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The Generalization of Szabó's Theorem for Rectangular Cuboids and an Application

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Dedicated to
Professor Dr. Oswald Giering on the occasion of his 80^{th} birthday
and to
Professor Dr. Hellmuth Stachel on the occasion of his 70^{th} birthday

Abstract. The reference system of central axonometry is based on the planar image of a three-dimensional cube. Szabó's Theorem provides a criterion on when this reference system is the central projection of a cube. However, it is more likely that in a picture or photo the image of a rectangular cuboid can be found than the image of a cube. This article provides the criterion on when the central axonometry of a rectangular cuboid with given dimensions is the central projection of a rectangular cuboid.

Key Words: central projection, central axonometry, image processing, 3D reconstruction

MSC 2010: 51N05, 94A08

1. Introduction

In 1910 E. Kruppa [8] established the projective generalization of a classical axonometry from the Euclidean 3-space onto a plane. Later F. Hohenberg [7] distinguished two such axonometries, the *parallel axonometry* and the *central axonometry*. While, due to Pohlke's Theorem[1], each parallel axonometry is similar to a parallel projection, a central axonometry in general is not similar or congruent to a central (or perspective) projection. For non-collinear main vanishing points a central axonometry is always affine to a central projection. Analogue to Pohlke's theorem there were many researchers dealing with conditions and criteria for a central axonometry being a central projection (see, e.g., [2], [3], [4], [5], [7], [9], [12], [13], [15] or [16]).

[1]For proofs of Pohlke's Theorem see, e.g., E. Kruppa [8], H. Brauner [1] or H. Stachel [11].

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Already in 1978 the first-named author started his investigations [14] and continued in [15]. The paper [16] can be seen as a turning point, and the analytical criteria formulated in [15] and [16] are now called *Szabó's Theorem*. A. DÜR's publication [3], M. HOFFMANN's papers [5] and [6] as well as T. SCHWARCZ's article [9] and his dissertation [10] are in close relation to [16].

All mentioned articles define central axonometries by an axonometric reference system which is the image of a unit cube. This article shows that the criterion gained in [16] can easily be modified, if one replaces the given image of a cube by that of a right cuboid of given dimensions. If a central axonometric image is not congruent to a central projection, one can determine an affine transformation such that the central axonometry is transformed into a central projection.

2. Szabó's Theorem

Let the reference system $O(E_x, E_y, E_z, V_x, V_y, V_z)$ be given with distances $e = OE_x$, $f = E_xV_x$, $g = OE_y$, $h = E_yV_y$, $i = OE_z$, and $j = E_zV_z$. The angles of the finite triangle $V_xV_yV_z$ of vanishing points are denoted by α, β and γ, respectively.

Then by [16] *this reference system is the central projection of an orthonormal base system, i.e., of a Cartesian frame and the ideal points of the axes, if and only if the equation*

$$\left(\frac{e}{f}\right)^2 : \left(\frac{g}{h}\right)^2 : \left(\frac{i}{j}\right)^2 = \tan\alpha : \tan\beta : \tan\gamma \tag{1}$$

holds.

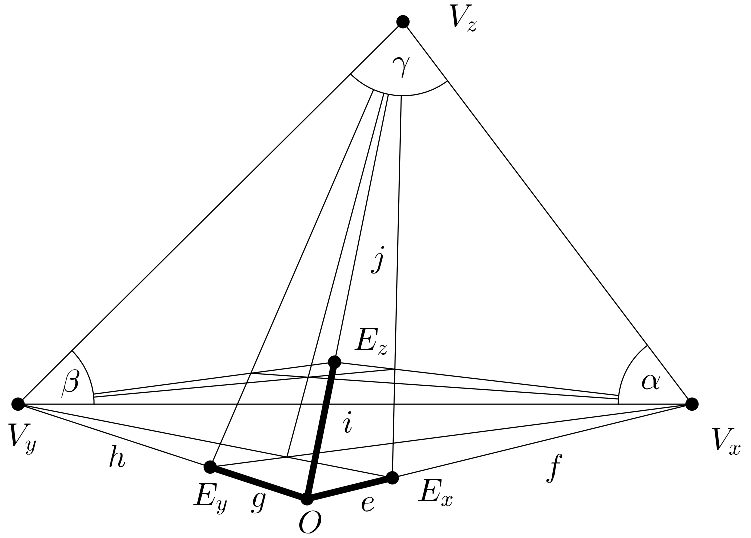

Figure 1: Szabó's Theorem for orthonormal bases

3. The new criterion

We modify Szabó's Theorem in the following way: We replace the unit cube by a right cuboid with given edge lengths a, b, c. For three edges with a common vertex, let a', b', c' be the lengths of their images with the common endpoint O (see Fig. 2). On the spanned lines let the finite points V_x, V_y and V_z be the images of the ideal points. We denote the distances of O from these vanishing points with $a' + a''$, $b' + b''$ and $c' + c''$. Again, α, β, γ are the interior angles in the triangle $V_x V_y V_z$.

