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Surprising Relations Between the Areas of Triangles in the Configuration of Routh's Theorem

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Abstract. The paper considers some extremum problems related to the ratio of the areas of triangles formed by the intersection of three cevians in connection with Routh's Theorem. The solution of the problems brings surprising results related to the golden ratio.

Key Words: Routh's Theorem, triangle's area, extremum problems, golden ratio
MSC 2010: 51M05, 51M16

1. Introduction

Given is a triangle ABC. We denote by D, E, F some three points that are located on the sides BC, AC and AB, respectively. We also denote by G, H and I the intersection points of the cevians CF, BE and AD (Fig. 1). Routh's Theorem [1–3] states that

$$\frac{S_{\Delta GHI}}{\Delta} = \frac{(1-\alpha\beta\gamma)^2}{(1+\alpha+\alpha\beta)(1+\beta+\beta\gamma)(1+\gamma+\gamma\alpha)}$$

where $\gamma = AF/FB$, $\beta = CE/EA$ and $\alpha = BD/DC$. $\Delta = S_{\Delta ABC}$ and $S_{\Delta GHI}$ denote the areas of the triangles ABC and GHI, respectively.

One can also write down the ratio of the areas of the triangles DEF and ABC as

$$\frac{S_{\Delta DEF}}{\Delta} = \frac{1+\alpha\beta\gamma}{(1+\alpha)(1+\beta)(1+\gamma)}\,.$$

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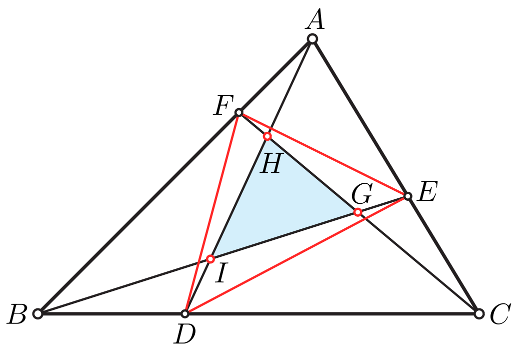

Figure 1: Routh's Theorem treats the area $S_{\Delta GHI}$

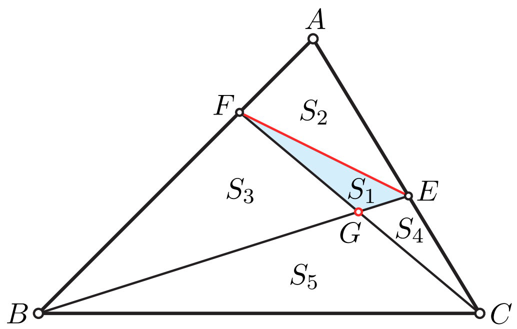

Figure 2: How to characterize the maximum area S_1?

When considering the areas' difference $W = S_{\Delta DEF} - S_{\Delta GHI}$, it is intuitively clear that W can vary between 0 (when the points D, E, F coincide with the vertices of the triangle ABC) and the maximal value $W_{\max}$. In other words, $0 \le W \le W_{\max} < \Delta$.

Since W is composed of the three areas $S_{\Delta FGE}$, $S_{\Delta FHD}$ and $S_{\Delta DIE}$, it is natural to seek the maximal value of each of these areas. To this end we consider the partial configuration, when only two cevians, BE and CF, are given in the triangle ABC which divide ABC into five triangles with the areas $S_1, \ldots, S_5$, as shown in Figure 2.

Intuitively, depending on the location of the points E and F, the areas $S_2, \ldots, S_5$ can have any value between 0 and Δ. On the other hand, the area S_1 cannot be as large as Δ. Therefore there is a maximal value $S_{1\max}$ of the area S_1, such that $0 \le S_1 \le S_{1\max} < \Delta$.

In the paper we will prove that $S_{1\max} = \frac{1}{\varphi^5}\Delta$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and that $W_{\max} = \frac{1}{4}\Delta$.

2. Finding $S_{1\max}$

Theorem 1. *The area of the triangle FGE is maximal iff the points F and E divide the sides AB and AC by the golden ratio φ, or in other words*

$$\frac{AF}{FB} = \frac{AE}{EC} = \frac{1+\sqrt{5}}{2} = \varphi\,.$$

Proof. We first prove the following lemma.

