Journal for Geometry and Graphics
Volume 18 (2014), No. 2, 173–182.

On Polygons with Coordinates of Vertices from Fibonacci and Lucas Sequences

Ravi Kumar Davala, Gopal Krishna Panda

Department of Mathematics, National Institute of Technology
Rourkela-769 008, Orissa, India
emails: davalaravikumar@gmail.com, gkpanda_nit@rediffmail.com

Abstract. Polygons in the plane with coordinates of points from the Fibonacci and Lucas sequences are considered. Certain area formulas for these polygons are explored. Restricting these polygons to triangles, some geometric properties are also studied.

Key words: Fibonacci sequence, Lucas sequence, area of triangles, orthologic triangles, paralogic triangles
MSC 2010: 51M25, 11B39

1. Introduction

The Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$ are defined by the recurrence relations

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 2$$

and

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n \geq 2.$$

The Binet forms of Fibonacci and Lucas sequences are

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \quad L_n = \alpha^n + \beta^n, \quad \text{where} \quad \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Using these Binet forms, one can easily have $F_{-n} = (-1)^{n+1} F_n$ and $L_{-n} = (-1)^n L_n$.

Čerin in [3] studied some geometric properties of triangles such as orthologic and paralogic triangles in the plane with coordinates of vertices from Fibonacci and Lucas sequences. He also studied certain area properties, Brocard angles and distances between circumcenters of these triangles. In this paper, we work on areas of polygons whose coordinates are from Fibonacci and Lucas sequences. Restricting the polygons to triangles, we explore some geometric properties like orthology and paralogy for these triangles.

ISSN 1433-8157/\$ 2.50

2. Area of polygons

For $k \in \mathbb{Z}^+$, let ABC and PQR denote the triangles with vertices $A = (F_k, F_{k+1})$, $B = (F_{k+1}, F_{k+2})$, $C = (F_{k+2}, F_{k+3})$ and $P = (L_k, L_{k+1})$, $Q = (L_{k+1}, L_{k+2})$, $R = (L_{k+2}, L_{k+3})$, respectively. The following theorem was proved by Čerin [3] using the Binet forms of Fibonacci and Lucas sequences.

Theorem 2.1. *For all $k \in \mathbb{Z}^+$ the areas $|ABC|$ and $|PQR|$ of the triangles ABC and PQR respectively are as follows:*

$$|ABC| = \frac{1}{2} \quad \text{and} \quad |PQR| = 5|ABC| = \frac{5}{2}\,.$$

In this section, we consider some results relating to area of the polygons involving Fibonacci and Lucas numbers as coordinates of vertices. The following index reduction formulas of Fibonacci and Lucas numbers (see [4]) will be frequently used in evaluating determinants occuring in the proofs of main theorems, without further reference.

Theorem 2.2 (Index reduction formulas of Fibonacci and Lucas numbers)**.**
If a, b, c, d and r are integers and $a + b = c + d$ then

$$\begin{array}{llcl} (a) & F_aF_b - F_cF_d & = & (-1)^r[F_{a+r}F_{b+r} - F_{c+r}F_{d+r}], \\ (b) & F_aL_b - F_cL_d & = & (-1)^r[F_{a+r}L_{b+r} - F_{c+r}L_{d+r}]. \end{array}$$

Let $\triangle_{n+2,k} = \{(F_{k+i}, F_{k+i+1}) : i = 0, 1, \cdots, (n+1)\}$ and $\nabla_{n+2,k} = \{(L_{k+i}, L_{k+i+1}) : i = 0, 1, \cdots, (n+1)\}$ for $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}$ denote the polygons with $(n+2)$ vertices and the area of these polygons be denoted by $|\triangle_{n+2,k}|$ and $|\nabla_{n+2,k}|$, respectively. The following theorem provides a formula for $|\triangle_{n+2,k}|$.

Theorem 2.3. *For $n \in \mathbb{Z}^+$ the area $|\triangle_{n+2,k}|$ is independent of k, and*

$$|\triangle_{n+2,k}| = \begin{cases} \frac{1}{2}\,F_{n+1} & \text{if } n \text{ is odd,} \\ \frac{1}{2}\,(F_{n+1} - 1) & \text{if } n \text{ is even.} \end{cases}$$

Proof. The proof is based on induction on n. By virtue of the Theorem 2.1, $|\triangle_{3,k}| = \frac{1}{2} = \frac{1}{2}F_2$ and hence the statement is true for $n = 1$. Let us assume that the statement be true for $n = p$. For $n = p + 1$, we distinguish two cases.