Then the given image defines a central projection of the cuboid if and only if

$$\left(\frac{a'}{a''}\right)^2 : \left(\frac{b'}{b''}\right)^2 : \left(\frac{c'}{c''}\right)^2 = a^2 \tan\alpha : b^2 \tan\beta : c^2 \tan\gamma. \tag{2}$$

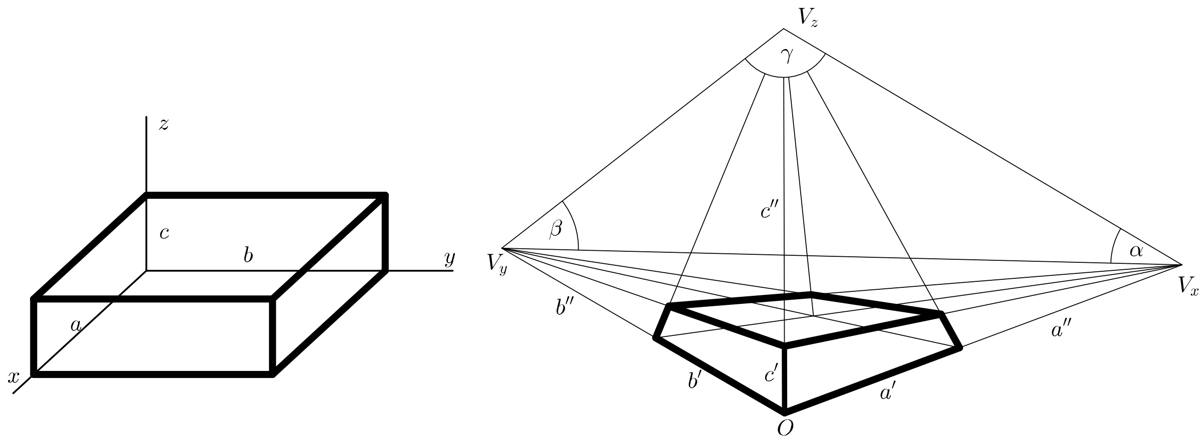

Figure 2: Szabó's Theorem for rectangular cuboids

Remark. When working with real photographs, the vanishing points of a modified reference system often exceed the limited space of a photo. However, once we can identify the images of two lines parallel to an axis, we can calculate the coordinates of the corresponding vanishing point. For the sake of simplicity, we always assume that the photos represent linear images, i.e., they are free of lens errors.

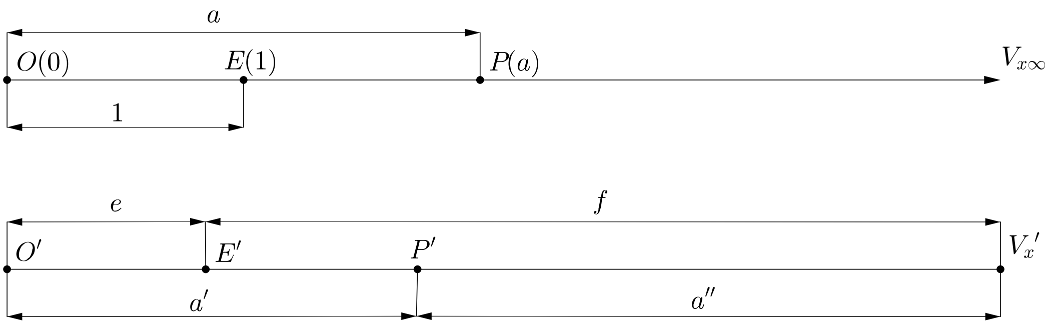

Figure 3: For cross-ratio calculations

Proof. The central axonometry maps each coordinate axis projectively onto its image. Therefore we are able to construct the images of the unit points. We will explain this in detail for the x-axis: Fig. 3 shows above the situation in space and below that in the image. Due to the projective invariance of cross-ratios we obtain

$$a = (PEOV_{x\infty}) = (PEO) = (P'E'O'V_x') = \frac{P'O'}{E'O'} : \frac{P'V'}{E'V'} = \frac{a'}{e} : \frac{a''}{f}\,,$$

$$a = \frac{a'f}{ea''} \quad \Longrightarrow \quad \frac{e}{f} = \frac{a'}{aa''} = \frac{1}{a} \cdot \frac{a'}{a''}\,.$$