Lemma 1.
$$\frac{S_{\Delta FGE}}{\Delta} = \frac{\beta\gamma}{(1+\beta)(1+\gamma)(1+\beta+\beta\gamma)}\,. \tag{1}$$

Proof of Lemma 1: $S_{\Delta FGE} = \Delta - (S_{\Delta AFE} + S_{\Delta BCE} + S_{\Delta BFG})$. It is easy to show that

$$S_{\Delta AFE} = \frac{\gamma}{(1+\beta)(1+\gamma)}\cdot\Delta \quad \text{and} \quad S_{\Delta BCE} = \frac{\beta}{1+\beta}\cdot\Delta\,.$$

In order to express $S_{\Delta BFG}$, we note that according to Menelaus' Theorem for the triangle AFC, there holds

$$\frac{AB}{BF}\cdot\frac{FG}{GC}\cdot\frac{CE}{EA} = 1 \quad \text{or} \quad \frac{FG}{GC} = \frac{1}{\beta(1+\gamma)}\,.$$

Therefore we have

$$\frac{S_{\Delta BFG}}{S_{\Delta BFC}} = \frac{FG}{FC} = \frac{1}{\beta(1+\gamma)+1},$$

and since

$$S_{\Delta BFC} = \frac{BF}{AB}\cdot\Delta = \frac{\Delta}{1+\gamma},$$

we obtain

$$S_{\Delta BFG} = \frac{1}{(1+\gamma)(1+\beta+\beta\gamma)}\cdot\Delta,$$

and therefore

$$S_{\Delta FGE} = \frac{\beta\gamma}{(1+\beta)(1+\gamma)(1+\beta+\beta\gamma)}\cdot\Delta. \qquad \square$$

Proof of Theorem 1: We denote $x = \frac{1}{\beta}$, $y = \gamma$; therefore

$$\frac{S_{\Delta FGE}}{\Delta} = f(x,y) = \frac{xy}{(1+x)(1+y)(1+x+y)}.$$

In order to find the maximum of $f(x,y)$ for $x, y > 0$, one must solve the following system of equations:

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

(this is a necessary condition for a function $f(x,y)$ to have an extremum at some point). The solution results in the following system of equations:

$$\begin{aligned} x^2 &= y+1 \\ y^2 &= x+1. \end{aligned}$$

The equations define two parabolas which intersect four times. Because of $x, y > 0$ we can conclude that only one solution is acceptable:

$$x = y = \frac{1+\sqrt{5}}{2} = \varphi.$$

It can be verified that for these values of x, y, the function $f(x,y)$ receives the maximum value

$$f_{\max} = \frac{5\sqrt{5}-11}{2} = \frac{1}{\varphi^5};$$

in other words $S_{1\max} = \frac{1}{\varphi^5}\,\Delta.$ $\square$

It is important to note that as a conseqence of Theorem 1

$$W = S_{\Delta DEF} - S_{\Delta GHI} < \frac{3}{\varphi^5}\,\Delta.$$

Later we show that W satisfies a stronger inequality which gives a maximum value for W:

$$W = S_{\Delta DEF} - S_{\Delta GHI} \le \frac{1}{4}\,\Delta < \frac{3}{\varphi^5}\,\Delta \approx 0.27\,\Delta.$$

Conclusion 1

It is easy to show that when S_1 receives its maximum value, the values of the other areas are

$$S_2 = \frac{1}{\varphi^2}\Delta\,, \quad S_5 = \frac{1}{\varphi^3}\Delta \quad \text{and} \quad S_3 = S_4 = \frac{1}{\varphi^4}\Delta\,.$$

This means that in this case the areas S_1, S_3, S_5, S_2 form a geometric progression with the quotient φ.

