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Design and Application Studies for a Cupola Forming Orbital Arrangement of Miura-Ori Basic Units

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Abstract. Deployable folded plate structures are a promising design principle for technical applications. Their two main abilities, providing stiffness in the folded or partially folded state and changing their shape between a folded and an unfolded state, make them predestinated especially for self-supporting convertible structures in architecture.

For reasons of cost effectiveness and assembly, structures based on repeating folding patterns like those from Tessellation Origami are the preferred source of inspiration. The probably most studied folding pattern is Miura-Ori. Its variations reach from fully regular patterns consisting of only one type of elements to completely irregular ones. Derivatives can cover tapered areas instead of rectangular ones or form arcs instead of remaining planar.

This paper presents variations of a pattern derived from Miura-Ori, whereby different configurations as well as changes of dimensional parameters are taken into account. The different variations are characterized by specifications like their numbers of elements and folds, the covered area or the folding sequence. The prospect of possible applications includes concepts using the unfolded but also concepts using the folded or partially folded state.

Key Words: Tessellation Origami, variations of Miura-Ori, orbital arrangement, application studies, foldable structures in architecture

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1. Deployable folded plate structures

Seemingly architecture consists of immobile buildings. However, even the most inconspicuous building comprises convertible components such as doors and windows. And on closer inspection, the field of architecture consists of a large variety of transformable structures like mobile pavilions, adaptive facades or even convertible roofs and retractable bridges. Because the demand towards sustainability increases, adaptive building parts as well as buildings for temporary use will be required even more so that the importance of transformable architecture is expected to increase. On the one hand foldable structures are adaptable and on the other hand they form a self-supporting structure. Due to these two key properties — strengthening and deployability — foldable structures are predestined for transformable constructions in architecture and other technical applications. Thereby the most promising foldable structures are based on the so called Origami Tessellations because these consist of repeating patterns [1] and for that reason only a limited number of modules is necessary.

In spite of these evident advantages realized folded plate structures are rare. In architecture complex foldable structures are found almost exclusively in subsystems, e.g., adaptable facade elements or smaller object architecture as shown in Figures 1 and 2.

Large-scaled convertible Origami-based folded plate structures are non-existent. This is probably due to a disproportional rising of challenges for the production process and specially designed connecting details with increasing complexity of the folding pattern.

2. Miura-Ori

The first step to reduce the complexity of a foldable structure is the use of a manageable folding pattern. A widespread solution is Miura-Ori, a pattern originally developed to compact large membrane structures for transportation to space [3]. Its regular arrangement, the fact that it also works with rigid plates and its constrained motion make it predestinated also for applications in architecture.

2.1. Variations of Miura-Ori

In the original Miura-Ori pattern, shown in Figure 3A, four congruent parallelograms are connected by their edges in a way that the four folding axes meet in one node. Such a basic

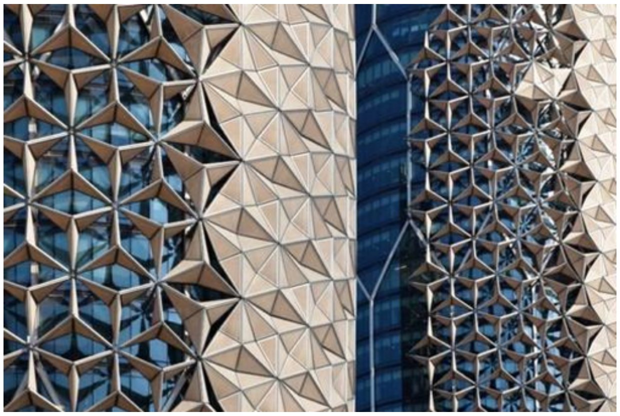


Figure 1: *Al Bahr Towers*,
Aedas

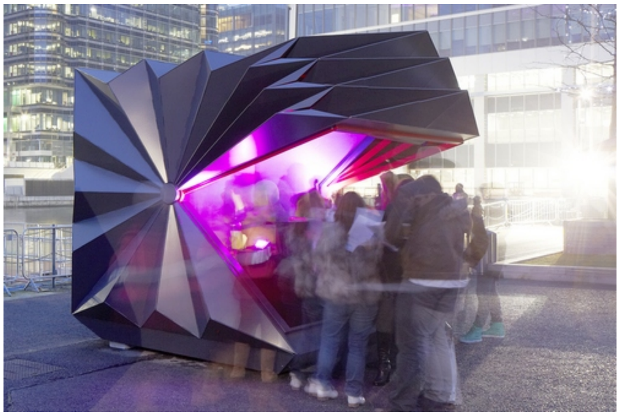


Figure 2: *Canary Wharf Kiosk*
Make Architects [2]

unit is deployable with one degree of freedom. Further it is developable and flat foldable.

