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Equicevian Points and Cubics of a Triangle

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Abstract. A point P in the plane of a given triangle ABC is said to be *equicevian* if the cevians AA_P, BB_P, and CC_P through P are of equal length. In this note, we see that the set Ω of equicevian points can be obtained via three cubic curves, and we give a complete description of Ω including also the imaginary solutions. There exist up to ten equicevian points, among them the four focal points of the Steiner circumellipse. Besides, we present properties of the so-called equicevian cubics which in the irreducible case are strophoids, i.e., rational and circular cubics with orthogonal tangents at their node.

Key words: Equicevian points, equicevian cubics, strophoid, Steiner's circumellipse, focal points, focal curves, pedal curves, Marden's Theorem, Euclidean construction

MSC 2010: 51N20, 51M25, 51M15

1. Introduction

It is well known that if any two of the four traditional centers (i.e., the centroid, the incenter, the circumcenter, and the orthocenter) coincide for a triangle, then the triangle must be equilateral (see, for example, [16]). The list can be extended to include other centers, such as the Fermat-Torricelli, the Gergonne, and Nagel points, as done in [12], and one would naturally feel that the list can be enlarged to include all *reasonable* or *natural* centers.

This paper is a result of our feeling that the celebrated Steiner-Lehmus Theorem should be placed in the context of coincidences of centers. The theorem states that *if the internal angle bisectors of two angles in a triangle ABC are equal, then the corresponding sides are equal.* A poor-man's version, which may turn out to be as strong, would state that *if the three*

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internal angle bisectors are equal, then the triangle is equilateral. Stating it in this manner, one is tempted to view this as saying that *if the incenter and the "equicevian center" of a triangle coincide, then it is equilateral*, where *the equicevian center* would be *the* point the cevians through which are of equal length.

This raises the question whether there exists, for an arbitrary triangle, a point through which the cevians are equal, and whether such a point, if it exists, is unique. It is the main aim of this paper to address this problem, which turned out to be rather complicated. We shall see that a triangle can have as many as two real and two conjugate complex equicevian points (not counting vertices and points on the sidelines), and that a triangle can have two real equicevian points that are interior. There are triangles with no equicevian point at all. It will be proved that there is an intimate connection with the Steiner circumellipse and with Marden's Theorem.

Related or not to the Steiner-Lehmus Theorem, the problem of existence and uniqueness of equicevian points is quite natural and interesting in its own. In fact, this problem has appeared in various forms at several places, and it would probably have appeared much more frequently had it been reasonably tractable.

2. The equicevian cubics

Definition 1. Let ABC be a triangle. We call a point P in the plane of ABC A-*equicevian* if the lengths $|BB_P|$ and $|CC_P|$ of the cevians through P are equal, and we define B-*equicevian* and C-*equicevian* points similarly.
A point P that is A- and B-equicevian (and hence also C-equicevian) is called *equicevian*. Thus, P is equicevian if $|AA_P| = |BB_P| = |CC_P|$.
If an equicevian point lies on one of the sidelines of ABC, we call it an *improper* and otherwise a *proper* equicevian point.

To study the set Ω_A of A-equicevian points of ABC, we place ABC in the cartesian plane in such a way that

$$B = (-1, 0), \quad C = (1, 0), \quad A = (u, v), \quad v > 0, \tag{1}$$

and we take $P = (X, Y)$ to be an arbitrary point (see Figure 1). For simplicity, we assume that

$$|AB| \geq |AC|, \text{ i.e., } u \geq 0.$$

The equations of AB, AC, AP, BP, and CP are given by

$$AB\colon\ y = \frac{(x+1)v}{u+1}, \quad AC\colon\ y = \frac{(x-1)v}{u-1},$$

$$AP\colon\ y = \frac{(Y-v)(x-u)}{X-u} + v, \quad BP\colon\ y = \frac{(x+1)Y}{X+1}, \quad CP\colon\ y = \frac{(x-1)Y}{X-1}.$$

Thus the intersection C_P of AB and CP, the intersection B_P of AC and BP, and the intersection A_P of BC and AP are

$$\begin{aligned} C_P &= \left(\frac{-vX + v - uY - Y}{vX - v - uY - Y}, \frac{-2vY}{vX - v - uY - Y}\right), \\ B_P &= \left(\frac{vX + v + uY - Y}{vX + v - uY + Y}, \frac{2vY}{vX + v - uY + Y}\right), \\ A_P &= \left(\frac{uY - vX}{Y - v}, 0\right). \end{aligned}$$

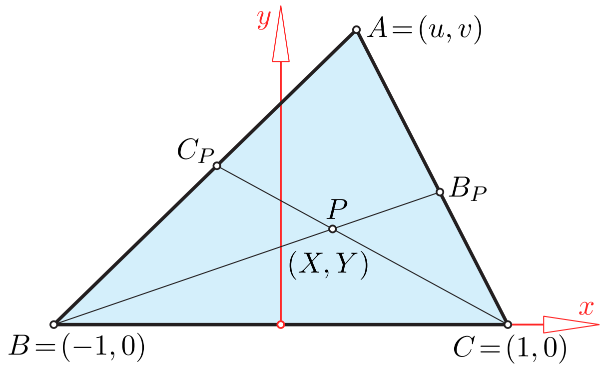

Figure 1: A-equicevian points P are characterized by $|BB_P| = |CC_P|$

This implies

$$|AA_P|^2 = \frac{v^2[(X-u)^2+(Y-v)^2]}{(Y-v)^2}, \quad |BB_P|^2 = \frac{4v^2[(X+1)^2+Y^2]}{[v(X+1)-(u-1)Y]^2}, \\ |CC_P|^2 = \frac{4v^2[(X-1)^2+Y^2]}{[v(X-1)-(u+1)Y]^2}, \tag{2}$$

and therefore $(BB_P)^2 - (CC_P)^2 = 16v^2 Y H_A$, where

$$\begin{aligned} H_A(X,Y) \;=\; & (vX-uY)(X^2+Y^2) + uv(X^2-Y^2) - (u^2-v^2+1)XY \\ & -(vX+uY) - uv. \end{aligned} \tag{3}$$

Hence the A-equicevian points lie on the line BC or on the cubic which satisfies the equation $H_A(X,Y) = 0$. We call this cubic the A-*equicevian cubic* associated with ABC.

Similarly, the B-equicevian points lie on the line AC or on the B-equicevian cubic $H_B(X,Y) = 0$, where

$$\begin{aligned} H_B(X,Y) \;=\; & ((u+3)Y - vX)\,(X^2+Y^2) + (2u+1)vX^2 - (2u+5)vY^2 \\ & -(2u^2-2v^2+6u+8)XY + (u^3+3u^2+(u+1)v^2+4u+4)\,Y \\ & -(u^2+v^2+2u-4)vX + (u^2+v^2-4)v. \end{aligned} \tag{4}$$

The C-equicevian points lie on the line AB or on the C-equicevian cubic $H_C(X,Y) = 0$, where

$$\begin{aligned} H_C(X,Y) \;=\; & ((u-3)Y - vX)\,(X^2+Y^2) + (2u-1)vX^2 - (2u-5)vY^2 \\ & -(2u^2-2v^2-6u+8)XY + (u^3-3u^2+(u-1)v^2+4u-4)\,Y \\ & -(u^2+v^2-2u-4)vX - (u^2+v^2-4)v. \end{aligned} \tag{5}$$

We summarize:

Theorem 1. *All A-equicevian points lie on the line BC or on the A-equicevian cubic $\mathcal{C}_A$ which is the zero-set of the polynomial $H_A(X,Y)$ defined in Eq. (3). Similarly, B-equicevian points lie on AC or on the B-equicevian cubic $\mathcal{C}_B\colon\ H_B(X,Y) = 0$ given by (4), and C-equicevian points lie on AB or on the C-equicevian cubic $\mathcal{C}_C\colon\ H_C(X,Y) = 0$ given by (5).*

Figures 7, 8, 6, 9, and 11 show the cubics $\mathcal{C}_A$, $\mathcal{C}_B$, and $\mathcal{C}_C$ corresponding to the respective values $(u,v) = (0.2000,\ 1.7636)$, $(0.620,\ 1.569)$, $(0.00,\ 1.56$ or $1.85)$, and $(0.75,\ 1.70)$.

It is not surprising that our search for equicevian points of a triangle results in an algebraic problem. Therefore it makes sense to extend Definition 1 to non-real points by calling (real or imaginary) points P *equicevian* if $|AA_P|^2 = |BB_P|^2 = |CC_P|^2$ — in the sense of Eq. (2).

3. Properties of the equicevian cubics

3.1. Particular A-equicevian points

For analyzing the A-equicevian cubic $\mathcal{C}_A$ it is useful to adopt the projective point of view. For this purpose we introduce homogeneous coordinates

$$(X_0, X_1, X_2) \text{ with } X = X_1/X_0 \text{ and } Y = X_2/X_0\,,$$

and we denote the triple of unknowns by $\mathbf{X} = (X_0, X_1, X_2)$. Homogenization of $H_A(X,Y)$ gives the polynomial

$$\begin{aligned} P(\mathbf{X}) \;=\; & (vX_1 - uX_2)(X_1^2 + X_2^2) + uvX_0(X_1^2 - X_2^2) \\ & - (u^2 - v^2 + 1)X_0X_1X_2 - X_0^2(vX_1 + uX_2) - uvX_0^3 \end{aligned} \tag{6}$$

with the zero set $\mathcal{C}_A^*$, which is the projective extension of $\mathcal{C}_A$. For the gradient $\operatorname{grad} P(\mathbf{X}) = (P_{X_0}(\mathbf{X}),\ P_{X_1}(\mathbf{X}),\ P_{X_2}(\mathbf{X}))$ of $P(\mathbf{X})$ we obtain

$$\begin{aligned} P_{X_0}(\mathbf{X}) &= uv(X_1^2 - X_2^2) + (v^2 - u^2 - 1)X_1X_2 - 2X_0(vX_1 + uX_2) - 3uvX_0^2, \\ P_{X_1}(\mathbf{X}) &= v(X_1^2 + X_2^2) + 2X_1(vX_1 - uX_2) + 2uvX_0X_1 - (u^2 - v^2 + 1)X_0X_2 - vX_0^2, \\ P_{X_2}(\mathbf{X}) &= -u(X_1^2 + X_2^2) + 2X_2(vX_1 - uX_2) - 2uvX_0X_2 - (u^2 - v^2 + 1)X_0X_1 - uX_0^2. \end{aligned} \tag{7}$$

We recall that at any regular point $\mathbf{p}\mathbb{R} = (p_0, p_1, p_2)\mathbb{R}$ of the projective algebraic curve $P(\mathbf{X}) = 0$ the tangent line satisfies the equation

$$P_{X_0}(\mathbf{p})X_0 + P_{X_1}(\mathbf{p})X_1 + P_{X_2}(\mathbf{p})X_2 = 0.$$

Singular points $\mathbf{s}\mathbb{R}$ are characterized by $\operatorname{grad} P(\mathbf{s}) = \mathbf{0}$.

From now on we denote by $\overline{A}$, $\overline{B}$, and $\overline{C}$ the respective images of A, B, and C under reflections in the midpoints of the opposite sides. $\overline{A}\,\overline{B}\,\overline{C}$ is called the *anticomplementary* triangle of ABC.

Theorem 2. *The A-equicevian cubic $\mathcal{C}_A$ passes through B and C and has the point $\overline{A} = (-u, -v)$ as a node (see Figure 2).*
For $u \neq 0$ the cubic $\mathcal{C}_A$ is irreducible and therefore rational. Otherwise it splits into the axis $A\overline{A}$ of symmetry and the circumcircle of the triangle $\overline{A}BC$.
As a converse of the statement in Theorem 1, all points of the line BC and the cubic $\mathcal{C}_A$ other than B, C, and $\overline{A}$ are A-equicevian.