Case I, p is odd:
The area of $|\triangle_{n+3,k}|$ is obtained by the sum of $|\triangle_{n+2,k}|$ and the area of the triangle formed by the vertices $(F_{k+p+1}, F_{k+p+2}), (F_{k+p+2}, F_{k+p+3})$ and (F_k, F_{k+1}).

$$\begin{aligned} |\triangle_{p+3,k}| &= |\triangle_{p+2,k}| + \frac{1}{2}\,\text{abs}\begin{vmatrix} F_{k+p+1} & F_{k+p+2} & F_k \\ F_{k+p+2} & F_{k+p+3} & F_{k+1} \\ 1 & 1 & 1 \end{vmatrix} \\ &= \frac{1}{2}F_{p+1} + \frac{1}{2}\,\text{abs}\begin{vmatrix} F_{k+p+1} & F_{k+p+2} & F_k \\ F_{k+p+2} & F_{k+p+3} & F_{k+1} \\ 1 & 1 & 1 \end{vmatrix} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}F_{p+1} + \frac{1}{2}\text{ abs}\begin{vmatrix} F_{k+p+1} & F_{k+p} & F_k \\ F_{k+p+2} & F_{k+p+1} & F_{k+1} \\ 1 & 0 & 1 \end{vmatrix} \\
&= \frac{1}{2}F_{p+1} + \frac{1}{2}\text{ abs}\left[F_{k+p}F_{k+1} - F_kF_{k+p+1} + F_{k+p+1}F_{k+p+1} - F_{k+p}F_{k+p+2}\right] \\
&= \frac{1}{2}F_{p+1} + \frac{1}{2}\text{ abs}\left[(-1)^kF_p + (-1)^{k+p}F_1\right] \\
&= \frac{1}{2}F_{p+1} + \frac{1}{2}(F_p - 1) \\
&= \frac{1}{2}(F_{p+2} - 1).
\end{aligned}$$

Case II, p is even:
The proof in this case is similar to that of case I; moreover,

$$\begin{aligned}
|\triangle_{p+3,k}| &= \frac{1}{2}(F_{p+1} - 1) + \frac{1}{2}\text{ abs}\begin{vmatrix} F_{k+p+1} & F_{k+p+2} & F_k \\ F_{k+p+2} & F_{k+p+3} & F_{k+1} \\ 1 & 1 & 1 \end{vmatrix} \\
&= \frac{1}{2}(F_{p+1} - 1) + \frac{1}{2}\text{ abs}\left[(-1)^kF_p + (-1)^{k+p}F_1\right] \\
&= \frac{1}{2}(F_{p+1} - 1) + \frac{1}{2}(F_p + 1) = \frac{1}{2}F_{p+2}.
\end{aligned}$$ □

The proof of the following theorem is similar to that of Theorem 2.3 hence it is omitted.

Theorem 2.4. *For $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}$ we have $|\nabla_{n+2,k}| = 5|\triangle_{n+2,k}|$.*

Let $\triangle'_{n+2,k} = \{(F_{k+2i}, F_{k+2i+1}) : i = 0, 1, \cdots, (n+1)\}$ and $\nabla'_{n+2,k} = \{(L_{k+2i}, L_{k+2i+1}) : i = 0, 1, \cdots, (n+1)\}$ for $n \in \mathbb{Z}^+$ denote the polygons with $n+2$ vertices and let the area of these polygons be denoted by $|\triangle'_{n+2,k}|$ and $|\nabla'_{n+2,k}|$, respectively. The following two theorems give the area of these polygons.

Theorem 2.5. *For $n \in \mathbb{Z}^+$ the area $|\triangle'_{n+2,k}|$ is independent of k, and*

$$|\triangle'_{n+2,k}| = \tfrac{1}{2}(F_{2n+2} - n - 1).$$

Proof. The area of the triangle formed by the vertices (F_k, F_{k+1}), (F_{k+2}, F_{k+3}) and (F_{k+4}, F_{k+5}) is

$$\begin{aligned}
\frac{1}{2}\text{ abs}\begin{vmatrix} F_k & F_{k+2} & F_{k+4} \\ F_{k+1} & F_{k+3} & F_{k+5} \\ 1 & 1 & 1 \end{vmatrix} &= \frac{1}{2}\text{ abs}\begin{vmatrix} F_k & F_{k+2} & F_{k+3} \\ F_{k+1} & F_{k+3} & F_{k+4} \\ 1 & 1 & 0 \end{vmatrix} \\
&= \frac{1}{2}\text{ abs}\left[F_{k+2}F_{k+4} - F_{k+3}^2 - F_kF_{k+4} + F_{k+1}F_{k+3}\right] \\
&= \frac{1}{2}\text{ abs}\left[(-1)^{k+1}F_1 - (-1)^{k+3}F_3\right] \\
&= \frac{1}{2} = \frac{1}{2}[F_4 - 1 - 1].
\end{aligned}$$

