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Geometric Constructions for Geometric Optics Using a Straightedge Only

Victor Oxman[1], **Moshe Stupel**[2], **Avi Sigler**[3]

[1] *Western Galilee College, P.O.B. 2125, Acre 24121, Israel*
email: victor.oxman@gmail.com

[2] *Shaanan College, P.O.B. 906, Haifa 26109 Israel*
email: stupel@bezeqint.net

[3] *Technical College for Aviation Professions, Haifa, Israel*
email: anysig@walla.co.il

Abstract. Geometric constructions occupy an important part in Euclidean geometry. An interesting branch of geometric constructions is construction with restrictions imposed on the drawing tools. In the following, five surprising construction tasks are presented, which are carried out using a straightedge only. All the tasks have to do with paths of light rays incident upon reflective planes (mirrors). We present the geometric theorems on which the constructions are based; some of them are given with a proof.

Key Words: geometric constructions, constructions using a straightedge only, light rays: incidence and reflection

MSC 2010: 51M15

1. Introduction

Classical constructions are carried out by means of a straightedge — as a tool for drawing straight lines, and a compass — as a tool for drawing arcs.

In the 17^{th} century the Danish mathematician Georg Mohr has published a book in which he carried out using a compass only all the geometric constructions described by Euclid in his book *"Elementa"* using a straightedge and a compass. Mohr's book was lost and it was found by accident only 250 years later.

In the 18^{th} century the Italian mathematician Mascheroni described a construction similar to Mohr's.

From Mohr's and Mascheroni's works it follows that any construction that can be carried out using a straightedge and a compass can also be carried out using a compass only [1, p.64]. In other words, use of the straightedge can be forgone. Of course, with such a

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construction it is not possible to draw a straight line; in this case it is said that the straight line has been constructed if two of its points have been fixed.

It therefore follows that where geometric constructions are concerned, the compass alone is equivalent to a straightedge and a compass used together, and the question is asked: is it also true with regards to the straightedge? Is the straightedge also equivalent to a straightedge and a compass together?

It turns out that this is not the case — one cannot forget the compass! It can be proven that if a straightedge only is given, which is only used for drawing straight lines, then most of the constructions in a plane cannot be carried out. For example, it was proven that if a circle with an unknown center is given in the plane, its center cannot be found using a straightedge only.

Some 30 years after Mascheroni, the Poncelet-Steiner theorem has stated that all constructions that can be carried out using a straightedge and a compass can be carried out using a straightedge only, provided that a circle and its center are given in the plane of construction [1, p.98].

However, even when there is no a circle with its center certain constructions can be carried out using a straightedge only [1–3].

The main result of the present paper is five new and unknown before tasks of construction of this kind. The constructions have to do with the incident ray, the reflected ray and the perpendicular to the reflective plane at the point of incidence (Figure 1). In order to prove the methods of construction we shall give lemmas in geometry with their proofs.

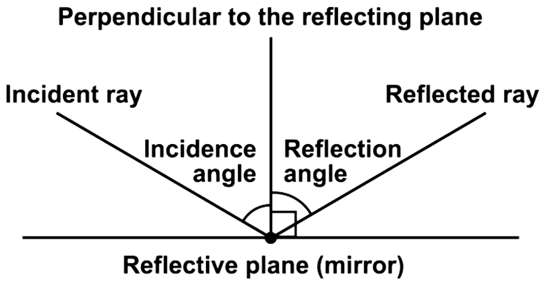

Figure 1: Light reflection

2. Constructing the angle of the reflected ray and related problems

2.1. Construction task 1

A straight line ℓ is given ("the reflective plane"), on which a point O is marked, and a perpendicular to ℓ is given at point O.

Given is a straight line FO ("the ray of light"), which forms an angle $\angle$FOH with HO (the incidence angle). Construct using a straightedge only the angle of reflection $\angle$HOG, i.e., construct a straight line OG, so that the angle $\angle$HOG is equal to the angle $\angle$FOH (as seen on Figure 2).