Proof. The vector $(1, -u, -v)$ is a common zero of all three derivatives P_{X_0}, P_{X_1} and P_{X_2}; therefore point $\overline{A}$ is singular. We will see later that for $u \neq 0$ the cubic is irreducible; in this case $\overline{A}$ is the only singular point of the projective cubic $\mathcal{C}_A^*$.

The line at infinity $X_0 = 0$ intersects $\mathcal{C}_A^*$ at the absolute circle-points $(0, 1, \pm i)\mathbb{R}$ and at the real point $A_\infty = (0, u, v)\mathbb{R}$, which is located on the median of A. The tangent line at A_∞ is the real asymptote of $\mathcal{C}_A$. It satisfies the equation

$$t_{A_\infty}\colon\ -uvX_0 + v(u^2 + v^2)X_1 - u(u^2 + v^2)X_2 = 0. \tag{8}$$

In order to obtain a rational parametrization of $\mathcal{C}_A$, we return to cartesian coordinates. All lines through $\overline{A}$ — up to the parallel to the y-axis — can be set up in parameter form by

$$X(\tau) = \tau - u, \quad Y(\tau) = k\tau - v, \quad \tau \in \mathbb{R}, \tag{9}$$

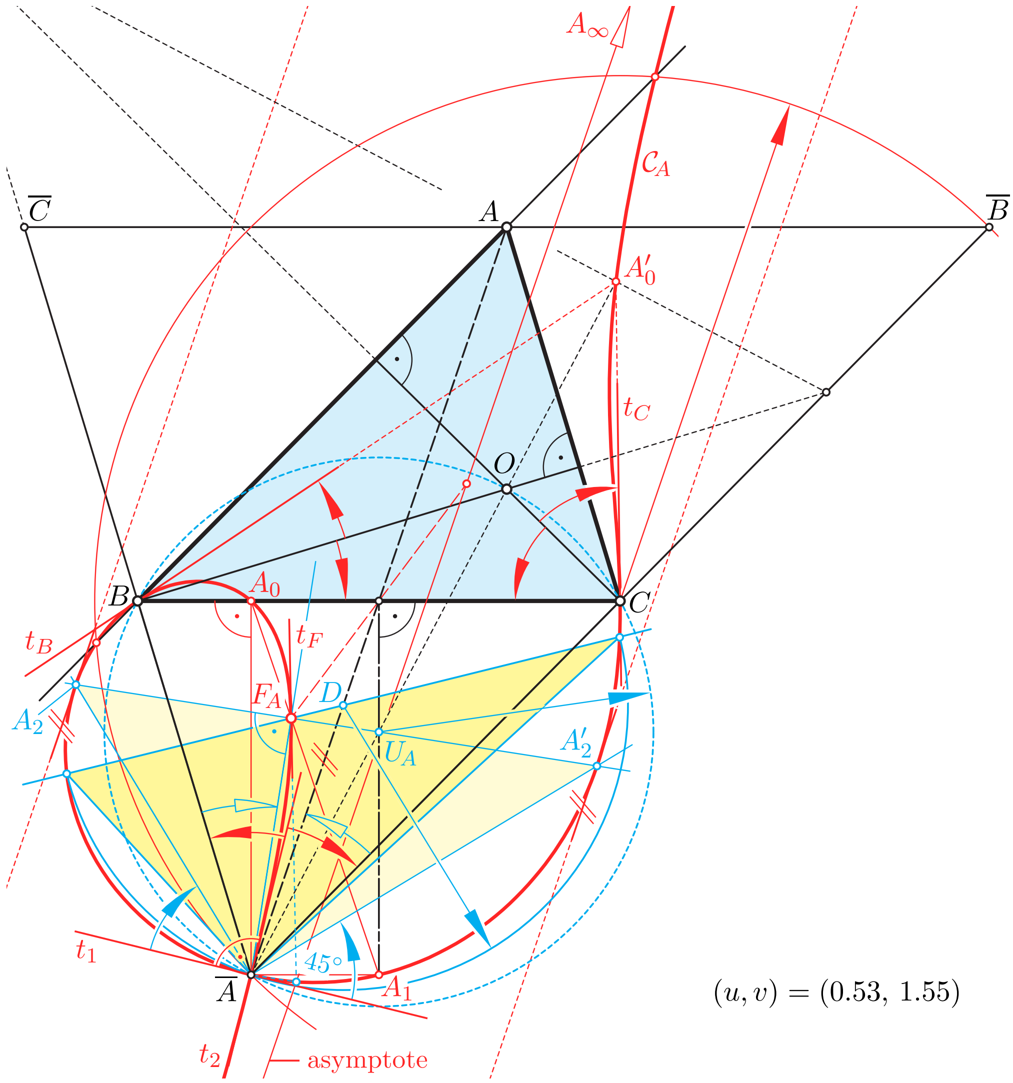

Figure 2: The A-equicevian cubic $\mathcal{C}_A$ is an oblique strophoid with node $\overline{A}$ and singular focal point F_A

where k denotes the tangent of the slope angle. Each single line intersects $\mathcal{C}_A^*$ beside $\overline{A}$ in a unique remaining point. The corresponding parameter τ is

$$\tau = \frac{uvk^2 + (u^2 - v^2 - 1)k - uv}{(uk - v)(1 + k^2)}. \tag{10}$$

This leads to the following rational representation of $\mathcal{C}_A$ in terms of the parameter $k \in \mathbb{R}$:

$$X = \frac{u^2k^3 - 2uvk^2 + (v^2 + 1)k}{(v - uk)(1 + k^2)}, \quad Y = \frac{(u^2 - 1)k^2 - 2uvk + v^2}{(uk - v)(1 + k^2)}. \tag{11}$$

A vanishing denominator characterizes points at infinity.

Whenever $\mathcal{C}_A$ is reducible, it must contain a line through $\overline{A}$. For the slope k of such a component the nominator and denominator in (10) must vanish simultaneously. However, there is no $k \in \mathbb{R}$ with this property. Only for $k \to \infty$, i.e.,

$$X = u, \quad Y = -\frac{u(v + 1)}{u},$$

we obtain in the case $u = 0$ an indeterminate point of intersection with $\mathcal{C}_A$.

Conversely, for $u = 0$ the polynomial $H_A(X, Y)$ from (3) factorizes into

$$H_A(X, Y) = X\left[v(X^2 + Y^2) + (v^2 - 1)Y - v\right].$$

The cubic $\mathcal{C}_A$ splits into $X = 0$ and into the circle through B, C and $\overline{A}$ (see Figure 6).

The points B and C are not A-equicevian since the cevians through B or C, respectively, are not determined. Also $\overline{A}$ is not A-equicevian because of $|BB_P| = |CC_P| = \infty$. Hence

$$\Omega_A = (\text{line } BC \cup \mathcal{C}_A) \setminus \{B, C, \overline{A}\} \tag{12}$$

is the set of A-equicevian points. □

All properties listed in the following theorem follow either from the geometric definition $|BB_P| = |CC_P|$ of $\mathcal{C}_A$ or from the equation $H_A = 0$ given in (3).

Theorem 3. *The remaining point of intersection between $\mathcal{C}_A$ and the line BC is $A_0 = (-u, 0)$, the pedal point of $\overline{A}$.*
The remaining intersection points of the cubic $\mathcal{C}_A$ with the line AB need not be real; they are located on the circle with center C and diameter $\overline{A}\,\overline{B}$. Similarily, $\mathcal{C}_A$ passes through the intersection points of sideline AC with the circle with center B and diameter $\overline{A}\,\overline{C}$.
$\mathcal{C}_A$ intersects the bisector[1] of BC at point $A_1 = (0, -v)$; the remaining two intersection points are conjugate complex with coordinates $(0, \pm i)$.

Remark 1. The intersection of the A-equicevian cubic with the internal angle bisector of A is investigated by V. Nicula and C. Pohoaţă in [20], by S. Abu-Saymeh and M. Hajja in [1], and by V. Oxman in [21]. The intersection with the median through A is investigated by S. Abu-Saymeh and M. Hajja in [2]. The intersection with the altitude through A is investigated by S. Abu-Saymeh and M. Hajja in [3]. Partial results on equicevian points for isosceles triangles have appeared in the Monthly problem in [9]. Results pertaining to the lengths of the cevians through an equicevian point have appeared as a Monthly problem in [13] and as a Mathematical Olympiad problem in [4]. In the Monthly problem in [22] the existence of two equicevian points is proved.
Also, partial results pertaining to what one may call *equi-semi-cevian* points of a triangle were obtained for isosceles triangles. These are the points P for which the cevians AA_P, BB_P, and CC_P have the property that $|PA_P| = |PB_P| = |PC_P|$. According to M. Fox in his *Feedback* [10] on J.A. Scott's note [25], this problem appears as early as the nineteenth century in [11], where it was realized that the problem is intractable even if one restricts attention to the case when the triangle is isosceles.

3.2. The equicevian cubics are strophoids

Definition 2. An irreducible cubic is called *circular* if it passes through the absolute circle-points $(0, 1, \pm i)\mathbb{R}$. A circular cubic is called *strophoid* if it has a node with orthogonal tangents. A strophoid without any axis of symmetry is called *oblique* (see, e.g., [15, p. 515] or [26, pp. 37–39]).

[1] The *bisector* of two points is the perpendicular bisector of the line segment joining them.

For any circular cubic, the tangent lines at the absolute circle-points meet at a point which is called *singular focal point* of the cubic.

Suppose the strophoid has an axis s of symmetry. Then the reflection in s, which maps the strophoid onto itself, must keep the node and the focal point fixed as well as the real ideal point together with its tangent, the asymptote. Since the asymptote t_{A_∞} with the equation (8) cannot contain any finite point of the curve, the axis s must be orthogonal to the asymptote and pass through the node and the focal point.

Theorem 4. *In the irreducible case, i.e., for $u \neq 0$, the A-equicevian cubic $\mathcal{C}_A$ has the following properties.*

1. *$\mathcal{C}_A$ is an oblique strophoid. The two tangent lines t_1, t_2 at the node $\overline{A}$ bisect the angle $\angle B\overline{A}C$ (Figure 2).*
2. *The singular focal point F_A of $\mathcal{C}_A$ lies on the cubic.*[2] *The line which connects F_A with the node $\overline{A}$ is a symmedian of the anticomplimentary triangle $\overline{A}\,\overline{B}\,\overline{C}$, i.e., it is symmetric to the median $\overline{A}A$ with respect to ('w.r.t.' in brief) the angle bisector of $\angle B\overline{A}C$. On this symmedian the point F_A is the pedal point w.r.t. the circumcenter U_A of the triangle $B\overline{A}C$. Also the line A_0A_1 (Theorem 3) passes through the focal point F_A.*
3. *The strophoid $\mathcal{C}_A$ is the pedal curve of a parabola $\mathcal{P}_A$ w.r.t. the point $\overline{A}$ (Figure 3). The directrix of $\mathcal{P}_A$ coincides with the median $A\overline{A}$. The focal point F_{pA} of $\mathcal{P}_A$ lies on the symmedian $\overline{A}F_A$ and is the mirror image of $\overline{A}$ under reflection in the line F_AU_A. The tangent of the parabola $\mathcal{P}_A$ at its vertex passes through F_A.*

Proof. By virtue of (7), the gradients of the polynomial $P(\mathbf{X})$ in (6) at the absolute circle-points are

$$\mathbf{g}_1 = (2uv + (v^2 - u^2 - 1)i,\ 2(v - iu),\ 2i(-ui + v)),$$
$$\mathbf{g}_2 = (2uv - (v^2 - u^2 - 1)i,\ 2(v + iu),\ -2i(ui + v)).$$

The point F_A of intersection of the two conjugate complex tangent lines at the absolute circle-points can be computed by the vector product $\mathbf{f} = \mathbf{g}_1 \times \mathbf{g}_2$. We obtain the homogeneous coordinates

$$F = \mathbf{f}\mathbb{R} = \big(-2(u^2 + v^2),\ u(u^2 + v^2 + 1),\ v(u^2 + v^2 - 1)\big)\,\mathbb{R}. \tag{13}$$

It can be verified by straightforward computation that $P(\mathbf{f}) = 0$. This means that the singular focal point F_A of $\mathcal{C}_A$ is located on the cubic.

