + g_{22}T^2)_{/1} - (g_{11}T^1 + g_{12}T^2)_{/2}}{w},$$

become (see (16) and (23))

$$\operatorname{div}^I \overline{T} = \frac{v\,(2\delta f' - \delta' f)}{2\delta^2 w^2}, \quad \operatorname{curl}^I \overline{T} = \frac{\delta\,(2\delta f'' - 3\delta' f') + f\,(2\delta'^2 - \delta\delta'')}{2\delta^3 w},$$

from which we obtain:

- *It is* $\operatorname{div}^I \overline{T} \equiv 0$ *iff* $f = c|\delta|^{1/2}$, $c \in \mathbb{R}^*$, or equivalently *iff* $\overline{T} = \overline{0}$.
- *It is* $\operatorname{curl}^I \overline{T} \equiv 0$ *iff* $\delta\,(2\delta f'' - 3\delta' f') + f\,(2\delta'^2 - \delta\delta'') = 0$, or, after standard calculation, *iff* $f = |\delta|^{1/2}\left(c_1 \int |\delta|^{1/2} du + c_2\right)$, $c_1, c_2 \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$.

Let $\operatorname{div}^G \overline{T}$ and $\operatorname{curl}^G \overline{T}$ be the divergence and the rotation of $\overline{T}$ with respect to the relative metric. In analogy to the computation above we get

$$\operatorname{div}^G \overline{T} \equiv 0, \quad \operatorname{curl}^G \overline{T} \equiv 0.$$

The relation $\operatorname{curl}^G \overline{T} \equiv 0$ agrees with $\overline{T} = \nabla^G \left(f|\delta|^{-1/2}, \overline{x}\right)$ (see (21)).

The support vector $\overline{Q}$ of an asymptotic normalization becomes (see (28))

$$\overline{Q} = w\frac{\kappa f v + \delta' f - \delta f'}{4\delta^2 f}\,\overline{e} + \frac{v}{4\delta w}\,(v\overline{n} + \delta\overline{z})\,. \tag{45}$$

We observe, that $\langle\overline{e},\overline{Q}\rangle = 0$ iff

$$\kappa f v + \delta' f - \delta f' = 0.$$

On differentiating twice relative to v we obtain the system

$$\kappa f = \delta' f - \delta f' = 0,$$

which implies $\kappa = 0$ and $f = c|\delta|,\ c \in \mathbb{R}^*$. The inverse also holds. So we have: *The support vectors $\overline{Q}$ are orthogonal to the generators iff Φ is conoidal and $f = c\,|\delta|$, $c \in \mathbb{R}^*$.* On account of (27) a direct computation yields

$$\operatorname{div}^I \overline{Q} = \frac{3\kappa f v^2 + (\delta' f - 2\delta f')\, v + \delta^2 f\,(\kappa - \lambda)}{4\delta^2 f w}, \tag{46}$$

$$\operatorname{curl}^I \overline{Q} = \frac{A_3 v^3 + A_2 v^2 + A_1 v + A_0}{4\delta^3 f^2 w^2}, \tag{47}$$

where

$$\begin{aligned}
A_3 &= f^2\,(\delta\kappa' - 2\delta'\kappa)\,, && \text{(48a)}\\
A_2 &= -2\delta'^2 f^2 + \delta f\,(\delta' f' + \delta'' f) + \delta^2\left[f'^2 - 2f^2\,(1+\kappa\lambda) - f f''\right], && \text{(48b)}\\
A_1 &= \delta^2 f\big[\delta\lambda f' + f\,[\delta\kappa' - \delta'\,(\kappa+\lambda)]\big], && \text{(48c)}\\
A_0 &= -\delta^2\big[f^2(\delta'^2 - \delta\delta'') + \delta^2[f f'' + f^2\,(1+\kappa\lambda) - f'^2]\big]. && \text{(48d)}
\end{aligned}$$

Also we have

$$\operatorname{div}^G \overline{Q} = \frac{2\kappa f v^4 + (\delta' f - 2\delta f')\, v^3 + 3\delta^2\kappa f v^2 - 2\delta^3 f' v + \delta^4 f\,(\kappa - \lambda)}{4\delta^2 f w^3}, \tag{49}$$

and

$$\operatorname{curl}^G \overline{Q} \equiv 0. \tag{50}$$

Let $\operatorname{div}^I \overline{Q} = 0$. Then by (46) we have

$$3\kappa f v^2 + (\delta' f - 2\delta f')\, v + \delta^2 f\,(\kappa - \lambda) = 0,$$

from which, by successive differentiations relative to v, we infer the system

$$\kappa f = \delta' f - 2\delta f' = \delta^2 f\,(\kappa - \lambda) = 0,$$

i.e., $\kappa = \lambda = 0$ and $f = c\,|\delta|^{1/2}$, $c \in \mathbb{R}^*$. The inverse also holds. So we have: *It is* $\operatorname{div}^I \overline{Q} \equiv 0$ *iff Φ is a right conoid and $f = c\,|\delta|^{1/2}, c \in \mathbb{R}^*$.* Treating the relations (47)–(50) similarly we obtain the following results:

- *It is* $\operatorname{curl}^I \overline{Q} \equiv 0$ *iff*
  - Φ *is an Edlinger surface with constant invariants and* $f = c \in \mathbb{R}^*$, or
  - Φ *is a right conoid,* $\delta = \dfrac{c_1}{u + c_2}$ *and* $f = \dfrac{c_1 c_3}{(u+c_2)\sqrt{e^{u(u+2c_2)}}}$, $c_1, c_3 \in \mathbb{R}^*$, $c_2 \in \mathbb{R}$, or
  - *the fundamental invariants of Φ fulfil the relations*

    $$c_1^2\delta^6 - 5c_3\,[\delta\,(u + c_1) + c_3] = 0, \quad \kappa = c_1\delta^2, \quad \lambda = \frac{-c_1\delta^4}{c_3^2 + c_1^2\delta^6}, \quad c_1, c_2, c_3 \in \mathbb{R}^*,$$

    and $f = c_2\,|\delta|\, e^{c_3 \int \frac{du}{\delta}}$.

- *It is* $\mathrm{div}^G \overline{Q} \equiv 0$ *iff* Φ *is a right helicoid and* $f = c \in \mathbb{R}^*$.

We consider now the following families of curves on Φ:

a) the curved asymptotic lines,

b) the curves of constant striction distance (u-curves) and

c) the $\widetilde{K}$-curves, i.e., the curves along which the Gaussian curvature is constant [7].

The corresponding differential equations of these families of curves are

$$\kappa v^2 + \delta' v + \delta^2 (\kappa - \lambda) - 2\delta v' = 0, \tag{51}$$

$$v' = 0, \tag{52}$$

$$2\delta v v' + \delta' \left(\delta^2 - v^2\right) = 0. \tag{53}$$

It will be our task to investigate necessary and sufficient conditions for the support vector field $\overline{Q}$ to be tangential or orthogonal to one of these families of curves. To this end we consider a directrix $\Lambda\colon\ v = v(u)$ of Φ. Then we have

$$\overline{x}' = (\delta\lambda + v')\,\overline{e} + v\overline{n} + \delta\overline{z}. \tag{54}$$

From (45) and (54) it follows: $\overline{x}'$ and $\overline{Q}$ are parallel or orthogonal iff

$$\kappa f v^3 + (\delta' f - \delta f')\, v^2 + \delta f \left[\delta (\kappa - \lambda) - v'\right] v + \delta^2 (\delta' f - \delta f') = 0 \tag{55}$$

or

$$(\kappa f v + \delta' f - \delta f')\,(\delta\lambda + v') + \delta f v = 0, \tag{56}$$

respectively. Then, from (51) and (55) (resp. (56)), we infer, that $\overline{Q}$ is tangential or orthogonal to the curved asymptotic lines iff

$$\kappa f v^3 + (\delta' f - 2\delta f')\, v^2 + \delta^2 f (\kappa - \lambda)\, v + 2\delta^2 (\delta' f - \delta f') = 0 \tag{57}$$

or

$$\kappa^2 f v^3 + \kappa\,(2\delta' f - \delta f')\, v^2 + [\delta^2 \kappa f (\kappa + \lambda) + \delta' (\delta' f - \delta f') + 2\delta^2 f] v + \delta^2 (\delta' f - \delta f') (\kappa + \lambda) = 0, \tag{58}$$

respectively. From (57) and (58), after successive differentiations relative to v, we obtain

$$\kappa f = \delta' f - 2\delta f' = \delta^2 f (\kappa - \lambda) = 2\delta^2 (\delta' f - \delta f') = 0$$

and

$$\kappa^2 f = \kappa\,(2\delta' f - \delta f') = \delta^2 \kappa f (\kappa + \lambda) + \delta' (\delta' f - \delta f') + 2\delta^2 f = \delta^2 (\delta' f - \delta f') (\kappa + \lambda) = 0,$$

respectively. Standard treatment of these systems leads to the following results:

- $\overline{Q}$ *is tangential to the curved asymptotic lines of* Φ *iff* Φ *is a right helicoid and* $f{=}c \in \mathbb{R}^*$.
- $\overline{Q}$ *is orthogonal to the curved asymptotic lines of* Φ *iff* Φ *is a right conoid and the function* f *is given by* $f = c\,|\delta|\, e^{2\int \frac{\delta}{\delta'} du}, c \in \mathbb{R}^*$.