Hence the statement is true for $n = 1$. Let us assume that the statement is true for $n = p$.

For $n = p+1$, we have

$$\begin{aligned}
|\triangle'_{p+3,k}| &= |\triangle'_{p+2,k}| + \frac{1}{2}\ \mathrm{abs}\begin{vmatrix} F_{k+2p+2} & F_{k+2p+4} & F_k \\ F_{k+2p+3} & F_{k+2p+5} & F_{k+1} \\ 1 & 1 & 1 \end{vmatrix} \\
&= \frac{1}{2}\,(F_{2p+2} - p - 1) + \frac{1}{2}\ \mathrm{abs}\begin{vmatrix} F_{k+2p+2} & F_{k+2p+3} & F_k \\ F_{k+2p+3} & F_{k+p+4} & F_{k+1} \\ 1 & 0 & 1 \end{vmatrix} \\
&= \frac{1}{2}\,(F_{2p+2}-p-1) + \frac{1}{2}\ \mathrm{abs}\,[F_{k+2p+3}F_{k+1} - F_kF_{k+2p+4} + F_{k+2p+2}F_{k+2p+4} - F^2_{k+2p+3}] \\
&= \frac{1}{2}\,(F_{2p+2} - p - 1) + \frac{1}{2}\ \mathrm{abs}\,[(-1)^k F_{2p+3} + (-1)^{k+2p+3}F_1] \\
&= \frac{1}{2}\,(F_{2p+2} - p - 1) + \frac{1}{2}[F_{2p+3} - F_1] \\
&= \frac{1}{2}\,[F_{2p+4} - p - 2].
\end{aligned}$$

So the statement is true for $n = p+1$, and by induction the statement is true for all n. □

Theorem 2.6. *For $k \in \mathbb{Z}$ $|\nabla'_{3,k}| = 5\,|\triangle'_{3,k}|$.*

Proof.

$$\begin{aligned}
|\nabla'_{3,k}| &= \frac{1}{2}\ \mathrm{abs}\begin{vmatrix} L_k & L_{k+2} & L_{k+4} \\ L_{k+1} & L_{k+3} & L_{k+5} \\ 1 & 1 & 1 \end{vmatrix} \\
&= \frac{5}{2}\ \mathrm{abs}\begin{vmatrix} L_k & L_{k+1} & L_{k+2} \\ F_{k+1} & F_{k+3} & F_{k+5} \\ 1 & 1 & 1 \end{vmatrix} \\
&= \frac{5}{2}\ \mathrm{abs}\begin{vmatrix} F_k & F_{k+2} & F_{k+4} \\ F_{k+1} & F_{k+3} & F_{k+5} \\ 1 & 1 & 1 \end{vmatrix} \\
&= 5|\triangle'_{3,k}|.
\end{aligned}$$

□

The proof of the following theorem is similar to that of Theorem 2.5; hence it is omitted.

Theorem 2.7. *For $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}$ holds $|\nabla'_{n+2,k}| = 5|\triangle'_{n+2,k}|$.*

Let $\triangle^*_{n+2,k} = \{(F_{k+i}, L_{k+i}) : i = 0, 1, \cdots, (n+1)\}$ for $n \in \mathbb{Z}^+$ and $k \in \mathbb{Z}$ denote the polygon with $(n+2)$ vertices, and let the area of this polygon be denoted by $|\triangle^*_{n+2,k}|$. In the following theorems we give the formulas for $|\triangle^*_{n+2,k}|$.