The y-coordinate of F_A is negative if and only if $v(u^2 + v^2 - 1)$ is positive, which is equivalent to $\angle A < \pi/2$. After reflection of F_A in the midpoint of BC it satisfies the equation (8) of the asymptote t_{A_∞} (Figure 2).

The slopes k_1 and k_2 of the two tangent lines t_1, t_2 at $\overline{A}$ satisfy $\tau = 0$, i.e., by (10)

$$uvk^2 + (u^2 - v^2 - 1)k - uv = 0. \tag{14}$$

We note that under $u \neq 0$ these two lines are orthogonal because of $k_1k_2 = -1$.

Two lines through $\overline{A}$ with slopes k and k' are in harmonic position w.r.t. the two tangent lines t_1, t_2 at the node $\overline{A}$ if and only if they are corresponding under the involution with t_1 and t_2 as fixed lines. This involution $k \mapsto k'$ can be expressed as

$$2uvkk' + (u^2 - v^2 - 1)(k + k') - 2uv = 0. \tag{15}$$

[2] It can be proved that a circular cubic with a node is a strophoid if and only if its singular focal point lies on the cubic. We will learn in the sequel that exactly in this case the absolute circle-points are associated points (see Definition 3 and Theorem 6, 3).

This is the inhomogeneous version of a symmetric bilinear form, and it has the property that for $k' = k$ it becomes proportional to (14), which means that the fixed lines of this involution coincide with t_1 and t_2.

The slopes $k = \dfrac{v}{u+1}$ of $\overline{A}C$ and $k' = \dfrac{v}{u-1}$ of $\overline{A}B$ satisfy Eq. (15), since

$$\frac{2uv^3}{u^2-1} + \frac{2uv(u^2-v^2-1)}{u^2-1} - 2uv = 0\,.$$

Therefore the tangent lines t_1, t_2 at the node $\overline{A}$ bisect the angle $\angle B\overline{A}C$.

The line $\overline{A}F_A$ with the equation

$$2uvX_0 + v(u^2+v^2+1)X_1 - u(u^2+v^2-1)X_2 = 0$$

has the slope $k = \dfrac{v(u^2+v^2+1)}{u(u^2+v^2-1)}$. Together with the slope $k' = \dfrac{v}{u}$ of the median $\overline{A}A$ the slope k satisfies Eq. (15) since

$$\frac{2uv^3(u^2+v^2+1)}{u^2(u^2+v^2-1)} + \frac{v(u^2-v^2-1)}{u(u^2+v^2-1)}\,2(u^2+v^2) - 2uv = 0\,.$$

Therefore, t_1 and t_2 bisect also the angle between the median $\overline{A}A$ and the line $\overline{A}F_A$, which characterizes the latter as a *symmedian* of the triangle $\overline{A}BC$. By the way, we never have $kk' = -1$; the line $\overline{A}F_A$ is never orthogonal to the asymptote $t_{A\infty}$. Hence the strophoid $\mathcal{C}_A$ is oblique as stated in item 2.

It can be verified that the line orthogonal to $\overline{A}F_A$ and passing through F_A contains the point

$$U_A = \left(0,\ \frac{1-u^2-v^2}{2v}\right)$$

which is the circumcenter of the triangle $\angle B\overline{A}C$ (see Figures 2 and 3).

In order to prove item 3, we first compute the equation of the line t which passes through an arbitrary point $T \in \mathcal{C}_A$ and is orthogonal to $\overline{A}T$ (Figure 3). Suppose point T is defined by the parameter k in the rational parametrization (11). Then we obtain

$$-\frac{1}{k}\left[X + \frac{u^2k^3 - 2uvk^2 + (v^2+1)k}{(uk-v)(1+k^2)}\right] = Y - \frac{(u^2-1)k^2 - 2uvk + v^2}{(uk-v)(1+k^2)}\,,$$

which after division by $(1+k^2)$ reduces to

$$t\colon\ (uk-v)X + k(uk-v)Y + k = 0. \tag{16}$$

The homogeneous line coordinates of t,

$$(U_0, U_1, U_2) = (k,\ (uk-v),\ (uk-v)k)\,,$$

satisfy the homogeneous quadratic equation

$$U_0(vU_1 - uU_2) + U_1U_2 = 0. \tag{17}$$

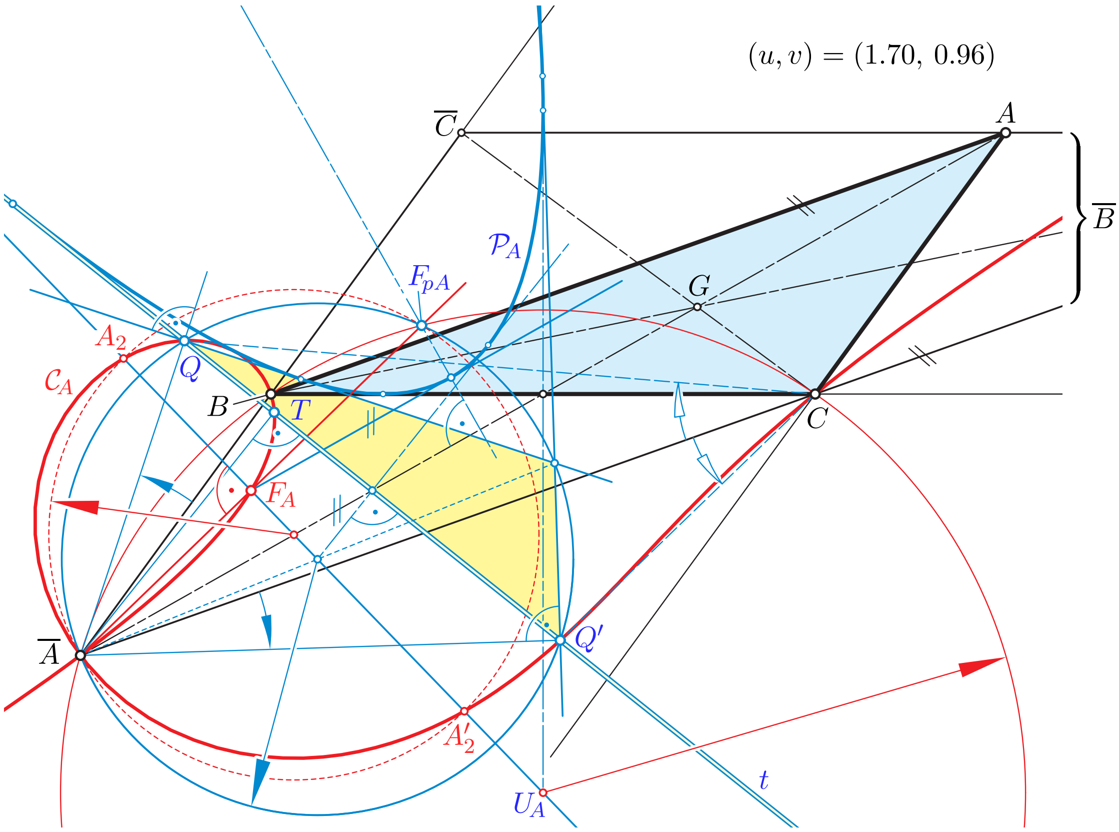

Figure 3: The strophoid $\mathcal{C}_A$ is the pedal curve of the parabola $\mathcal{P}_A$ w.r.t. $\overline{A}$, which is a point of the parabola's directrix $\overline{A}A$. The points Q and Q' are associated on $\mathcal{C}_A$.

This is the tangential equation of a parabola, since the polynomial on the left hand side is irreducible and contains $(U_0, U_1, U_2) = (1, 0, 0)$ as a zero.[3] After inversion of the symmetric coefficient matrix we obtain the (point) equation of the parabola

$$\mathcal{P}_A\colon\ X_0^2 + 2X_0(uX_1 - vX_2) + u^2X_1^2 + 2uvX_1X_2 + v^2X_2^2 = 0.$$

Its point of contact with the line at infinity is $(0, v, -u)\mathbb{R}$. Hence the axis of $\mathcal{P}_A$ is orthogonal to the median $A\overline{A}$. On the other hand, the axis as the polar of A_∞ satisfies

$$(u^2 - v^2) + u(u^2 + v^2)X + v(u^2 + v^2)Y = 0.$$

The line BC and the bisector of B and C are mutually orthogonal tangents of $\mathcal{P}_A$. Therefore the midpoint of BC is a point of the directrix, and the directrix coincides with the median $A\overline{A}$. The finite tangent drawn from A_∞ to $\mathcal{P}_A$ contacts at the vertex and has the equation

$$-uv + (u^2 + v^2)(vX + uY) = 0.$$

By virtue of (13), this tangent passes through F_A.

The respective second tangents through the absolute circle-points have the equation

$$\pm i + (\pm iu + v)X \mp i(\pm iu + v)Y = 0.$$

[3] In the reducible case $u = 0$ the lines t which satisfy (17) form two pencils; the lines t are parallel to the sideline BC or they pass through the orthocenter O of ABC.

Their point of intersection is the parabola's focal point F_{pA} with the homogeneous coordinates $((u^2+v^2),\,-u,\,v)\,\mathbb{R}$. Point F_{pA} lies on the symmedian $\overline{A}F_A$; the second tangent drawn from F_A to $\mathcal{P}_A$ is the bisector of F_{pA} and $\overline{A}$, which passes through the circumcenter U_A of the triangle $\overline{A}BC$. □

Many properties of the equicevian cubic $\mathcal{C}_A$ are related to a particular involutive point-pairing on the cubic and of projective origin. This will be addressed in the following two subsections.

3.3. Associated points

Definition 3. Tangents t of the parabola $\mathcal{P}_A$ intersect the cubic $\mathcal{C}_A^*$ beside the pedal point T w.r.t. $\overline{A}$ in two real or conjugate complex points Q and Q'. We call them *associated points* of the cubic $\mathcal{C}_A$.[4]

In Figures 2 and 3 we find several pairs of associated points on $\mathcal{C}_A^*$. The tangential equation of $\mathcal{P}_A$ shows that the two tangents t_1, t_2 of $\mathcal{C}_A$ at the node $\overline{A}$ are also tangent to $\mathcal{P}_A$. Hence point $\overline{A}$ is self-associated. When the tangent t of $\mathcal{P}_A$ tends to infinity, the pedal point tends to A_∞; therefore the absolute circle-points are associated, too. We summarize

Theorem 5. *On the equicevian cubic $\mathcal{C}_A^*$, the following pairs of points are associated: (B,C), (F_A, A_∞), (A_2, A_2') on the line F_AU_A, the conjugate complex points on the bisector of BC (Theorem 3), the absolute circle-points, and $(\overline{A},\overline{A})$.*

Theorem 6. *Pairs (Q,Q') of associated points define on the A-equicevian cubic $\mathcal{C}_A^*$ an involutive one-to-one correspondance with $\overline{A}\mapsto\overline{A}$ and the following properties:*

1. *The lines which connect $\overline{A}$ with a pair (Q,Q') of associated points, $Q,Q'\neq\overline{A}$, are symmetric w.r.t. the bisectors t_1,t_2 of $\angle B\overline{A}C$.*
2. *The midpoint of associated points Q,Q' lies on the median $\overline{A}A$.*
3. *The tangents of $\mathcal{C}_A$ at associated points meet each other at the point $T'\in\mathcal{C}_A$, which is associated to the pedal point T on the line $t=QQ'$ w.r.t. $\overline{A}$.*
4. *For any point $P\in\mathcal{C}_A$, the lines PQ and PQ' are symmetric w.r.t. $P\overline{A}$.*
5. *Associated points share the lengths $|BB_Q|=|BB_{Q'}|$ and $|CC_Q|=|CC_{Q'}|$ of cevians.*