Accordingly, the criterion (1) can be modified to

$$\left(\frac{1}{a} \cdot \frac{a'}{a''}\right)^2 : \left(\frac{1}{b} \cdot \frac{b'}{b''}\right)^2 : \left(\frac{1}{c} \cdot \frac{c'}{c''}\right)^2 = \tan\alpha : \tan\beta : \tan\gamma\,.$$

Or, in a shorter version

$$\left(\frac{a'}{a''}\right)^2 : \left(\frac{b'}{b''}\right)^2 : \left(\frac{c'}{c''}\right)^2 = a^2 \tan\alpha : b^2 \tan\beta : c^2 \tan\gamma. \qquad \square$$

Remark. There are typing errors in some of the mathematical expressions in [16]. Here, we would like to correct them[2].

4. Affine transformation of a central axonometry into a central projection

Obviously, the left hand side of Eq. (2) consists of proportions of ratios. Therefore it is invariant under affine transformations. This means that, to given a, b, c, the right hand side of (2) only depends on the angles of the triangle of the vanishing points. These angles will vary when one applies an affine transformation of the image plane. Our goal is to find a new triangle of vanishing points named by V'_x, V'_y and V'_z such that the criterion (2) is fulfilled. For this purpose we use the abbreviations

$$\left(\frac{a'}{a''}\right)^2 = A, \quad \left(\frac{b'}{b''}\right)^2 = B, \quad \left(\frac{c'}{c''}\right)^2 = C, \quad \tan\alpha' = u, \quad \tan\beta' = v,$$

where α', β' and γ' are the interior angles in the new triangle at points V'_x, V'_y and V'_z, respectively.

At the triangle of the new vanishing points the tangent of the third angle can be expressed by the tangents of the other two as

$$\tan\gamma' = \tan\left(180^\circ - (\alpha' + \beta')\right) = -\tan\left(\alpha' + \beta'\right) = \frac{u+v}{uv-1}\,.$$

[2] On [16, page 9] the equations before Section 4 should read

$$\left(\frac{j}{i}\right)^2 \tan\gamma \frac{\cos\alpha}{\sin\alpha} \sin^2\alpha \cos 2\beta + \left(\frac{j}{i}\right)^2 \tan\gamma \frac{\cos\beta}{\sin\beta} \sin^2\beta \cos 2\alpha$$
$$= \frac{1}{2}\left(\frac{j}{i}\right)^2 \tan\gamma \left(\sin 2\alpha \cos 2\beta + \sin 2\beta \cos 2\alpha\right) = \frac{1}{2}\left(\frac{j}{i}\right)^2 \tan\gamma(-\sin 2\gamma) = -\left(\frac{j}{i}\right)^2 \sin^2\gamma\,.$$

On [16, page 10] the second equation in (6) should read $k = \sqrt{\left(\frac{f}{e}\frac{h}{g}\right)^2 + \left(\frac{f}{e}\frac{j}{i}\right)^2 + \left(\frac{h}{g}\frac{j}{i}\right)^2}$.

It can be concluded from the criterion that

$$\begin{aligned}
A:B &= a^2u : b^2v\,; & A:C &= a^2u : c^2\tan\gamma' = a^2u : c^2\frac{u+v}{uv-1}\,;\\
Ab^2v &= Ba^2u\,; & Ac^2\frac{u+v}{uv-1} &= Cua^2 \implies c^2Au + c^2Av = Ca^2u^2v - Ca^2u\,;\\
v &= \frac{Ba^2}{Ab^2}u\,; & Ac^2u + Ca^2u &= v(Ca^2u^2 - Ac^2) = \frac{Ba^2}{Ab^2}u(Cu^2a^2 - Ac^2)\,;\\
&& A^2b^2c^2u + ACa^2b^2u &= BCa^4u^3 - ABa^2uc^2;\\
&& A^2b^2c^2 + ACa^2b^2 &= BCa^4u^2 - ABa^2c^2;\\
&& u^2 &= \frac{A^2b^2c^2 + ACa^2b^2 + ABa^2c^2}{BCa^4}\,.
\end{aligned}$$