Theorem 2.8. *For $n \in \mathbb{Z}^+$ $|\triangle^*_{n+2,k}|$ is independent of k, and*

$$|\triangle^*_{n+2,k}| = \begin{cases} F_{n+1} - 1 & \text{if } n \text{ is even}, \\ F_{n+1} & \text{if } n \text{ is odd}. \end{cases}$$

Proof. The area of the triangle formed by the vertices (F_k, L_k), (F_{k+1}, L_{k+1}) and (F_{k+2}, L_{k+2}) is

$$\begin{aligned}
\frac{1}{2}\,\text{abs}\begin{vmatrix} F_k & F_{k+1} & F_{k+2} \\ L_k & L_{k+1} & L_{k+2} \\ 1 & 1 & 1 \end{vmatrix} &= \frac{1}{2}\,\text{abs}\begin{vmatrix} F_k & F_{k+1} & F_{k+2} \\ F_{k+1} & F_{k+2} & F_{k+3} \\ 1 & 1 & 0 \end{vmatrix} \\
&= \frac{1}{2}(2F_2) \qquad \text{(using Theorem 2.3)} \\
&= 1 = F_{1+1}.
\end{aligned}$$

Hence the statement is true for $n = 1$. Let us assume that the statement is true for $n = m$. For $n = m + 1$, we distinguish two cases.

Case I, m is even:
In this case

$$\begin{aligned}
|\triangle^*_{m+3,k}| &= |\triangle^*_{m+2,k}| + \frac{1}{2}\,\text{abs}\begin{vmatrix} F_{k+m+1} & F_{k+m+2} & F_k \\ L_{k+m+1} & L_{k+m+2} & L_k \\ 1 & 1 & 1 \end{vmatrix} \\
&= F_{m+1} - 1 + \frac{1}{2}\,\text{abs}\begin{vmatrix} F_{k+m+1} & F_{k+m+2} & F_k \\ L_{k+m+1} & L_{k+m+2} & L_k \\ 1 & 1 & 1 \end{vmatrix} \\
&= F_{m+1} - 1 + \text{ abs}\begin{vmatrix} F_{k+m+1} & F_{k+m+2} & F_k \\ F_{k+m+2} & F_{k+m+3} & F_{k+1} \\ 1 & 1 & 1 \end{vmatrix} \\
&= F_{m+1} - 1 + \text{ abs}\begin{vmatrix} F_{k+m+1} & F_{k+m} & F_k \\ F_{k+m+2} & F_{k+m} & F_{k+1} \\ 1 & 0 & 1 \end{vmatrix} \\
&= F_{m+1} - 1 + F_m + 1 \\
&= F_{m+2}.
\end{aligned}$$

Case II, m is odd:
In this case

$$\begin{aligned}
|\triangle^*_{m+3,k}| &= F_{m+1} + \frac{1}{2}\,\text{abs}\begin{vmatrix} F_{k+m+1} & F_{k+m+2} & F_k \\ L_{k+m+1} & L_{k+m+2} & L_k \\ 1 & 1 & 1 \end{vmatrix} \\
&= F_{m+1} + [F_m - 1] \\
&= F_{m+2} - 1.
\end{aligned}$$

So the assertion is true for $n = m + 1$ and by induction the statement is true for all n. □

Let $\Phi_{n+2,k,p} = \{(F_{k+i}, F_{k+p+i}) : i = 0, 1, \cdots, (n+1)\}$ and $\Phi'_{n+2,k,p} = \{(L_{k+i}, L_{k+p+i}) : i = 0, 1, \cdots, (n+1)\}$ for $n \in \mathbb{Z}^+$ and $k, p \in \mathbb{Z}$ denote the polygons with $n+2$ vertices and let the area of these polygons be denoted by $|\Phi_{n+2,k,p}|$ and $|\Phi'_{n+2,k,p}|$ respectively. The following two theorems present formulas for $|\Phi_{n+2,k,p}|$ and $|\Phi'_{n+2,k,p}|$.

Theorem 2.9. *For $n \in \mathbb{Z}^+$ and $p \in \mathbb{Z}$ the area $|\Phi_{n+2,k,p}|$ is independent of k, and*

$$|\Phi_{n+2,k,p}| = \begin{cases} \frac{1}{2} F_p F_{n+1} & \text{if } n \text{ is odd,} \\ \frac{1}{2} F_p (F_{n+1} - 1) & \text{if } n \text{ is even.} \end{cases}$$

Proof. The area of the triangle formed by the vertices (F_k, F_{k+p}), (F_{k+1}, F_{k+p+1}) and (F_{k+2}, F_{k+p+2}) is

$$\begin{aligned} \frac{1}{2} \operatorname{abs} \begin{vmatrix} F_k & F_{k+1} & F_{k+2} \\ F_{k+p} & F_{k+p+1} & F_{k+p+2} \\ 1 & 1 & 1 \end{vmatrix} &= \frac{1}{2} \operatorname{abs} \begin{vmatrix} F_k & F_{k+1} & 0 \\ F_{k+p} & F_{k+p+1} & 0 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \frac{1}{2} F_p = \frac{1}{2} F_p F_2. \end{aligned}$$

Hence the statement is true for $n = 1$. Let us assume that the statement is true for $n = m$. For $n = m + 1$, we distinguish two cases.