Proof, item 1. Let point $Q'\in\mathcal{P}_A$ with parameter k' be one of the remaining intersection points of the line $t=QQ'\perp\overline{A}T$ with the cubic $\mathcal{C}_A$. Then Eq. (16) implies

$$\begin{aligned}k(uk'-v)(1+k'^2)-(uk-v)\left[u^2k'^3-2uvk'^2+(v^2+1)k'\right]\\ +k(uk-v)\left[(u^2-1)k'^2-2uvk'+v^2\right]=0,\end{aligned}$$

which after some computations and division by $(k'-k)$ reduces to the quadratic equation in k':

$$\left[uk(u^2-1)-u^2v\right]k'^2-2uv(uk-v)k'-v\left(v^2+1-uvk\right)=0.$$

The two zeros k_1' and k_2' with

$$k_1'+k_2'=\frac{2uv(uk-v)}{uk(u^2-1)-u^2v},\quad k_1'k_2'=\frac{v(uvk-v^2-1)}{uk(u^2-1)-u^2v}\tag{18}$$

[4]It can be proved that on the line t the point of contact with the parabola $\mathcal{P}_A$ and point T separate the associated points (Q,Q') harmonically.

satisfy the equation

$$2uvk_1'k_2' + (u^2 - v^2 - 1)(k_1' + k_2') - 2uv = 0\,.$$

This means by (15) that the lines connecting $\overline{A}$ with a pair (Q, Q') of associated points of $\mathcal{C}_A$ are symmetric w.r.t. the two node tangents t_1 and t_2.

We continue the proof of Theorem 6 after recalling some projective properties of cubics with a node.[5]

3.4. Involutions on cubics with a node

Let $\mathcal{C}$ be a cubic with the node N (Figure 4).[6] Each line through N intersects $\mathcal{C}$ beside N (multiplicity 2) in a single point. This defines a map of the pencil N of lines onto $\mathcal{C}$ which is one-to-one for lines which differ from the tangents t_1, t_2 at N, while both tangents t_1 and t_2 are sent to N. The following definition refers to this correspondance.

Definition 4. Let $\mathcal{C}$ be a cubic with the node N. The involution in the pencil N with the two tangents t_1, t_2 at N as fixed lines induces an *involution α of type 1* on $\mathcal{C}$. Any involution in the pencil N which interchanges t_1 and t_2 induces an *involution β of type 2* on $\mathcal{C}$.

Of course, involutions of both types keep the node N fixed. Note that these involutions do not preserve collinearity of triples of points on $\mathcal{C}$.

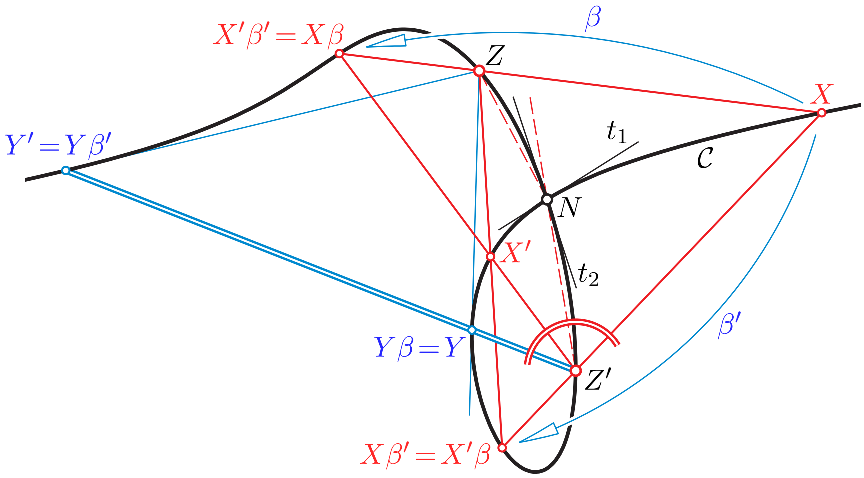

Figure 4: On the cubic $\mathcal{C}$ the involutions α, β and $\beta' = \alpha \circ \beta$ commute pairwise

Lemma 7. *1. For any involution β of type 2 on $\mathcal{C}$, all lines which connect corresponding points $X, X\beta \in \mathcal{C}$, $X \neq X\beta$, have a point $Z \in \mathcal{C} \setminus \{N\}$ in common, the 'center' of β. Also the tangent lines at the fixed points Y, Y' of β pass through the center Z (see Figure 4).*

2. Each involution β of type 2 commutes with the involution α of type 1. Therefore α maps each pair $(X, X\beta)$ of points corresponding under β again onto such a pair, and vice versa. In particular, the fixed points Y, Y' of β are corresponding under α.

[5]There is a vast literature on nonsingular cubics, i.e., cubics without any singularity. In particular, the commutative group of points on the cubic is often addressed (e.g., [6]). Less attention is paid to cubics with a node.

[6]The statements that follow are also valid for cubics with an isolated double point, i.e., in the case of conjugate complex node tangents.

3. *Each involution β of type 2 defines another involution $\beta' = \alpha \circ \beta = \beta \circ \alpha$ of type 2, and the involutions α, β and β' commute pairwise. The centers Z of β and Z' of β' are corresponding under α.*
4. *The lines connecting Z' with corresponding points $X, X\beta \in \mathcal{C}_A \setminus \{N, Z'\}$ constitute an involution in the pencil Z'. This involution keeps the line $Z'N$ fixed as well as the line through the fixed points of β.*
5. *For each point $Z' \in \mathcal{C}\backslash\{N\}$, the lines connecting Z' with corresponding points $(X,\ X')$ of α constitute an involution which keeps the line $Z'N$ fixed. For each quadrangle formed by pairs of points (X, X') and (Y, Y') corresponding under α, the diagonal points $XY \cap X'Y'$ and $XY' \cap X'Y$ lie on $\mathcal{C}_A$, and they are corresponding under α, as well.*

Proof. Two different involutions α, β in the pencil N commute if and only if the fixed lines of one involution are corresponding under the other involution. Exactly in this case the composition $\beta' = \alpha \circ \beta$ is an involution, too.

In order to reduce the computational cost, we set up the given projective cubic $\mathcal{C}$ in normal form [8]

$$\mathcal{C}\colon\ X_0(X_1^2 - X_2^2) + X_1^3 = 0.$$

Point $N = (1, 0, 0)\mathbb{R}$ is the node with the tangents $X_1 \pm X_2 = 0$. The cubic $\mathcal{C}$ admits the rational parametrization

$$\mathbf{X}(t) = \big(t^3,\ -t(t^2-1),\ (t^2-1)\big)$$

with $\mathbf{X}(1) = \mathbf{X}(-1) = (1, 0, 0)$ as the node. The parameter t serves as a projective coordinate in the pencil N. Hence $t \mapsto t' = 1/t$ induces the involution

$$\alpha\colon\ \mathcal{C} \to \mathcal{C},\quad \big(t^3,\ -t(t^2-1),\ (t^2-1)\big)\mathbb{R} \ \mapsto\ \big(1,\ (t^2-1),\ -t(t^2-1)\big)\mathbb{R}.$$

The mapping $t \mapsto \dfrac{at-c}{ct-a}$, $a^2 \neq c^2$, induces on $\mathcal{C}$ an involution β of second kind:

$$\beta\colon\ \big(t^3,\ t(1-t^2),\ (t^2-1)\big)\mathbb{R} \ \mapsto\ \big((at-c)^3,\ (at-c)(c^2-a^2)(t^2-1),\ (ct-a)(a^2-c^2)(t^2-1)\big)\mathbb{R}.$$

For each $t \in \mathbb{R}$ the line connecting X with $X\beta$ has the line coordinate vector

$$\mathbf{X}(t) \times \mathbf{X}(t)\beta = \big((c^2-a^2)(t^2-1),\ (c^2-a^2)t^2 + act - c^2,\ ct(at-c)\big).$$

All these lines pass through the point

$$Z = \big(-c^3,\ c(c^2-a^2),\ a(c^2-a^2)\big)\mathbb{R},$$

which lies on $\mathcal{C}$ and belongs to the parameter $t = -c/a$.

When X converges towards a fixed point Y of β, the line $XX\beta$ converges against the tangent of $\mathcal{C}$ at the fixed point. Cubics with a node are of class 4 (see [8]). Therefore, conversely, through point $Z \in \mathcal{C}$ at most two tangents of $\mathcal{C}$ can be drawn which differ from the tangent at Z.

The α-image Z' corresponds to $t = -a/c$ and has the homogeneous coordinates

$$Z' = \big(a^3,\ a(c^2-a^2),\ c(c^2-a^2)\big)\mathbb{R}.$$

For the lines connecting Z' with points X and $X\beta$ we obtain the homogeneous line coordinates

$$\begin{aligned} Z'X &= \big((c^2-a^2)(t^2-1),\ a^2 - act + (c^2-a^2)t^2,\ at(a-ct)\big)\mathbb{R},\\ Z'X\beta &= \big((a^2-c^2)(t^2-1),\ a^2t^2 - act + (c^2-a^2),\ a(at-c)\big)\mathbb{R}. \end{aligned}$$

Now we can verify that there is a projective transformation mapping $Z'X$ onto $Z'X\beta$, since in matrix form

$$\begin{pmatrix} (a^2-c^2)(t^2-1) \\ a^2t^2-act+(c^2-a^2) \\ a(at-c) \end{pmatrix} = \begin{pmatrix} a^2-c^2 & 0 & 0 \\ 2a^2-c^2 & c^2-a^2 & 0 \\ ac & 0 & c^2-a^2 \end{pmatrix} \begin{pmatrix} (c^2-a^2)(t^2-1) \\ a^2-act+(c^2-a^2)t^2 \\ at(a-ct) \end{pmatrix}.$$

This means for the pencil Z', there is a projectivity $Z'X \mapsto Z'X\beta$ which by definition is involutory and fixes the lines through N and through the fixed points of β.

Since β and $\beta' = \alpha \circ \beta$ commute, the line connecting the fixed points Y, Y' of β (see Figure 4) passes through Z', and vice versa. For any point $X \in \mathcal{C} \setminus \{N\}$ with $X \neq X\beta, X\beta'$, the four points X, $X\beta$, $X\beta'$, and $X' = X\alpha$ form a quadrangle with Z and Z' as diagonal points. We note coinciding lines $Z'X\beta = Z'X'$ and $ZX\beta' = ZX'$.

Item 5 is just a consequence of the fact that for any pair (X, Y), $X, Y \in \mathcal{C} \setminus \{N\}$, there is an involution β of the second kind with $X \mapsto Y$. But also for each given point $Z \in \mathcal{C} \setminus \{N\}$ there is such an involution with the center Z.

By the way, the plane cubic $\mathcal{C}$ with a node can also be seen as the image of a spatial cubic under central projection. This offers a second way to prove Lemma 7 (note the analysis of a "strophoidal" spatial cubic in [7, 318–332]). □

Remark 2. If point Z is chosen as the unit point E of a commutative group which can be defined on $\mathcal{C} \setminus \{N\}$ (see, e.g., [6]), then by Lemma 7, 1. and 2. the image Z' under α is the only point other than $Z = E$ which satisfies $Z' \circ Z' = E$. The involution α sends point $X \in \mathcal{C} \setminus \{N\}$ onto $X' = Z' \circ X$.

3.5. Properties of equicevian cubics ctd.

According to Theorem 6, 1., associated points (X, X') of the A-equicevian cubic $\mathcal{C}_A$ are corresponding under the involution α of type 1, as defined in Definition 4. Hence we can directly apply Lemma 7.