Since the triangle $V_x'V_y'V_z'$ is an acute one, $0 < u$ is true. Consequently,

$$u = \sqrt{\frac{A^2b^2c^2 + ACa^2b^2 + ABa^2c^2}{BCa^4}}\,, \tag{3}$$

$$v = \frac{Ba^2}{Ab^2}u \implies v = \frac{Ba^2}{Ab^2}\sqrt{\frac{A^2b^2c^2 + ACa^2b^2 + ABa^2c^2}{BCa^4}}\,. \tag{4}$$

The tangent functions u and v define angles α' and β' of the wanted new triangle of vanishing points. It can be built based on edge $V_x'V_y'$ in the following convenient way. Let us consider an arbitrary Cartesian frame then position V_x' and V_y' at $(0,0)$ and $(l,0)$ respectively, where $l \in \mathbb{R}^+$ is arbitrary. So edge $V_x'V_y'$ will be on the x-axis. It is proposed to set the value of l to the distance of V_x and V_y, because this way we can avoid significant difference in size among the old and the new triangle. Here we note that if we keep the vanishing points $V_x = V_x'$ and $V_y = V_y'$ fixed, it implies that our affine transformation will be perspective.

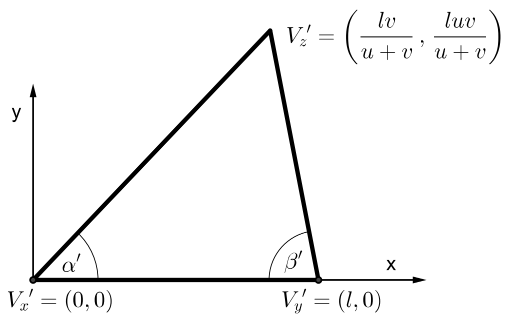

Figure 4: The new triangle of the vanishing points

It follows from this choice that the equations of the two remaining edges of the triangle are

$$\begin{aligned}
y &= ux\\
y &= -v(x-l) = -vx + lv,
\end{aligned}$$

using u and v as the slopes of the lines. Then solving this simple system of equations for x and y we get

$$V_z' = (x,y) = \left(\frac{lv}{u+v},\ \frac{luv}{u+v}\right). \tag{5}$$

Let V_i^j and $V_i'^j$ denote the j-coordinate of V_i and V_i', respectively, $i,j \in \{x,y\}$. If A denotes the matrix of the affinity which transforms the triangle $V_xV_yV_z$ onto $V_x'V_y'V_z'$, i.e.,

$$\begin{pmatrix} V_x^x & V_x^y & 1 \\ V_y^x & V_y^y & 1 \\ V_z^x & V_z^y & 1 \end{pmatrix} \cdot A = \begin{pmatrix} V_x'^x & V_x'^y & 1 \\ V_y'^x & V_y'^y & 1 \\ V_z'^x & V_z'^y & 1 \end{pmatrix},$$

then A can be calculated in the following way:

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} = \begin{pmatrix} V_x^x & V_x^y & 1 \\ V_y^x & V_y^y & 1 \\ V_z^x & V_z^y & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} V_x'^x & V_x'^y & 1 \\ V_y'^x & V_y'^y & 1 \\ V_z'^x & V_z'^y & 1 \end{pmatrix}. \tag{6}$$

5. An application

In this section we show an example of how we can use the new theorem (2).

On the left side of Fig. 5 we can see a photo of a computer tower holder, which is a rectangular cuboid. The real outside dimensions of the holder are 52 cm × 26 cm × 12.3 cm. In the following we denote these values by b, a and c, respectively.

Let us suppose that we have lost our original photo, but we have one, which shows it — or just a part of it. (For example, somebody has left it on the table in a room, where a family photo was taken.) On the right side of Fig. 5 we can see the photo of the photo. Our goal is to get a central projection of the computer tower holder, using only the remaining photo on the right side of Fig. 5 (which in general is different from a central projection) and the dimensions of the holder. We can achieve this goal using the method shown in Section 4. The necessary steps are as follows.

Figure 5: The original photo (left), which is a central projection, and its photo (right), which is not

The first step is the localization of the changed central axonometric reference system (see Fig. 2) in the image. Since the real dimensions of the box are known, we only need to measure the necessary distances — a', a'', b', b'', c' and c'' — to be able to get u and v by substituting into equations (3) and (4).