Case I, m is odd:
In this case

$$\begin{aligned} |\Phi_{m+3,k}| &= |\Phi_{m+2,k}| + \frac{1}{2} \operatorname{abs} \begin{vmatrix} F_{k+m+1} & F_{k+m+2} & F_k \\ F_{k+p+m+1} & F_{k+p+m+2} & F_{k+p} \\ 1 & 1 & 1 \end{vmatrix} \\ &= \frac{1}{2} F_p (F_{m+1}) + \frac{1}{2} \operatorname{abs} \begin{vmatrix} F_{k+m+1} & F_{k+m+2} & F_k \\ F_{k+p+m+1} & F_{k+p+m+2} & F_{k+p} \\ 1 & 1 & 1 \end{vmatrix} \\ &= \frac{1}{2} F_p (F_{m+1}) + \frac{1}{2} \operatorname{abs} \begin{vmatrix} F_{k+m+1} & F_{k+m} & F_k \\ F_{k+p+m+1} & F_{k+p+m} & F_{k+p} \\ 1 & 0 & 1 \end{vmatrix} \\ &= \frac{1}{2} F_p (F_{m+1}) + \frac{1}{2} [F_m F_p - F_p] \\ &= \frac{1}{2} F_p [F_{m+2} - 1]. \end{aligned}$$

Case II, m is even:
In this case

$$\begin{aligned} |\Phi_{m+3,k}| &= \frac{1}{2} F_p (F_{m+1} - 1) + \frac{1}{2} \operatorname{abs} \begin{vmatrix} F_{k+m+1} & F_{k+m+2} & F_k \\ F_{k+p+m+1} & F_{k+p+m+2} & F_{k+p} \\ 1 & 1 & 1 \end{vmatrix} \\ &= \frac{1}{2} F_p (F_{m+1} - 1) + \frac{1}{2} [F_m F_p + F_p] \\ &= \frac{1}{2} F_p F_{m+2}. \end{aligned}$$

So the assertion is true for $n = m + 1$ and by induction the statement is true for all n. □

The proof of the following theorem is similar to that of Theorem 2.9; hence it is omitted.

Theorem 2.10. *For $n \in \mathbb{Z}^+$ and $k, p \in \mathbb{Z}$ we have $|\Phi'_{n+2,k,p}| = 5\,|\Phi_{n+2,k,p}|$.*

For $n \in \mathbb{Z}^+$, $k \in \mathbb{Z}$ and $p \in \mathbb{Z}$ let $\Psi_{n+2,k,p} = \{(F_{k+i}, L_{k+p+i}) : i = 0, 1, \cdots, (n+1)\}$ and $\Psi'_{n+2,k,p} = \{(L_{k+i}, F_{k+p+i}) : i = 0, 1, \cdots, (n+1)\}$ denote the polygons with $n+2$ vertices and let the area of these polygons be denoted by $|\Psi_{n+2,k,p}|$ and $|\Psi'_{n+2,k,p}|$, respectively. The following two theorems give the formulas for $|\Psi_{n+2,k,p}|$ and $|\Psi'_{n+2,k,p}|$.

Theorem 2.11. *For $n \in \mathbb{Z}^+$ and $p \in \mathbb{Z}$, $|\Psi_{n+2,k,p}|$ is independent of k and*

$$|\Psi_{n+2,k,p}| = \begin{cases} \frac{1}{2} L_p F_{n+1} & \text{if } n \text{ is odd,} \\ \frac{1}{2} L_p (F_{n+1} - 1) & \text{if } n \text{ is even.} \end{cases}$$

Proof. The area of the triangle formed by the vertices (F_k, L_{k+p}), (F_{k+1}, L_{k+p+1}) and (F_{k+2}, L_{k+p+2}) is

$$\begin{aligned} \frac{1}{2} \text{ abs} \begin{vmatrix} F_k & F_{k+1} & F_{k+2} \\ L_{k+p} & L_{k+p+1} & L_{k+p+2} \\ 1 & 1 & 1 \end{vmatrix} &= \frac{1}{2} \text{ abs} \begin{vmatrix} F_k & F_{k+1} & 0 \\ L_{k+p} & L_{k+p+1} & 0 \\ 1 & 1 & -1 \end{vmatrix} \\ &= \frac{1}{2} L_p = \frac{1}{2} L_p F_{1+1}. \end{aligned}$$

Hence the statement is true for $n = 1$. Let us assume that the statement is true for $n = m$. For $n = m + 1$, we distinguish two cases.