Proof of Theorem 6, items 2–5. Let β be the involution on $\mathcal{C}_A$ induced by pairs of orthogonal lines through $\overline{A}$. Then the absolute circle-points are the two fixed points of β. By virtue of Lemma 7, 1., the center Z of β coincides with the singular focal point F_A, and point A_∞ serves as point Z'. By Lemma 7, 4. and 5., for each finite pair (X, X') of associated points the connecting lines with A_∞ are symmetrical w.r.t. $A_\infty\overline{A} = A\overline{A}$. Hence the midpoint of (X, X') lies on the median $A\overline{A}$ (Figure 3).

Let t be the line connecting the associated points $Q, Q' \in \mathcal{C}_A \setminus \{\overline{A}\}$. There is an involution β_1 of second kind which fixes Q. By Lemma 7, 2., β_1 fixes Q' as well, and by Lemma 7, 1. the tangent lines of $\mathcal{C}_A$ at Q and Q' meet at the center of β_1. This center is associated to the pedal point of t w.r.t. $\overline{A}$. Note in Figure 2, e.g., the points (B, C) with pedal point A_0 and the associated meeting point A_0' of the tangents at B and C, or the pair (A_2, A_2') with pedal point F_A and $F_A' = A_\infty$ (Theorem 5).

In order to prove Theorem 6, 4. and 5., we choose two pairs of associated points of $\mathcal{C}_A$, the pair (Q, Q') and the two absolute circle-points. By virtue of Lemma 7, 5., the connecting lines with any point $Z' \in \mathcal{C}_A$ belong to an involution which fixes $Z'N$. The second fixed line must be orthogonal to $Z'N$ since also the isotropic lines are corresponding. Consequently, the other two corresponding lines, $Z'Q$ and $Z'Q'$, must be symmetric w.r.t. $Z'N$. Applied to the choice $Z' = C$ (see Figure 3), the lines CQ and CQ' are symmetric w.r.t. to $\overline{A}\,\overline{B}$ and the altitude through C. The latter guarantees that the cevians CC_Q and $CC_{Q'}$ have the same length. The same holds for vertex B.[7] □

Remark 3. We refer to Theorem 6, 4.: Let (Q, Q') be fixed. Then the strophoid $\mathcal{C}_A$ is the locus of points P such that the lines PQ and PQ' have a bisector passing through $\overline{A}$. If the three given points Q, Q' and $\overline{A}$ were collinear and $\overline{A}$ different from the midpoint of QQ', the requested locus of points P is the Apollonius circle together with the diameter line QQ', hence a reducible strophoid.

The following two properties of strophoids are related to circles. The first is often used to characterize strophoids as particular sets of points (see, e.g., [17, pp. 95, 155], [18, pp. 59–70], or [24]).

Theorem 8. *1. Let a point D vary on the median $A\overline{A}$. Then the equicevian cubic $\mathcal{C}_A$ is the locus of intersection points of the line DF_A and the circle centered at D and passing through the node $\overline{A}$ (Figure 2).*

2. Each pair (Q, Q') of associated points of $\mathcal{C}_A$ lies on a circle which is centered on the line F_AU_A and passes through $\overline{A}$ and F_{pA} (Figure 3). The circumcircle of the triangle $\overline{A}BC$ with center U_A is a particular case; it intersects $\mathcal{C}_A$ at B and C.

3. The two pencils of circles mentioned before share a circle (dotted in Figure 3) which passes through the points A_2 and A_2' on the bisectors of the two tangent lines t_1, t_2 at the node $\overline{A}$ (see also Figure 2).

Proof. We recall the involution β induced by pairs of orthogonal lines through $\overline{A}$ (note page 145): The circle with diameter $XX\beta$ must pass through $\overline{A}$ because of the right angle there. Since by virtue of Lemma 7, 4. the lines $A_\infty X$ and $A_\infty X\beta$ are harmonic w.r.t. $A_\infty\overline{A}$ and the line at infinity, the center of this circle lies on the median $\overline{A}A$. Due to Lemma 7, 1., the line $XX\beta$ passes through F_A. In Figure 2, e.g., the pedal point A_0 of BC and the point A_1 on the bisector of B and C are corresponding in β. The two intersection points A_2, A_2' on the line F_AU_A, which is perpendicular to $\overline{A}F_A$, are not only corresponding under β, but also associated, and therefore fixed points of the involution $\beta' = \alpha \circ \beta$ with the center A_∞.

In order to prove item 2, we start with the associated points (Q, Q') on line t with pedal point T (Figure 3); the midpoint of QQ' lies on the median $\overline{A}A$. Beside t, also the line through Q orthogonal to $\overline{A}Q$ and the line through Q' orthogonal to $\overline{A}Q'$ are tangent to the parabola $\mathcal{P}_A$. These three tangent lines define a triangle (shaded in Figure 3) which according to a wellknown theorem passes through the parabola's focal point F_{pA}. On the other hand, due to the right angles at Q and Q', this circle passes through $\overline{A}$, too. Therefore, its center lies on the bisector F_AU_A of F_{pA} and $\overline{A}$, but also on the bisector of Q and Q', which intersects the median $\overline{A}A$ at the midpoint of QQ' (Theorem 6, 2.). □

3.6. Equicevian cubics are focal curves

Theorem 9. *Pairs (Q, Q') of associated points on the A-equicevian cubic $\mathcal{C}_A \setminus \{B, C, \overline{A}\}$ are the focal points of conics $\mathcal{N}$ which pass through B and C with the respective tangents $B\overline{A}$ and $C\overline{A}$ (Figure 5).*

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[7] By the way, the second part of Lemma 7, 5. implies that the bisector of QQ' intersects $\mathcal{C}_A$ in two conjugate complex points. They are the finite intersection points of the isotropic lines through Q and those through Q'.

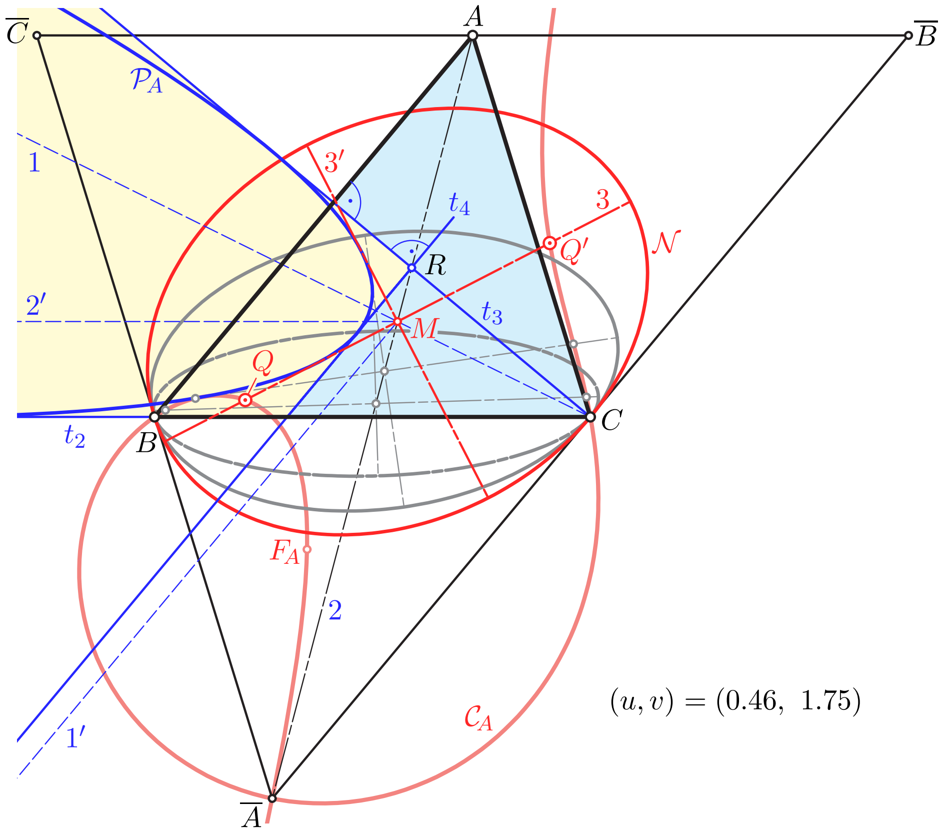

Figure 5: The equicevian curve $\mathcal{P}_A$ is the locus of focal points (Q, Q') of conics $\mathcal{N}$ which contact line $B\overline{A}$ at B and line $C\overline{A}$ at C.

Proof. For each conic $\mathcal{N}$ of the said set, the median $A\overline{A}$ is a diameter. We can specify any conic in this set by choosing its center M on the median $A\overline{A}$. Any choice different from $\overline{A}$ and from the midpoint of BC guarantees a regular conic.

The median $\overline{A}A$ is the directrix of the negative pedal curve $\mathcal{P}_A$. We show first that the axes of the conic $\mathcal{N}$ coincide with the tangents $3, 3'$ drawn from M to the parabola $\mathcal{P}_A$.

For this purpose we recall a theorem which is attributed to G. Desargues: Given any quadrilateral, there is a range of conics tangent to the four sidelines. Let P be a point which is not placed on any of these lines. Then the tangents drawn from P onto the conics of this range are pairs of an involution. Also the lines connecting P with opposite vertices of the given quadrilateral belong to this involution.

We pick out four tangents $t_1, \dots, t_4$ of the parabola $\mathcal{P}_A$ (note the shaded quadrangle in Figure 5): the line t_1 at infinity, the sideline $t_2 = BC$, the altitude t_3 at C, which intersects the median $\overline{A}A$ at R, and the second tangent t_4 drawn from R to $\mathcal{P}_A$. This tangent t_4 is orthogonal to t_3 and therefore parallel to AB.

Point $C = t_2 \cap t_3$ and the ideal point $t_1 \cap t_4$ of AB are opposite vertices as well as $R = t_3 \cap t_4$ together with the ideal point $t_1 \cap t_2$ of BC. By virtue of Desargues' theorem, in the pencil M the lines $1 = MC$ and $1'$ parallel to AB make a pair of the said involution. The same holds for $2 = MR = \overline{A}A$ and $2'$ parallel to BC. The pairs $(1, 1')$ and $(2, 2')$ are at the same time conjugate diameters of the conic $\mathcal{N}$, which contacts the line $C\overline{A}$ at C and has $\overline{A}$ as the pole of BC. The tangents 3 and $3'$ of $\mathcal{P}_A$ form the orthogonal pair in the said involution; therefore they are the axes of $\mathcal{N}$.

On line 3, the two associated points Q, Q' of $\mathcal{P}_A$ are symmetric w.r.t. M (Theorem 6, 2.), and their connections with C are symmetric w.r.t. the altitude t_3 of ABC (Theorem 6, 4.). This defines the pair (Q, Q') uniquely.

It is well known that at each point C of a conic the tangent and the normal bisect the angle between the connections with the focal points, which of course are also symmetric w.r.t. the center. Consequently, the associated points Q, Q' on line 3 are identical with the focal points of $\mathcal{N}$.

We know already from Footnote 7 (page 146) that exactly on one of these two orthogonal tangents $3, 3'$ of $\mathcal{P}_A$ the points of intersection with the cubic are real. However, all four focal points are points of the A-equicevian cubic $\mathcal{C}_A$.

Conversely, each pair (Q, Q') of associated points other than $(\overline{A}, \overline{A})$ and (B, C) has a midpoint $M \in A\overline{A}$, where a conic $\mathcal{N}$ of the requested kind can be centered (note [15, p 515, footnote 235]). The particular pair (F_A, A_∞) defines a parabola. Hence F_A is the focal point of the *parabola*, which is tangent to $B\overline{A}$ and $C\overline{A}$ at B and C, respectively. □

4. The equicevian points of a triangle

Points P which are A-equicevian and B-equicevian, satisfy the property

$$|AA_P|^2 = |BB_P|^2 = |CC_P|^2.$$

Hence they are C-equicevian, too.