Then we need to use an arbitrarily given Cartesian coordinate system in the image plane in order to determine the coordinates of the vanishing points V_x, V_y and V_z. In the case of digital images, this reference frame can be the original coordinate system of an image manipulation program. We used GIMP [18] for determining these necessary distances and

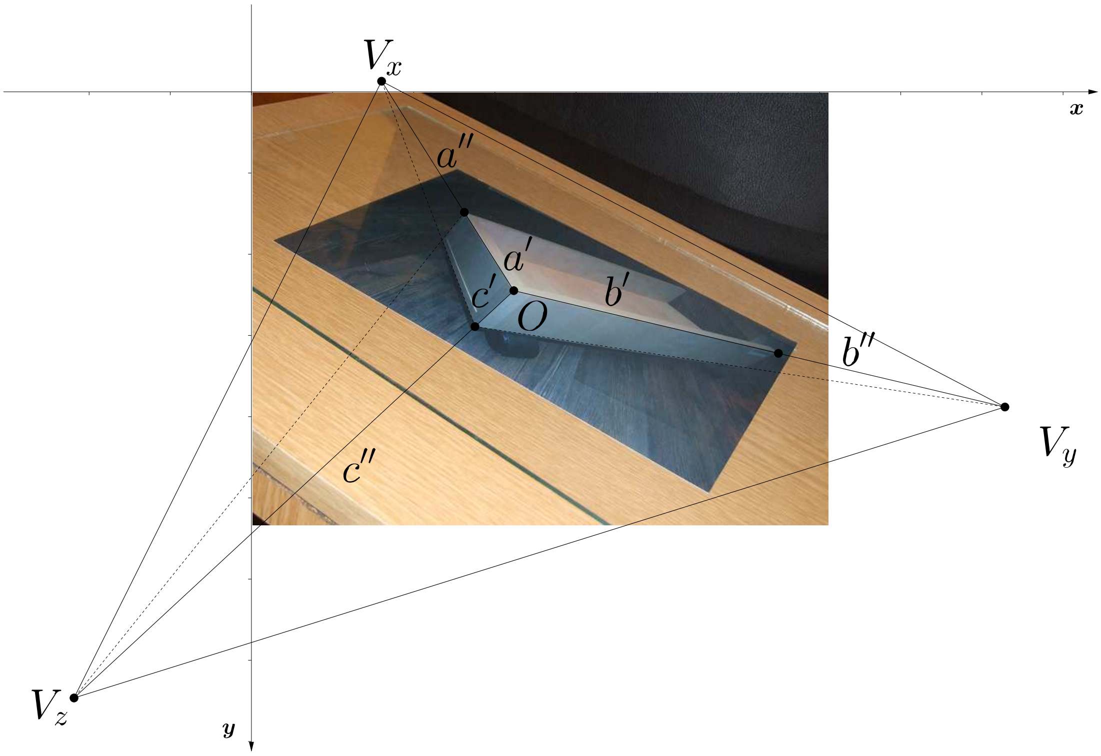

Figure 6: The triangle of the vanishing points on the photo of the photo and the necessary lenghts for the calculations

positions. The default coordinate system of this software has its origin at the top-left corner of the image, its positive x-axis points to the right and the positive y-axis points downward.

After specifying the distance l of V_x and V_y (but it can be arbitrary) we can construct the triangle of the new vanishing points V_x', V_y' and V_z' by substituting into Eq. (5). The result of this step can be seen in Fig. 7.

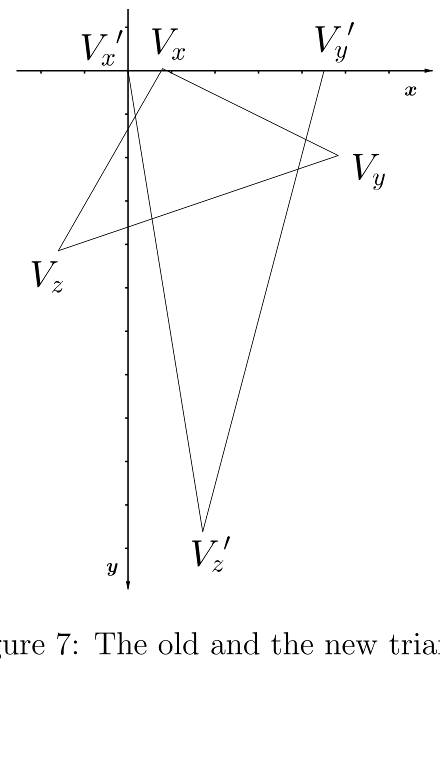

Figure 7: The old and the new triangle

Now we need to describe the affinity that transforms the triangle $V_x V_y V_z$ onto the triangle $V'_x V'_y V'_z$. We can determine the matrix of this affinity using Eq. (6).