Case I, m is odd:

$$\begin{aligned} |\Psi_{m+3,k,p}| &= |\Psi_{m+2,k,p}| + \frac{1}{2} \text{ abs} \begin{vmatrix} F_{k+m+1} & F_{k+m+2} & F_k \\ L_{k+p+m+1} & L_{k+p+m+2} & L_{k+p} \\ 1 & 1 & 1 \end{vmatrix} \\ &= \frac{1}{2} L_p(F_{m+1}) + \frac{1}{2} \text{ abs} \begin{vmatrix} F_{k+m+1} & F_{k+m+2} & F_k \\ L_{k+p+m+1} & L_{k+p+m+2} & L_{k+p} \\ 1 & 1 & 1 \end{vmatrix} \\ &= \frac{1}{2} L_p(F_{m+1}) + \frac{1}{2} \text{ abs} \begin{vmatrix} F_{k+m+1} & F_{k+m} & F_k \\ L_{k+p+m+1} & L_{k+p+m} & L_{k+p} \\ 1 & 0 & 1 \end{vmatrix} \\ &= \frac{1}{2} L_p(F_{m+1}) + \frac{1}{2}[F_m L_p - L_p] \\ &= \frac{1}{2} L_p[F_{m+2} - 1] \end{aligned}$$

Case II, m is even:

$$\begin{aligned} |\Psi_{m+3,k,p}| &= \frac{1}{2} L_p(F_{m+1} - 1) + \frac{1}{2} \text{ abs} \begin{vmatrix} F_{k+m+1} & F_{k+m+2} & F_k \\ L_{k+p+m+1} & L_{k+p+m+2} & L_{k+p} \\ 1 & 1 & 1 \end{vmatrix} \\ &= \frac{1}{2} L_p(F_{m+1} - 1) + \frac{1}{2}[F_m L_p + L_p] \\ &= \frac{1}{2} L_p F_{m+2} \end{aligned}$$

So the assertion is true for $n = m + 1$ and by mathematical induction the statement is true for all n. ∎

Theorem 2.12. *For $n \in \mathbb{Z}^+$ and $k, p \in \mathbb{Z}$ holds* $|\Psi'_{n+2,k,p}| = |\Psi_{n+2,k,p}|$.

Proof. Recall that the area of the triangle formed by the vertices (a_1, a_2), (b_1, b_2), (c_1, c_2) is equal to the area of the triangle formed by the vertices (a_2, a_1), (b_2, b_1), (c_2, c_1). Now the proof follows from Theorem 2.10 and 2.11. □

3. Geometric properties

Recall that the triangles ABC and XYZ are called *orthologic* triangles when the perpendiculars at vertices of ABC onto the corresponding sides of XYZ are concurrent. Let the point of concurrence be $[x, y]$. It is also well known that the relation of *orthology* for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of XYZ onto the corresponding sides of ABC are concurrent at the point $[y, x]$.

By replacing perpendiculars with parallels in the above definition we get the analogous notion of *paralogic* triangles. In this section, we frequently use formulas stated in Theorem 2.2.

Theorem 3.1. *For $k \in \mathbb{Z}$ the triangles $\triangle'_{3,k}$ and $\nabla'_{3,k}$ are orthologic.*