A regular array of these basic units is characterized by parallel lengthwise folds with alternating folding directions (mountain/valley) and crosswise folds with a zigzag course which are either mountain or valley folds over the complete width of the array. In this assembly all nodes and their folding angles are identical.

From this base, variations of the pattern can be derived as studied in [4] and [5]. Changes in the dimensions or angles of parallelograms affect the dimensions of the folded and unfolded structure but do not influence the global appearance. Replacing parallelograms by trapezia can lead to surfaces folding into a curve instead of contracting in a plane (Figure 3B according to [4]).

If the lengthwise folds intersect each other instead of being parallel a tapered version of the pattern appears (Figure 3C according to [4]). Allowing changes in the direction of these folds can lead to irregular patterns (Figure 3D).

The pattern even works when the plate's angles at a node are not equal to 360°. In these cases the resultant structure is not flat foldable or not developable.

All these variations are transformable with only one degree of freedom.

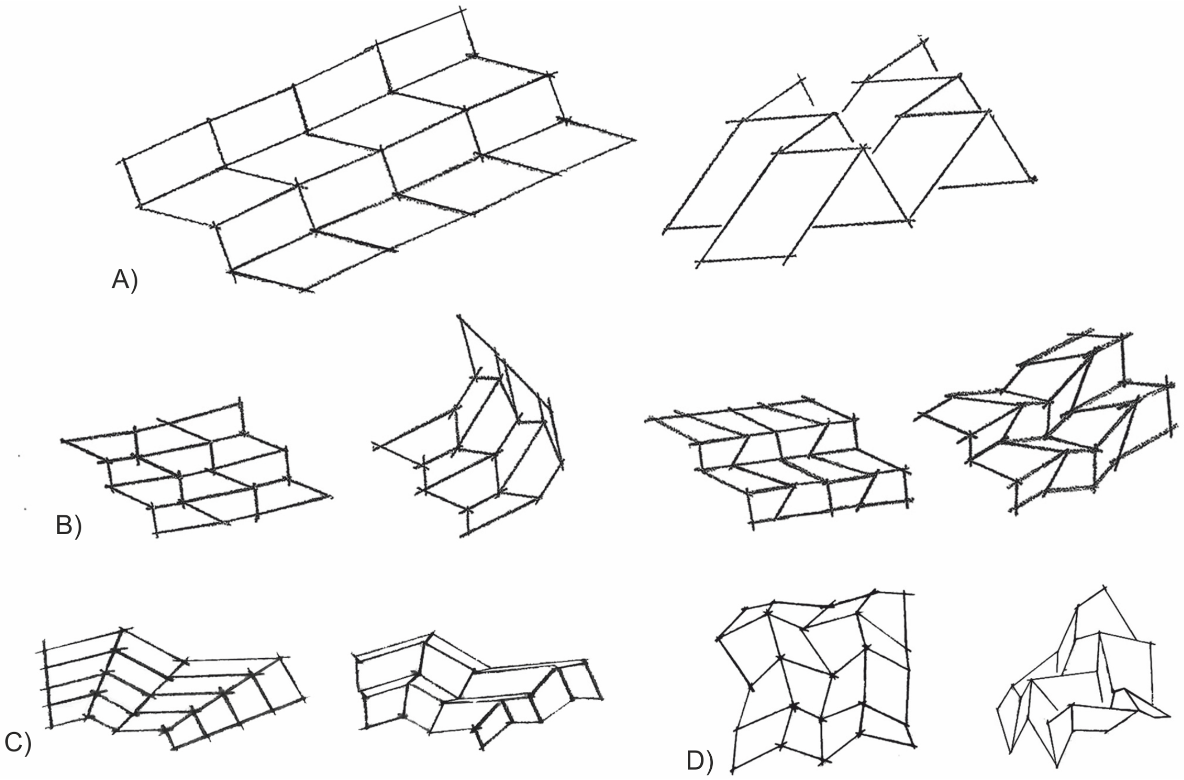


Figure 3: Miura-Ori

2.2. Cupola forming orbital arrangement

While planar or curve forming variations of the Miura-Ori pattern can be build up using only one type of elements, the tapered arrangement requires bigger elements for every orbital row. An attempt to assemble a tapered pattern with only one type of elements is to arrange Miura-Ori basic units of symmetric trapezia as depicted in Figure 4. Although a mesh of