It is interesting to note that the converse is also true, i.e., if S_1, S_3, S_5, S_2 form a geometric progression with a ratio q, then $q = \varphi$ and $\gamma = \frac{1}{\beta} = \varphi$. Really, in this case we obtain

$$\frac{S_{\Delta AFE}}{S_{\Delta BFE}} = \gamma = \frac{S_2}{S_1+S_3} = \frac{S_1 q^3}{S_1+S_1 q} = \frac{q^3}{1+q}\,, \quad \frac{S_{\Delta FGE}}{S_{\Delta AFE}} = \frac{\beta}{1+\beta+\gamma} = \frac{S_1}{S_2} = \frac{S_1}{S_1 q^3} = \frac{1}{q^3}\,,$$

$$\frac{S_{\Delta BFG}}{S_{\Delta BFC}} = \frac{1}{1+\beta(1+\gamma)} = \frac{S_3}{S_3+S_5} = \frac{S_1 q}{S_1 q + S_1 q^2} = \frac{1}{1+q}\,.$$

From these three equations we obtain that

$$\gamma = \frac{q^3}{1+q}\,, \quad \beta = \frac{q+1}{q^3}$$

and therefore $q^4 - q^3 - q - 1 = 0$, or $(q^2+1)(q^2-q-1) = 0$. From the last equations follows $q = \varphi$ and then $\gamma = \frac{1}{\beta} = \varphi$.

3. Finding the value of $W_{\max}$

In order to find the value of $W_{\max}$, we first prove the following theorem:

Theorem 2. *The following algebraic inequality holds for any three positive real numbers x, y, z:*

$$\frac{xy}{(1+x)(1+y)(1+x+xy)} + \frac{yz}{(1+y)(1+z)(1+y+yz)} + \frac{zx}{(1+z)(1+x)(1+z+zx)} \le \frac{1}{4}\,. \quad (3)$$

The equality holds only if $x = y = z = 1$.

Proof. For the proof of this theorem we need another lemma.

Lemma 2. *If a, b, c are three positive real numbers and $abc = 1$ then*

$$\frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c} \le 1\,.$$

The equality holds only if $a = b = c = 1$.

Proof of Lemma 2: Indeed

$$1 - \frac{1}{2+a} - \frac{1}{2+b} - \frac{1}{2+c} = \frac{ab+bc+ca+abc-4}{(2+a)(2+b)(2+c)} \ge 0\,,$$

since $abc = 1$ and according to the average equality $ab+bc+ca \ge 3\sqrt[3]{a^2b^2c^2} = 3$. Equality holds only if $a = b = c = 1$. □

Proof of Theorem 2: Based on the relations $1+x \geq 2\sqrt{x}\,,\ 1+y \geq 2\sqrt{y}\,,\ 1+xy \geq 2\sqrt{xy}$ we obtain

$$\frac{xy}{(1+x)(1+y)(1+x+xy)} \leq \frac{xy}{4\sqrt{xy}\,(x+2\sqrt{xy})} = \frac{1}{4}\cdot\frac{1}{2+\sqrt{\frac{x}{y}}}\,.$$

A similar inequality exists for the other two terms of (3). The claim of Theorem 2 follows from this and from Lemma 2. □

Theorem 3. $W = S_{\Delta DEF} - S_{\Delta GHI} \leq \frac{1}{4}\,\Delta\,.$

Proof. Formulas that are similar to (1) can be written for $S_{\Delta DFH}$ and $S_{\Delta EID}$. Therefore

$$W = S_{\Delta DEF} - S_{\Delta GHI} = S_{\Delta FGE} + S_{\Delta DHF} + S_{\Delta EID} =$$
$$\left(\frac{\beta\gamma}{(1+\beta)(1+\gamma)(1+\beta+\beta\gamma)} + \frac{\gamma\alpha}{(1+\gamma)(1+\alpha)(1+\gamma+\gamma\alpha)} + \frac{\alpha\beta}{(1+\alpha)(1+\beta)(1+\alpha+\alpha\beta)}\right)\cdot\Delta\,.$$

Hence, from Theorem 2 we obtain $W \leq \frac{1}{4}\,\Delta\,.$ Equality holds only if the points D, E, F are the middles of the sides of the triangle ABC, and in this case $W = W_{\max} = \frac{1}{4}\,\Delta\,.$

Conclusion 2
If $\alpha\beta\gamma = 1$, then $S_{\Delta DEF} \leq \frac{1}{4}\,\Delta$.

Indeed, in this case according to the inverse of Ceva's theorem, the cevians AD, BE and CF are concurrent and $S_{\Delta GHI} = 0\,.$

Conclusion 3
If $\alpha\beta\gamma = 1$ (that is, if the cevians AD, BE and CF are concurrent), then $S_{\Delta DEF} = \frac{1}{4}\,\Delta$ only if ΔDEF is the medial triangle of the triangle ABC.

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