Lemma 1. *If in any triangle* ΔABC *given are the altitude* AH*, and two segments* BN *and* CM*, which intersect at point* E *on the altitude. Then* $\angle$MHE $=$ $\angle$NHE *(see Figure 3).*

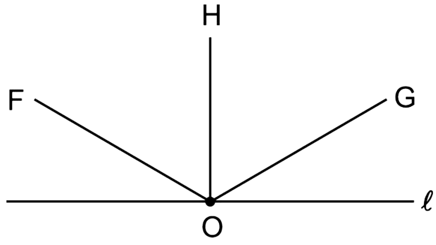

Figure 2: Construction task 1

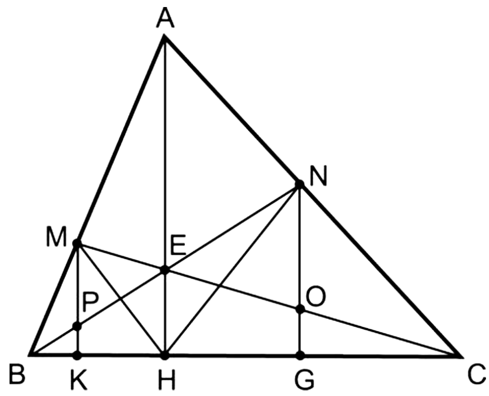

Figure 3: Lemma 1

Proof: We construct the perpendiculars MK ⊥ BC and NG ⊥ BC. Therefore the triangles ΔMEP and ΔOEN are similar, and

$$\mathrm{MP}/\mathrm{NO} = \mathrm{ME}/\mathrm{EO} = \mathrm{KH}/\mathrm{HG}. \qquad (*)$$

In addition, we have NO/NG = AE/AH = MP/MK. Therefore, MK/NG = MP/NO, and together with (∗) we obtain that MK/NG = KH/HG, which means that the triangles ΔMKH and ΔNGH are similar and ∠MHK = ∠NHG. So ∠MHE = ∠NHE, which is the required proof. □

Description of the construction of task 1:

A point A on OH is selected, along with any two points B and C on the line ℓ on two different sides of the point O (as shown in Figure 4).

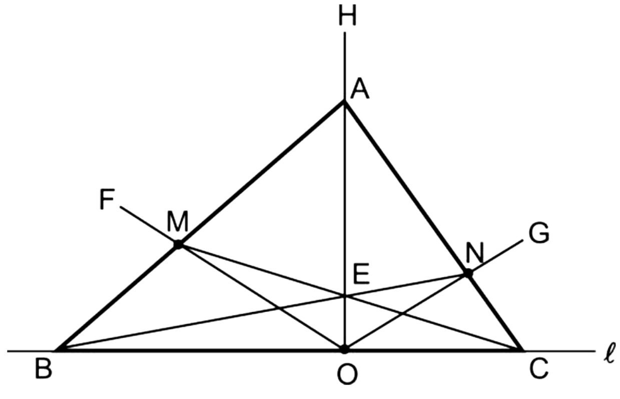

Figure 4: The construction task 1

We denote by M the point of intersection of AB with FO. We connect the point M with the point C and denote by E the point of intersection of MC with OH.
In the triangle ΔBAC we draw a straight line connecting the point B with the point E. The continuation of the line BE intersects the side of the triangle at point N. We connect the point O with N.
According to Lemma 1, $\angle$FOH = $\angle$GOH, and therefore OG is the reflected ray.

2.2. Construction task 2

Given is a straight line ℓ, with a point O on it, and two rays FO and GO, such that the angle between ℓ and FO is equal to the angle between ℓ and GO. Construct a perpendicular to ℓ at point O using a straightedge only.