From (52) and (55), resp. (56), we obtain: $\overline{Q}$ is tangential or orthogonal to the u-curves iff

$$\kappa f v^3 + (\delta' f - \delta f')\, v^2 + \delta^2 f\,(\kappa - \lambda)\, v + \delta^2\,(\delta' f - \delta f') = 0$$

or

$$f\,(1 + \kappa\lambda)\, v + \lambda\,(\delta' f - \delta f') = 0,$$

respectively. Treating these polynomials in the same way we result:

- *$\overline{Q}$ is tangential to the u-curves of Φ iff Φ is a right conoid and $f = c\,|\delta|, c \in \mathbb{R}^*$.*
- *$\overline{Q}$ is orthogonal to the u-curves of Φ iff the striction curve of Φ is an Euclidean line of curvature and $f = c\,|\delta|, c \in \mathbb{R}^*$.*

From (53) and (55), resp. (56), we obtain: $\overline{Q}$ is tangential or orthogonal to the $\widetilde{K}$-curves iff

$$2\kappa f v^3 + (\delta' f - 2\delta f')\, v^2 + 2\delta^2 f\,(\kappa - \lambda)\, v + \delta^2\,(3\delta' f - 2\delta f') = 0$$

or

$$\delta'\kappa f v^3 + \left[2\delta^2 f\,(1+\kappa\lambda) + \delta'\,(\delta' f - \delta f')\right] v^2 + \delta^2\left[\delta' f\,(2\lambda - \kappa) - 2\delta\lambda f'\right] v - \delta^2\delta'\,(\delta' f - \delta f') = 0,$$

respectively. Treating analogously these polynomials we easily obtain:

- *$\overline{Q}$ is tangential to the $\widetilde{K}$-curves of Φ iff Φ is a right helicoid and $f = c \in \mathbb{R}^*$.*
- *$\overline{Q}$ is orthogonal to the $\widetilde{K}$-curves of Φ iff Φ is an Edlinger surface and $f = c \in \mathbb{R}^*$.*

To complete this work we consider the Euclidean lines of curvature of Φ. Their differential equation, initially being

$$g_{12}h_{11} - g_{11}h_{12} + (g_{22}h_{11} - g_{11}h_{22})\, v' + (g_{22}h_{12} - g_{12}h_{22})\, v'^2 = 0,$$

becomes, on account of (16) and (17),

$$\delta\left[w^2\,(1 + \kappa\lambda) + \delta'\lambda v\right] + \left[\kappa w^2 + \delta' v - \delta^2\lambda\right] v' - \delta v'^2 = 0,$$

from which, by virtue of (55), we infer, that $\overline{Q}$ is tangent to the one family of the lines of curvature of Φ iff

$$-\kappa f f' v^3 + \left[\delta f'^2 - \delta f^2\,(1 + \kappa\lambda) - \delta' f f'\right] v^2 + \delta f\,(\kappa - \lambda)\,(\delta' f - \delta f')\, v + \delta\,(\delta f' - \delta' f)^2 = 0.$$

It results the system

$$\kappa f f' = \left[\delta f'^2 - \delta f^2\,(1 + \kappa\lambda) - \delta' f f'\right] = \delta f\,(\kappa - \lambda)\,(\delta' f - \delta f') = \delta\,(\delta f' - \delta' f)^2 = 0,$$

from which we get

$$\delta' = 1 + \kappa\lambda = f' = 0.$$

Hence Φ *is an Edlinger surface and the function f is constant.* Moreover, we can easily confirm, that the Euclidean principal directions at a point P of an Edlinger surface read

$$v' = 0 \quad \text{and} \quad v' = \frac{\delta^2 + \kappa^2 w^2}{\delta\kappa}.$$

Since the second of these relations verifies (55), we have: *When Φ is an Edlinger surface and the function f is constant, then the support vector field $\overline{Q}$ is tangent to those Euclidean lines of curvature of Φ, which are orthogonal to the striction curve of Φ.*

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Received Juli 3, 2013; final form November 3, 2013