Proof. It is well-known (see [1]) that the triangles ABC and XYZ with coordinates (a_1, a_2), (b_1, b_2), (c_1, c_2), (x_1, x_2), (y_1, y_2), and (z_1, z_2) are orthologic if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ x_1 & y_1 & z_1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 & c_2 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Since

$$\begin{aligned}
&\begin{vmatrix} F_k & F_{k+2} & F_{k+4} \\ L_k & L_{k+2} & L_{k+4} \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} F_{k+1} & F_{k+3} & F_{k+5} \\ L_{k+1} & L_{k+3} & L_{k+5} \\ 1 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} F_k & F_{k+2} & F_{k+3} \\ L_k & L_{k+2} & L_{k+3} \\ 1 & 1 & 0 \end{vmatrix} + \begin{vmatrix} F_{k+1} & F_{k+3} & F_{k+4} \\ L_{k+1} & L_{k+3} & L_{k+4} \\ 1 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} F_k & F_{k+1} & F_{k+3} \\ L_k & L_{k+1} & L_{k+3} \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} F_{k+1} & F_{k+2} & F_{k+4} \\ L_{k+1} & L_{k+2} & L_{k+4} \\ 1 & 0 & 0 \end{vmatrix} \\
&= [F_{k+1}L_{k+3} - F_{k+3}L_{k+1}] + [F_{k+2}L_{k+4} - F_{k+4}L_{k+2}] \\
&= (-1)^k[F_1L_3 - F_3L_1] + (-1)^{k+1}[F_1L_3 - F_3L_1] = 0,
\end{aligned}$$

$\triangle'_{3,k}$ and $\nabla'_{3,k}$ are orthologic triangles. □

Theorem 3.2. *For $k \in \mathbb{Z}$ the triangles $\triangle'_{3,k}$ and $\nabla'_{3,k}$ are paralogic.*

Proof. Recall that the triangles ABC and XYZ with coordinates (a_1, a_2), (b_1, b_2), (c_1, c_2), (x_1, x_2), (y_1, y_2), and (z_1, z_2) are paralogic if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} a_2 & b_2 & c_2 \\ x_1 & y_1 & z_1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Since

$$\begin{aligned}
&\begin{vmatrix} F_k & F_{k+2} & F_{k+4} \\ L_{k+1} & L_{k+3} & L_{k+5} \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} F_{k+1} & F_{k+3} & F_{k+5} \\ L_k & L_{k+2} & L_{k+4} \\ 1 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} F_k & F_{k+2} & F_{k+3} \\ L_{k+1} & L_{k+3} & L_{k+4} \\ 1 & 1 & 0 \end{vmatrix} - \begin{vmatrix} F_{k+1} & F_{k+3} & F_{k+4} \\ L_k & L_{k+2} & L_{k+3} \\ 1 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} F_k & F_{k+1} & F_{k+3} \\ L_{k+1} & L_{k+2} & L_{k+4} \\ 1 & 0 & 0 \end{vmatrix} - \begin{vmatrix} F_{k+1} & F_{k+2} & F_{k+4} \\ L_k & L_{k+1} & L_{k+3} \\ 1 & 0 & 0 \end{vmatrix} \\
&= [F_{k+1}L_{k+4} - F_{k+3}L_{k+2}] - [F_{k+2}L_{k+3} - F_{k+4}L_{k+1}] \\
&= (-1)^k[F_1L_4 - F_3L_2] - (-1)^k[F_2L_3 - F_4L_1] = \ 0,
\end{aligned}$$

$\triangle'_{3,k}$ and $\nabla'_{3,k}$ are paralogic triangles for every k. □

Theorem 3.3. *For $k \in \mathbb{Z}$ the triangles $\triangle'_{3,k}$ and $\nabla'_{3,k}$ are reversely similar.*

Proof. It is well known that two triangles are reversely similar if and only if they are orthologic and paralogic (see [2]). Hence the proof follows from Theorems 3.1 and 3.2. □

Theorem 3.4. *For $k \in \mathbb{Z}$ and $p \in \mathbb{Z}$ the triangles $\Phi_{3,k,p}$ and $\Phi'_{3,k,p}$ are orthologic if and only if p is odd.*

Proof. Since

$$\begin{aligned}
&\begin{vmatrix} F_k & F_{k+1} & F_{k+2} \\ L_k & L_{k+1} & L_{k+2} \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} F_{k+p} & F_{k+p+1} & F_{k+p+2} \\ L_{k+p} & L_{k+p+1} & L_{k+p+2} \\ 1 & 1 & 1 \end{vmatrix} \\
&= [F_{k+1}L_k - F_kL_{k+1}] + [F_{k+p+1}L_{k+p} - F_{k+p}L_{k+p+1}] \\
&= (-1)^k[F_1L_0] + (-1)^{k+p}[F_1L_0] \\
&= (-1)^k 2[1 + (-1)^p] \ = \ 0
\end{aligned}$$

if and only if p is odd, the result follows. □

Theorem 3.5. *For $k \in \mathbb{Z}$ and $p \in \mathbb{Z}$ the triangles $\Phi_{3,k,p}$ and $\Phi'_{3,k,p}$ are paralogic for all p.*