Lemma 2. *In the triangle* ΔABC, *let* AD *be the bisector of the interior angle of the triangle at the vertex* A, *and let* AD_1 *be the bisector of the exterior angle at the vertex* A. *Then the point* D *is a harmonic point to* D_1 *with respect to the points* B *and* C.

Proof: According to the theorem of the angle bisector of an interior angle in a triangle, we have BD/DC = AB/AC. According to the theorem of the angle bisector of an exterior angle in a triangle, we have $\mathrm{BD}_1/\mathrm{CD}_1$ = AB/AC. Therefore BD/DC = $\mathrm{BD}_1/\mathrm{CD}_1$.
It should be noted that $\mathrm{AD}_1 \perp \mathrm{AD}$ (bisectors of supplementary adjacent angles). □

Corollary 3. *If in triangle* ΔABC *the segment* AD *is the bisector of the interior angle at vertex* A, *and* D_1 *is a harmonic point to* D *with respect to* B *and* C, *then* AD_1 *is the bisector of the exterior angle at the vertex* A, *and there holds* $\mathrm{AD}_1 \perp \mathrm{AD}$.

Auxiliary construction:
In order to carry out task 2, we describe the known method of construction of a harmonic point using a straightedge only [2], which will be followed by an elegant proof of its correctness.

Description of the auxiliary construction:
Three points A, B, C are marked on the straight line ℓ, and a point D that is harmonic to B with respect to A and C is to be constructed (see Figure 5).
Some point K outside the line AC is selected and connected to the points A, B, C. On the segment KB we select some point E and construct the straight lines AE and CE, which intersect the segments AK and CK at the points G and F respectively.
A line GF is drawn, which intersects the line AC at the point D. The point D is the sought harmonic point.

Proof of the correctness of the auxiliary construction:
According to Menelaus' Theorem for triangle ΔAKC and line GD ([4, Section 3.4]) there holds

$$\frac{\mathrm{AD}}{\mathrm{CD}} \cdot \frac{\mathrm{CF}}{\mathrm{FK}} \cdot \frac{\mathrm{KG}}{\mathrm{GA}} = 1\,, \tag{$*$}$$

and according to Ceva's Theorem for the same triangle ([4, Section 1.2]) there holds

$$\frac{\mathrm{AB}}{\mathrm{BC}} \cdot \frac{\mathrm{CF}}{\mathrm{FK}} \cdot \frac{\mathrm{KG}}{\mathrm{GA}} = 1\,. \tag{$**$}$$

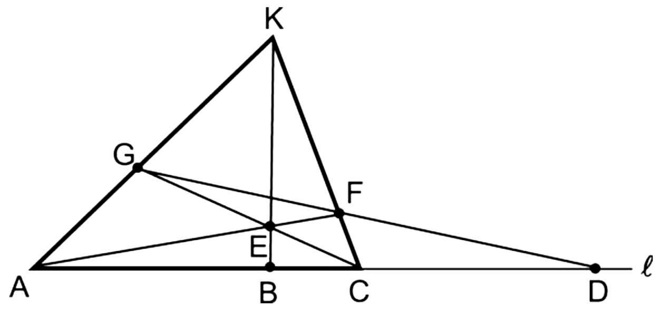

Figure 5: Constructing a point D, that is harmonic to B with respect to A and C

From (∗) and (∗∗) we obtain that

$$\frac{\mathrm{AD}}{\mathrm{CD}} = \frac{\mathrm{AB}}{\mathrm{BC}}\,.$$

Description of the construction for task 2:
We select some point A on the ray FO, and some point B on the continuation of the ray GO beyond the point O (see Figure 6). We connect the points A and B by a straight line and denote by D the point of intersection of the segment AB with the line ℓ.
In accordance with the auxiliary construction, we construct the point D_1 that is harmonic to the point D with respect to points A and B.
Hence, from the conclusion of Lemma 2, we obtain that $\mathrm{OD}_1 \perp \ell$, i.e., we obtained the perpendicular to the reflective plane at the incidence point.