In order to find all equicevian points for a given triangle ABC, we have to intersect the sets Ω_A and Ω_B where according to Theorem 1 and by (12)

$$\Omega_A = (BC \cup \mathcal{C}_A) \setminus \{B, C, \overline{A}\} \;\text{ and analogously }\; \Omega_B = (AC \cup \mathcal{C}_B) \setminus \{A, C, \overline{B}\}.$$

According to Theorem 3, on each sideline of the triangle there are at most two real equicevian points P. Those on BC are characterized by the conditions $|AP| = |BC|$ and $P \neq B, C$ (note Figure 7).

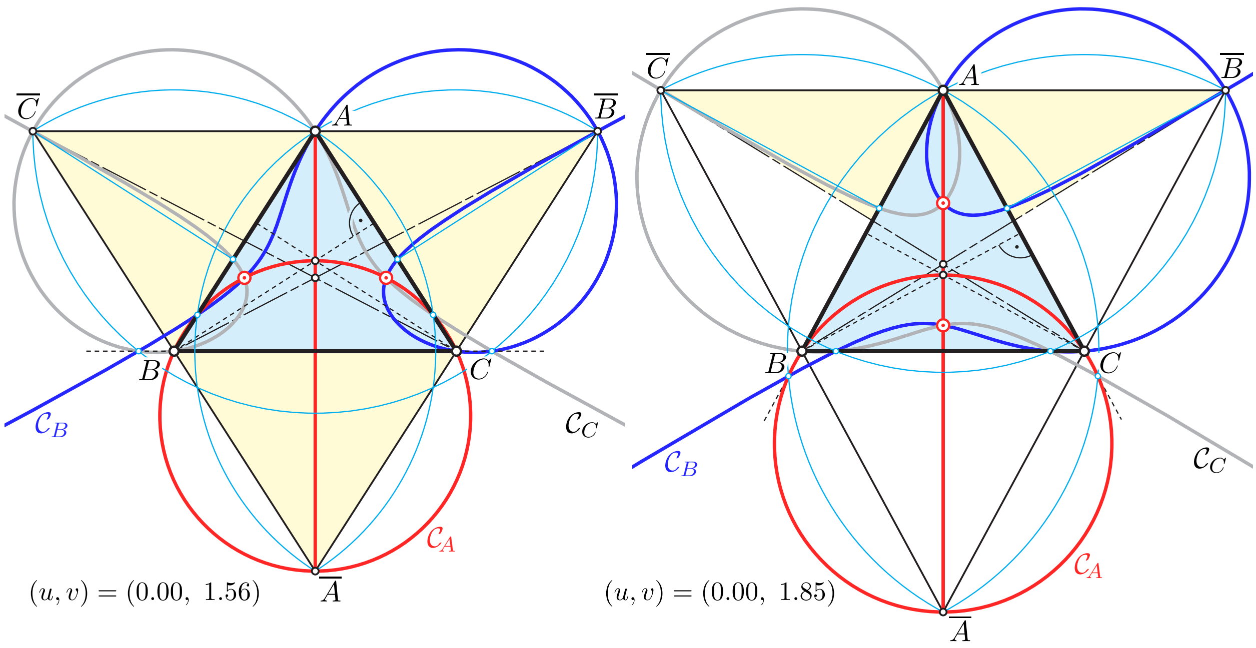

Figure 6: At isosceles triangles, two improper real equicevian points are either on the axis of symmetry or symmetric to it. At equilateral triangles the two become coincident.

On at least two sidelines the equicevian points must be real, because if the altitude h_A of A is longer then $|BC|$, we obtain

$$h_B \leq |BC| < h_A \leq |AC|$$

and analogously $h_C < |AB|$.

Suppose one of the equicevian points P on the sideline BC coincides with the vertex B or C. Then the triangle must be isosceles. Figure 6 shows the case $u = 0$ with $|AB| = |AC|$: By virtue of Theorem 2, the cubic $\mathcal{C}_A$ splits into the axis of symmetry $A\overline{A}$ and the circumcircle of $\overline{A}BC$; the second node of $\mathcal{C}_A$ is the orthocenter O of ABC. Exactly for equilateral triangles the intersection $\Omega_A \cap \Omega_B \cap \Omega_C$ consists of a single point, the center of ABC, which is a node for all three cubics.

4.1. Remaining equicevian points

Proper equicevian points are among the intersection points between the cubics $\mathcal{C}_A$ and $\mathcal{C}_B$, where $\mathcal{C}_B$ — the zero-set of the polynomial $H_B(X, Y)$ in (4) — has properties analogous to those of $\mathcal{C}_A$ listed in Theorems 2, 3, 4, 6, and 8. These two cubics share the absolute circle-points, the vertex C and the real or conjugate complex points of intersection of the line AB

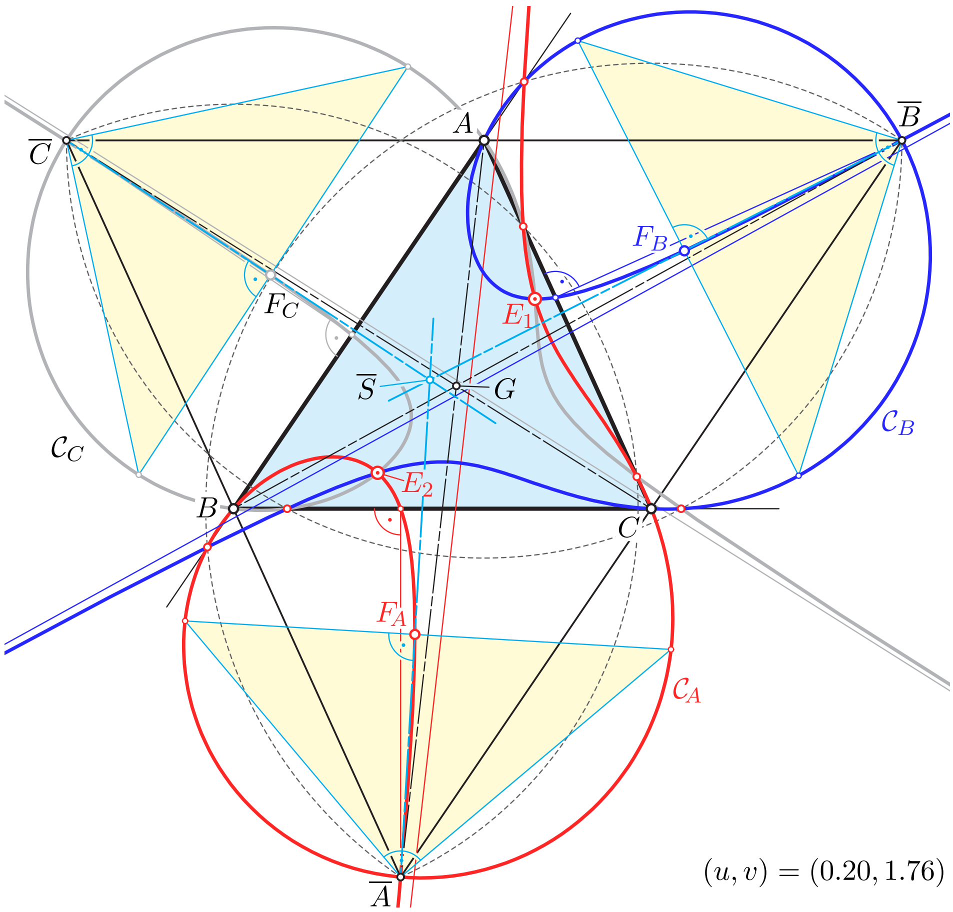

Figure 7: This triangle ABC has 6 improper equicevian points and two proper equicevian points E_1 and E_2 in the interior. The displayed strophoids $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ have the focal points F_A, F_B, F_C, respectively.

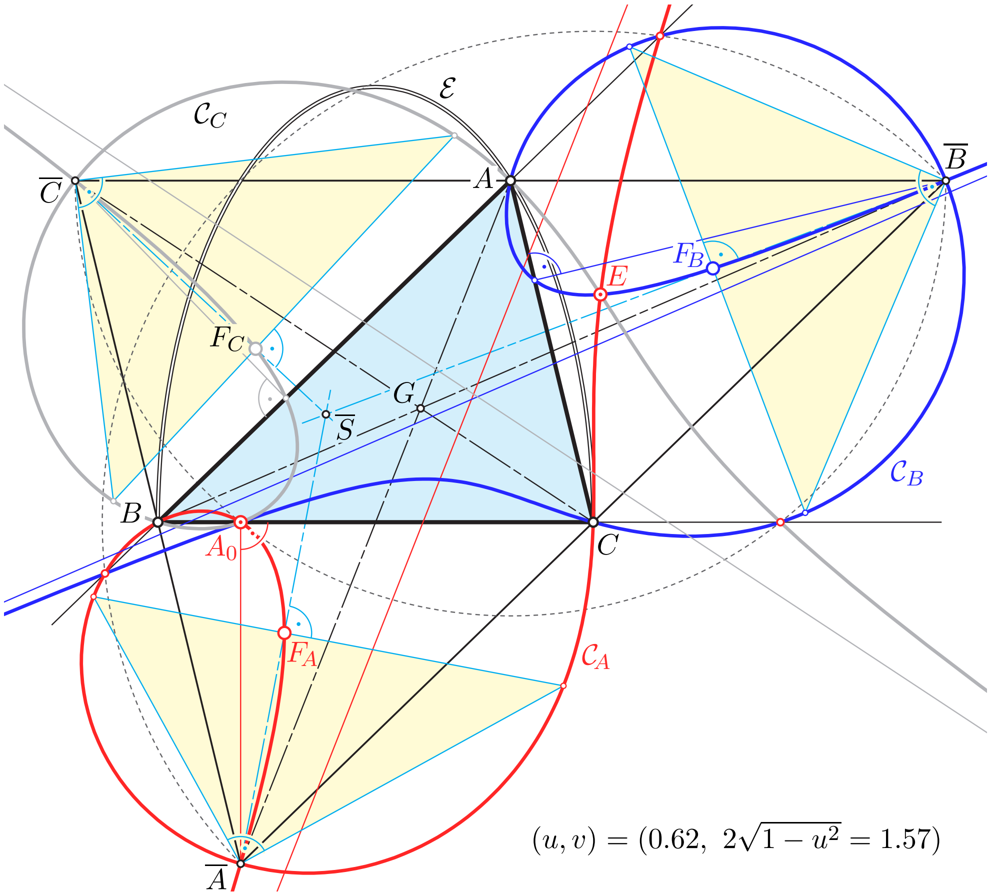

Figure 8: This triangle ABC has 4 improper equicevian points and one proper equicevian points E in the exterior. The displayed strophoids $\mathcal{C}_A$, $\mathcal{C}_B$, $\mathcal{C}_C$ have the focal points F_A, F_B, F_C, respectively.

and the circle with center C and radius $|AB|$ (Figures 7 and 8). Point C and the absolute circle-points are intersection points of multiplicity 1 for the following reasons:

- The tangents of $\mathcal{C}_A$ and $\mathcal{C}_B$ at C are the respective mirror images of BC and AC under reflection in the altitude of C (note Figure 2). Therefore they must be different; they include an angle which is congruent to $\angle ACB$.
- The tangents of $\mathcal{C}_A$ and $\mathcal{C}_B$ at the absolute circle-points have the respective singular focal points F_A and F_B as their only real points. A coincidence of F_A and F_B would mean by Theorems 9 and 5 that there is a parabola which contacts all three sides of $\overline{A}\,\overline{B}\,\overline{C}$ at their respective midpoints C, A and B;[8] the parabola would have a symmetry center at the centroid G of ABC.