Proof. Since

$$\begin{aligned}
&\begin{vmatrix} F_k & F_{k+1} & F_{k+2} \\ L_{k+p} & L_{k+p+1} & L_{k+p+2} \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} F_{k+p} & F_{k+p+1} & F_{k+p+2} \\ L_k & L_{k+1} & L_{k+2} \\ 1 & 1 & 1 \end{vmatrix} \\
&= [F_{k+1}L_{k+p} - F_kL_{k+p+1}] - [F_{k+p+1}L_k - F_{k+p}L_{k+1}] \\
&= (-1)^k[F_1L_p] - (-1)^k[F_{p+1}L_0 - F_pL_1] \\
&= (-1)^k[L_p - 2F_{p+1} + F_p] \ = \ 0,
\end{aligned}$$

the result follows. □

Theorem 3.6. *For $k \in \mathbb{Z}$ and $p \in \mathbb{Z}$ the triangles $\Phi_{3,k,p}$ and $\Phi'_{3,k,p}$ are reversely similar if and only if p is odd.*

Proof. The proof follows from Theorems 3.4 and 3.5. □

Theorem 3.7. *For $k \in \mathbb{Z}$ and $p \in \mathbb{Z}$ the triangles $\Psi_{3,k,p}$ and $\Psi'_{3,k,p}$ are orthologic if and only if p is even.*

Proof.

$$\begin{aligned}
&\begin{vmatrix} F_k & F_{k+1} & F_{k+2} \\ L_k & L_{k+1} & L_{k+2} \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} L_{k+p} & L_{k+p+1} & L_{k+p+2} \\ F_{k+p} & F_{k+p+1} & F_{k+p+2} \\ 1 & 1 & 1 \end{vmatrix} \\
&= [F_{k+1}L_k - F_kL_{k+1}] + [L_{k+p+1}F_{k+p} - F_{k+p+1}L_{k+p}] \\
&= (-1)^k[F_1L_0 - 0] + (-1)^{k+p}[0 - F_1L_0] \\
&= (-1)^k 2[1 - (-1)^p] \ = \ 0
\end{aligned}$$

if and only if p is even. □

Theorem 3.8. *For $k, p \in \mathbb{Z}$ the triangles $\Psi_{3,k,p}$ and $\Psi'_{3,k,p}$ are paralogic if and only if $p = 0$.*

Proof. Since

$$\begin{aligned}
&\begin{vmatrix} F_k & F_{k+1} & F_{k+2} \\ F_{k+p} & F_{k+p+1} & F_{k+p+2} \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} L_{k+p} & L_{k+p+1} & L_{k+p+2} \\ L_k & L_{k+1} & L_{k+2} \\ 1 & 1 & 1 \end{vmatrix} \\
&= [F_{k+1}F_{k+p} - F_{k+p+1}F_k] - [L_{k+p+1}L_k - L_{k+p}L_{k+1}] \\
&= (-1)^k[F_1F_p] - (-1)^k[L_{p+1}L_0 - L_pL_1] \\
&= (-1)^k[F_p - 2L_{p+1} + L_p] \\
&= (-1)^{k+1}4F_p \ = \ 0
\end{aligned}$$

if and only if $p = 0$, the result follows. □

Theorem 3.9. *For $k \in \mathbb{Z}$ the triangles $\Psi_{3,k,0}$ and $\Psi'_{3,k,0}$ are reversely similar.*

Proof. The proof follows from Theorems 3.7 and 3.8. □

Acknowledgement

It is a pleasure to thank the anonymous referee for his valuable comments and suggestions which resulted in an improved presentation of this paper.

References

[1] Z. Cerin: *Hyperbolas, orthology and antipedal triangles.* Glasnik Mat. **33**, 143–160 (1998).

[2] Z. Cerin: *On propellers from triangles.* Beiträge zur Algebra and Geometrie **42**, 575–582 (2001).

[3] Z. Cerin: *On Triangles with Fibonacci and Lucas numbers as Coordinates.* Sarajevo J. of Math. **3**(15), 3–7 (2007).

[4] `www.home.zonnet.nl/LeonardEuler/fiboe1d.htm.`

Received March 7, 2014; final form November 2, 2014