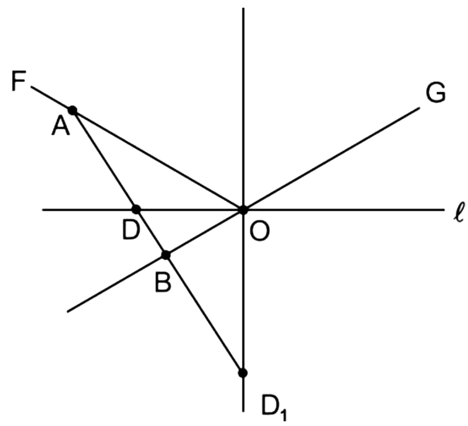

Figure 6: Construction task 2

2.3. Construction task 3

Given are a straight line ℓ with a point O on it, and two rays FO and GO, such that the angle between the line ℓ and the ray FO is equal to the angle between the line ℓ and the ray GO. Also given is another ray KO.
Construct another ray MO, such that the angle between the line ℓ and the ray MO would be equal to the angle between the line ℓ and the ray KO.

Description of the construction for task 3:
Based on the construction task 2, we construct a perpendicular to the line ℓ at the point O, and then, using construction task 1, we construct the required ray.

2.4. Construction task 4

Given are a straight line ℓ (as a reflective plane), a ray GO incident on the line and two perpendiculars ℓ_1 and ℓ_2 to the straight line ℓ (see Figure 7). Construct the reflected ray MO.

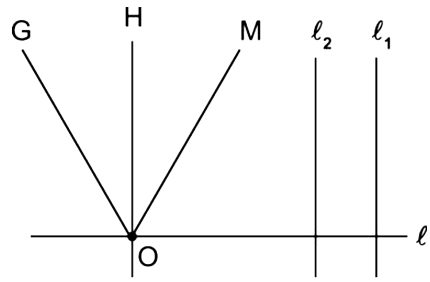

Figure 7: Construction task 4

Description of the construction for task 4:
ℓ_1 and ℓ_2 are two parallel lines. It is known that when two parallel lines are given, it is possible to draw a straight line through any point so that it is parallel to the two straight lines, using a straightedge only.
The construction is based on Steiner's Theorem for a trapezoid: the straight line connecting the point of intersection of the trapezoid's diagonals with the point of intersection of the continuations of the sides of the trapezoid bisects its bases [2, problem 15]. According to this construction it is possible to draw a straight line HO that is parallel to ℓ_1 and ℓ_2. Since HO is perpendicular to ℓ, then according to construction task 1, it is possible to construct the required ray MO.

2.5. Construction task 5

Given are a straight line ℓ, and a pair of rays MO_1 and NO_1, which are the incident and the reflected ray at point O_1. Similarly, given are the rays GO_2 and KO_2, which are the incident and the reflected ray at point O_2 (see Figure 8).
In addition, given is the ray FO, which is incident at point O. Construct the ray reflected

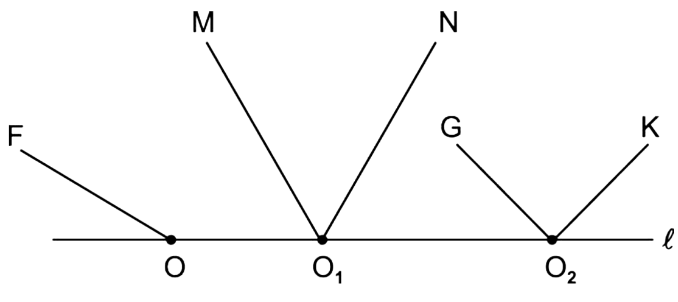

Figure 8: Construction task 5

from this point.
When the perpendiculars to ℓ at points O_1 and O_2 are obtained (task 2), the reflected ray at point O is constructed in accordance with the construction in task 4.

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