A higher intersection multiplicity between $\mathcal{C}_A$ and $\mathcal{C}_B$ can only appear at the points on line AB (note, e.g., the cubics $\mathcal{C}_B$ and $\mathcal{C}_C$ in Figure 8 at A_0). According to Bézout's Theorem, there are at most four points of intersection remaining, hence four real or pairwise conjugate complex proper equicevian points. We call them *remaining equicevian points*. They all must be finite as the medians through A and B can never be parallel. Note that all remaining equicevian points must be common to all three cubics (see Figures 7 and 8).

[8]The lines $\overline{A}F_A$, $\overline{B}F_B$ and $\overline{C}F_C$ are symmedians of the triangle $\overline{A}\,\overline{B}\,\overline{C}$; they meet at the symmedian point $\overline{S}$ (= Lemoine- or Grebepoint, X_6 in the list of triangle centers given in [14]).

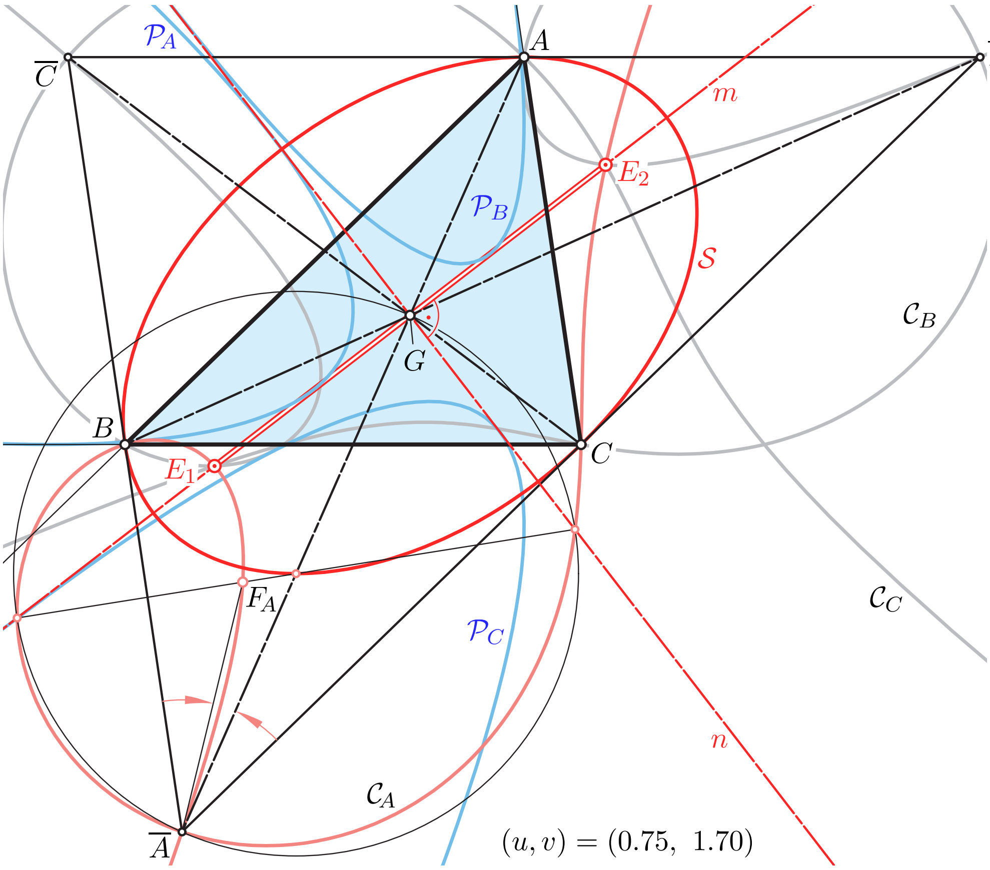

Figure 9: The negative pedal curves $\mathcal{P}_A$, $\mathcal{P}_B$, and $\mathcal{P}_C$ of the three equicevian cubics $\mathcal{C}_A$, $\mathcal{C}_B$, and $\mathcal{C}_C$ share the tangents m and n passing through the centroid G. These tangents are the principal axes of the Steiner circumellipse $\mathcal{S}$ of ABC. The real remaining equicevian points E_1 and E_2 are the focal points of $\mathcal{S}$.

4.2. The main theorem

Theorem 10. *For any triangle ABC, the remaining four equicevian points, i.e., the finite intersection points of the three equicevian cubics $\mathcal{C}_A$, $\mathcal{C}_B$ and $\mathcal{C}_C$, are identical with the two real and two conjugate complex focal points of the Steiner circumellipse $\mathcal{S}$ of the triangle ABC. For non-isosceles triangles the axes of $\mathcal{S}$ are common tangents of the three negative pedal curves, the parabolas $\mathcal{P}_A$, $\mathcal{P}_B$ and $\mathcal{P}_C$.*

Proof. We recall Theorem 9 and choose the centroid G as the center of a conic $\mathcal{S}$ which contacts $B\overline{A}$ at B and $C\overline{A}$ at C. Since G lies also on the median $\overline{B}B$ and its conjugate diameter w.r.t. $\mathcal{S}$ is parallel to AC (note the lines 2 and 2′ in Figure 5), the same conic $\mathcal{S}$ will contact $A\overline{B}$ at A. Hence $\mathcal{S}$ will be the Steiner circumellipse of the triangle ABC (Figure 9). The two real focal points of $\mathcal{S}$ as well as its two conjugate complex focal points are associated points not only of $\mathcal{C}_A$, but also of $\mathcal{C}_B$ and $\mathcal{C}_C$ and hence *the* remaining four equicevian points.

For non-equilateral triangles, the axes m, n of $\mathcal{S}$ are unique. They contain pairs of associated points w.r.t. $\mathcal{C}_A$, $\mathcal{C}_B$, and $\mathcal{C}_C$. According to Definition 3, the axes m, n of the Steiner ellipse $\mathcal{S}$ are common tangents of the three negative pedal curves $\mathcal{P}_A$, $\mathcal{P}_B$ and $\mathcal{P}_C$ (Figure 9). □

Lemma 11. *Given a non-equilateral triangle, among the two axes m, n of the Steiner ellipse $\mathcal{S}$, the major axis does not meet the interior of the longest side.*

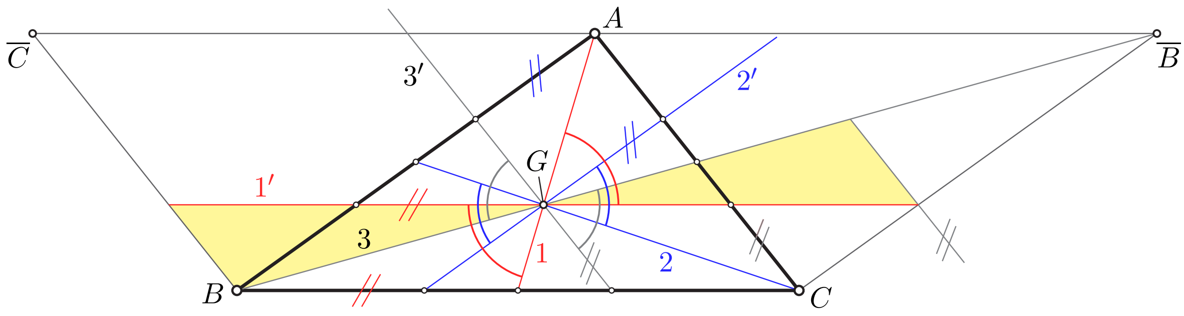

Figure 10: The pairs $(1, 1')$, $(2, 2')$ and $(3, 3')$ are conjugate diameters of the Steiner circumellipse. In the displayed case $|BC| \geq |AB| \geq |AC|$ the real remaining equicevian points are located in the shaded closed area.

Under $|BC| \geq |AB| \geq |AC|$ segments of the median through B and the parallel to BC through the centroid belong to the boundary of the centrally symmetric closed area where the real focal points of $\mathcal{S}$ are located (see Figure 10).

Proof. It is well known that the major axis of an ellipse lies within the acute angle formed by any two conjugate diameters. There are three pairs $(1, 1')$, $(2, 2')$, $(3, 3')$ of conjugate diameter lines of the Steiner ellipse $\mathcal{S}$ with its center at the centroid G (see Figure 10): each median is conjugate to the parallel of the respective side. The six involved lines through G meet the sides either at the vertices or at the midpoints or they intersect the sides in the ratio $1 : 2$.

We first focus on non-isosceles triangles and assume $|BC| > |AB| > |AC|$. For each side, the acute angle between the corresponding pair of conjugate diameters contains the common vertex with the neighbor side which is longer. Therefore the common intersection of the three acute angles cannot contain any point of the longest side BC (note shaded in Figure 10) .

In the case of an isosceles triangle with either $|BC| = |AB| > |AC|$ or $|BC| > |AB| = |AC|$ either $3, 3'$ or $1, 1'$ are already the respective axes of $\mathcal{S}$. For the same reason as before in the first case 3 is the major axis, in the second case $1'$ (compare Figure 6).

Since $\mathcal{S}$ is the Steiner inellipse of the triangle $\overline{A}\,\overline{B}\,\overline{C}$, the real remaining equicevian points, which are symmetric w.r.t. G, must lie in the interior of $\overline{A}\,\overline{B}\,\overline{C}$ (Figure 10). □

We express the main theorem also in coordinates:

Theorem 12. *Based on the coordinates (1) of the triangle ABC (see Figure 1), the remaining equicevian points $E_1, \ldots, E_4$ are as below:*

$$\textit{for } u \neq 0 \left\{ \begin{array}{l} E_{1,2} \;=\; \left((1 \pm 2k)\,\frac{u}{3},\; \left(1 \pm \frac{2}{k}\right)\frac{v}{3} \right), \\ E_{3,4} \;=\; \left(\frac{1}{3}\left(u \mp \frac{2iv}{k}\right),\; \frac{1}{3}\,(v \pm 2iku) \right), \\ \textit{where } k = \frac{1}{u\sqrt{2}}\sqrt{u^2 - v^2 + 3 + \sqrt{(u^2+v^2+3)^2 - 12v^2}}, \end{array} \right. \tag{19}$$

$$\textit{for } u = 0 \left\{ \begin{array}{l} \textit{and } v^2 \leq 3: \;\; E_{1,2} = \frac{1}{3}\left(u \pm 2\sqrt{3-v^2},\, v\right), \;\; E_{3,4} = \frac{1}{3}\left(u,\, v \pm 2i\sqrt{3-v^2}\right), \\ \textit{and } v^2 \geq 3: \;\; E_{1,2} = \frac{1}{3}\left(u,\, v \pm 2\sqrt{v^2-3}\right), \;\; E_{3,4} = \frac{1}{3}\left(u \pm 2i\sqrt{v^2-3},\, v\right). \end{array} \right.$$

Proof. Due to the following corollary, which is equivalent to Marden's theorem[9], the computation of the real remaining equicevian points E_1, E_2 is straightforward. The two imaginary solutions E_3, E_4 are the finite points of intersection of the isotropic lines passing through E_1 and E_2. □

Corollary 13. *Let $a, b, c \in \mathbb{C}$ be the complex coordinates of the vertices of the triangle ABC w.r.t. a Cartesian coordinate system with the origin at the centroid G. Then the complex coordinates of the two real remaining equicevian points of the triangle ABC are*

$$e_{1,2} = \pm\sqrt{\frac{2}{3}\,(a^2+b^2+c^2)}, \quad \textit{where } a+b+c=0.$$

Proof. We start with recalling a wellknown lemma: For any ellipse centered at the origin let $p, q \in \mathbb{C}$ be the complex coordinates of the endpoints of two conjugate diameters. Then $f_{1,2} = \pm\sqrt{p^2+q^2}$ are the complex coordinates of the real focal points.
This can be verified in the following way: Since a rotation about the origin through an angle ψ, i.e., the multiplication with $e^{i\psi}$, acts on $f_{1,2}$ in the same way as on p and q, we can set

$$p = r\cos\varphi + is\sin\varphi \quad \text{and} \quad q = -r\sin\varphi + is\cos\varphi,$$

where $r, s \in \mathbb{R}$ are the semiaxes of the given ellipse. Then $p^2+q^2 = r^2-s^2$ which confirms the statement.