ImageMagick [17] is an open source software for image manipulation, with which we can apply the affinity on the images based on the matrix of the transformation. It uses the same type of coordinate system like GIMP.[3]

Below we can see an example command of ImageMagick, which generates the result. `aij` denotes the element of the wanted transformation matrix at row i and column j, where $i, j \in \{1, 2, 3\}$. Of course we must replace these aij symbols with concrete numbers. For more details see [19].

```
convert PhotoOfThePhoto.JPG -matte -virtual-pixel Transparent
-affine a11,a12,a21,a22,a31,a32 -transform +repage Result.JPG
```

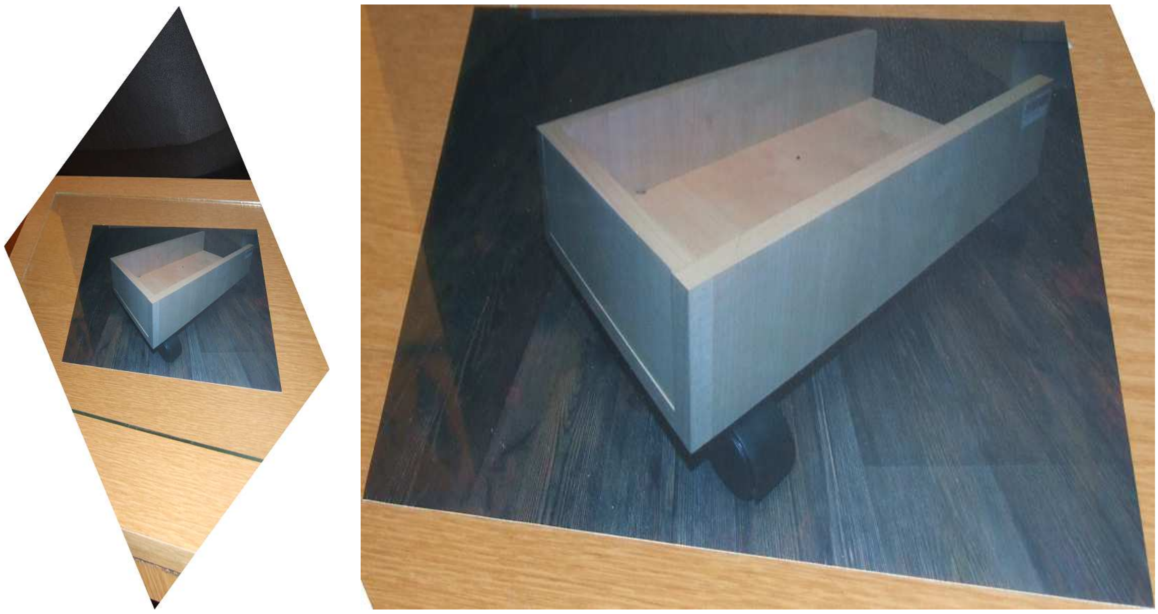

Figure 8: The result image, which shows a central projection of the holder

On the left side of Fig. 8 we see the result immediately after applying the calculated affine transformation on the photo of the photo, so it shows a central projection of the holder. On the right side of Fig. 8 we see a cropped version of the result.

It is important to note that not only one result exists, because there are *infinitely many* positions of the origin and the axes for a given triangle of vanishing points (like in the case of the spatial Möbius grid). But all obtained results are central projections of the computer tower holder. The underlying projections *share the projection center* since for any spatial line the property of being mapped onto a point is preserved. For each central projection the orthocenter of the triangle of vanishing points is the *principal point*.

Remark. In order to simplify the procedure, a computer program has been created which can be used freely for non-commercial use [20].

[3]An alternative approach is based on the *singular value decomposition* of the matrix A. Each two-dimensional affine transformation is the product of a rotation about the origin, a scaling of the x- and y-coordinates, one more rotation about the origin and a translation.

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