For the Steiner circumellipse we can set $p = a$ and $q = \frac{1}{\sqrt{3}}(c-b)$. This implies for the real focal points and equicevian points

$$e_{1,2}^2 = a^2 + \frac{1}{3}(b^2+c^2-2bc).$$

We subtract from the right hand side the vanishing term $\frac{1}{3}[a^2-(b+c)^2]$ and obtain

$$e_{1,2}^2 = a^2 + \frac{1}{3}(b^2+c^2-2bc-a^2+b^2+2bc+c^2) = \frac{2}{3}(a^2+b^2+c^2),$$

which is the statement of Corollary 13. □

Remark 4. Corollary 13 is also the basis for a new proof of Marden's Theorem: The vertices $\overline{A}$, $\overline{B}$ and $\overline{C}$ with the complex coordinates $-2a$, $-2b$ and $-2c$, respectively, are the zeros of the polynomial $p(z) = (z+2a)(z+2b)(z+2c)$. Because of $a+b+c=0$ it reduces to

$$p(z) = z^3 - 2(a^2+b^2+c^2)z - 8abc.$$

As a consequence, the derivative $p'(z) = 3z^2 - 2(a^2+b^2+c^2)$ has the two roots e_1 and e_2 as given above.

4.3. Ruler and compass construction

Having Theorem 10 in mind, it is not hard to find ruler-and-compass constructions for the vertices and the focal points of the Steiner circumellipse. Nevertheless, also a direct construction of the remaining equicevian points deserves publication, as on the one hand it is rather short, and on the other hand there are no apparent relations to constructions for ellipses — like, e.g., to the construction of the axes from conjugate diameters.

[9] This theorem states (see, e.g., [19, 5]): If in the complex plane the roots of a third-degree polynomial $p(z)$ are the vertices of a triangle $\overline{A}\,\overline{B}\,\overline{C}$, then the roots of the first derivative $p'(z)$ are the focal points of the Steiner inellipse of $\overline{A}\,\overline{B}\,\overline{C}$.

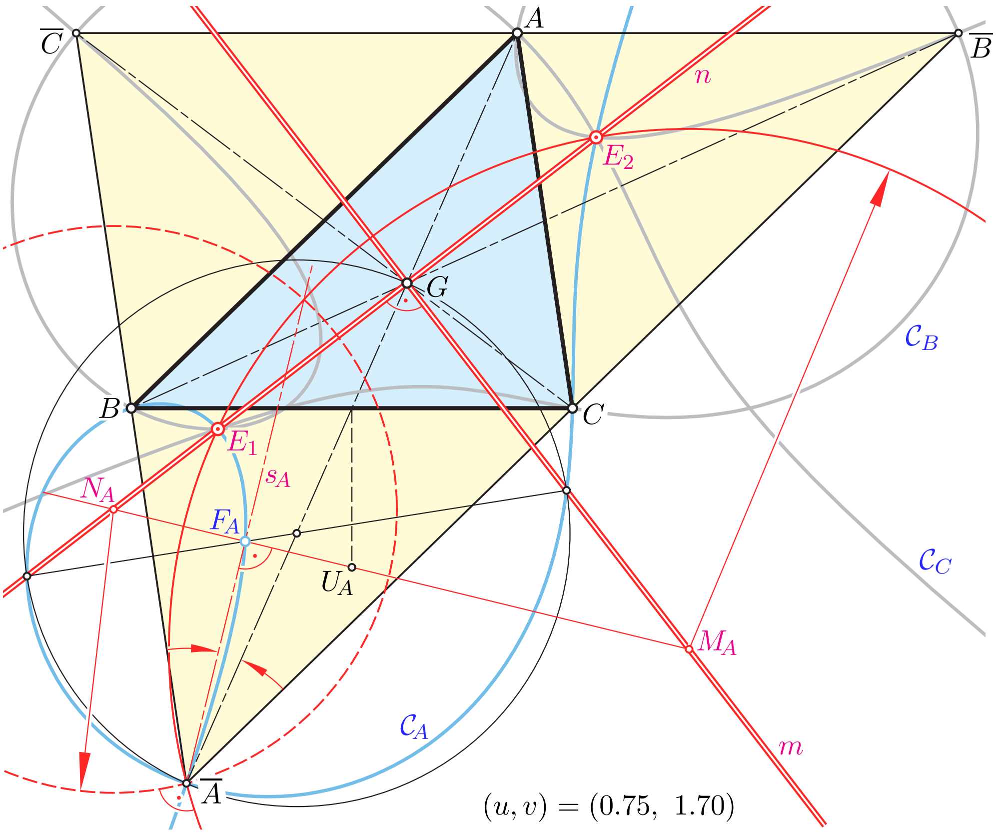

Figure 11: Ruler and compass construction of the remaining real equicevian points E_1 and E_2. The displayed equicevian cubics $\mathcal{C}_A, \mathcal{C}_B, \mathcal{C}_C$ are only used as an illustration.

Theorem 14. *The remaining real equicevian points E_1 and E_2 of a given triangle ABC can be constructed by ruler and compass in the following way (Figure 11):*

1. *Draw the anticomplementary triangle $\overline{A}\,\overline{B}\,\overline{C}$ and the medians $A\overline{A}$, $B\overline{B}$ and $C\overline{C}$ meeting at the centroid G of both the triangles ABC and $\overline{A}\,\overline{B}\,\overline{C}$.*

2. *Determine the circumcenter U_A of the triangle $\overline{A}BC$ and reflect the median $A\overline{A}$ in one angle bisector at $\overline{A}$ in order to obtain the symmedian s_A of $\overline{A}\,\overline{B}\,\overline{C}$. The pedal point of s_A w.r.t. U_A is denoted by F_A (=focal point of $\mathcal{C}_A$).*

3. *Draw the circle with diameter $G\overline{A}$ and intersect it with the diameter line through F_A. The connections of these points with G give two orthogonal lines m and n.*

4. *The line F_AU_A intersects m in point M_A and n in N_A. Draw circles with centers M_A and N_A, respectively, which pass through $\overline{A}$. The remaining equicevian points are the intersection points of the first circle with n and of the second circle with m. By Lemma 11, the real solutions are found on the line which does not meet the interior of the triangle's longest side.*

Proof. Concerning item 3, the circle with diameter $\overline{A}G$ must on the one hand pass through the pedal points of m and n w.r.t. $\overline{A}$. On the other hand, this is a particular circle of the set explained in Theorem 8, 1. The circles in item 4 belong to the pencil of circles described in Theorem 8, 2. □

4.4. Real proper equicevian points in the interior

Theorem 15. *Each triangle ABC has 0, 1 or 2 real proper and 0 or $2, \dots, 6$ real improper equicevian points.*

1. *Only equilateral triangles have no real improper equicevian point; the center is the unique proper equicevian point.*
2. *For non-equilateral triangles, the number of proper real equicevian points is less than two if and only if one vertex lies on the ellipse $\mathcal{E}$ which has the opposite side as its minor axis and the ratio $2:1$ of semiaxes.*
3. *The number of proper equicevian points is zero if and only if the triangle is isosceles with side lengths $2:\sqrt{3}:\sqrt{3}$.*

Proof. Suppose a real equicevian point E lies on the sideline BC. Since the focal points of $\mathcal{S}$ differ from the vertices ABC, point E must coincide with the pedal point $A_0 = (-u, 0)$ of $\overline{A}$ (see Figures 2 and 8). However, A_0 is equivecian if and only if $|A_0A| = |BC|$, i.e., $4u^2+v^2 = 4$. This means, vertex A lies on the ellipse $\mathcal{E}$ with B and C as endpoints of the minor axis and the ratio $2:1$ of semiaxes (Figure 8).

When both real remaining equicevian points lie on sidelines, then according to Figure 10 one of them belongs to the side's third next to the longest side while the other one lies in the neighboring half but outside the first third. On the other hand, the two points must be symmetric with respect to the centroid. There remains as the only possibility that the major axis of $\mathcal{S}$ is parallel to the longest side. The triangle must be isosceles, and if BC is not the longest side then point A_0 must separate BC in the ratio $1:2$. This gives $u = \dfrac{1}{3}$ and $v = 2\sqrt{1-u^2} = \frac{4}{3}\sqrt{2}$. □

The final theorem gives necessary and sufficient conditions for the existence of proper equicevian points in the interior of the triangle ABC. These criteria are expressed in terms of the constant k which is used in Theorem 12 and can also be written as

$$k = \frac{1}{u\sqrt{2}}\sqrt{u^2 - v^2 + 3 + \sqrt{(u^2-v^2+3)^2 + 4u^2v^2}}. \tag{20}$$

The following geometric interpretation of k plays also a role in [22].

Lemma 16. *Let S and S' denote the respective third vertices of the two regular triangles with the basis BC. Then for each triangle ABC with A not located on the line SS' the constant k equals the ratio between major and minor axis of the unique ellipse which has the focal points S, S' and passes through A.*

Proof. Let a, b be the major and minor semiaxes of the ellipse with focal points S, S', which w.r.t. the coordinate system used in (1) have the coordinates $(0, \pm\sqrt{3})$. In terms of $k = \frac{a}{b} > 1$ these semiaxes satisfy

$$a^2 = \frac{3k^2}{k^2-1} \quad \text{and} \quad b^2 = \frac{3}{k^2-1}\,.$$

The ellipse passes through A if and only if

$$(k^2u^2 + v^2)(k^2-1) = 3k^2.$$

The only positive solution of this equation is k as given in (20). □

Theorem 17. *Let ABC be a non-regular triangle with coordinates (1) and $u, v > 0$.*

1. *The real equicevian point E_1 from Theorem 12 lies in the interior of the triangle ABC if and only if the constant k from (20) satisfies $u < \dfrac{1}{k+1}$.*
2. *Point E_2 from Theorem 12 lies in the interior of the triangle ABC if and only if $k > 2$ and $u < \dfrac{1}{k-1}$.*
3. *Both real equicevian points E_1, E_2 from Theorem 12 are in the interior of ABC if and only if $k > 2$ and $u < \dfrac{1}{k+1}$.*

Proof. We embed the plane of the triangle ABC as the plane $z = 0$ into the Euclidean 3-space. Then point P is an interior point of the triangle ABC with coordinates $A = (u, v, 0)$, $B = (-1, 0, 0)$ and $C = (1, 0, 0)$ and $u, v > 0$ if and only if all three vector products $\overrightarrow{BC} \times \overrightarrow{BP}$, $\overrightarrow{CA} \times \overrightarrow{CP}$ and $\overrightarrow{AB} \times \overrightarrow{CP}$ have a positive z-coordinate. By virtue of Theorem 12, for $P = E_1$ and $P = E_2$ we obtain the following triple of z-coordinates:

$$\frac{2(k \pm 2)v}{3k}, \quad \frac{2(k \mp 1)v}{3k}(1 \mp (k+1)u), \quad \frac{2(k \mp 1)v}{3k}(1 \pm ku + u).$$

For E_1 the upper signs are valid; therefore the first and third terms are always positive (also in agreement with Lemma 11). The condition stated in the theorem characterizes a positive middle term.

For E_2 the lower signs are valid, and in this case the stated condition is equivalent to the requirement that the first and third terms are positive.

Item 3 follows from $\dfrac{1}{k+1} < \dfrac{1}{k-1}$, which holds for all $k > 1$. □

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