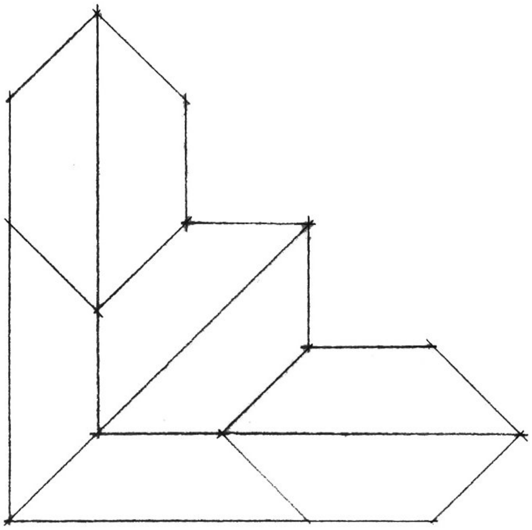


Figure 4: Crease pattern

lengthwise and crosswise folds cannot be identified, also this pattern is deployable with only one degree of freedom.

In this crease pattern the long bases of the trapezia are always connected to other long bases, but short bases and legs need to be connectable to each other. Hence symmetry is not sufficient as criterion but the trapezia additionally need to be triamonds.

Like other tapered patterns the presented one requires a radial cut for folding. Starting from an unfolded or developed state its motion sequence (Figure 5 A-C) is a simultaneous combination of an angular folding around a vertical axis and a coiling up around a horizontal one. Assuming a negligible thickness the final state is flat folded.

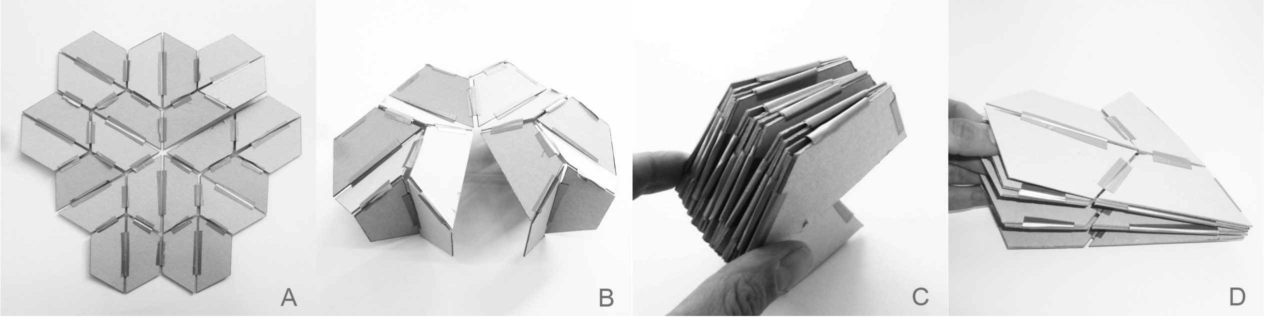


Figure 5: Motion sequence

In general three states are regarded for Origami structures:

- unfolded or developed,
- partially folded and
- folded or if possible flat folded.

For practical reasons like space requirements for transport or storage an additional state may be necessary (Figure 5 D):

- folded parts or folded in an alternative way.

For this state the structure is disassembled into sub-structures. The folding sequence of the substructure is based on the chosen crease pattern, but it does not necessarily use all folds or valley folds may turn into mountain folds or the other way round.

3. Design study

Based on the crease pattern described in Figure 4 different designs are possible. These can be achieved by changing geometrical parameters of single elements or by adding elements in angular direction and thereby extending the structure.

3.1. Parameterization and extension

Since the crease pattern is a tessellation of triamonds for every element three of its edge lengths are identical and the fourth one is dependent on them according to $b = a(1+2\cos\alpha)$ as depicted in Figure 6. Hence there are only two independent parameters for each trapezium:

- one angle (α) and
- one edge length (a or b).

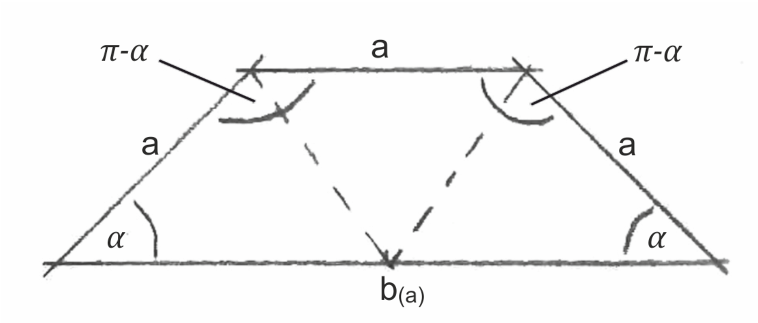


Figure 6: Parameterization of trapezia

Depending on the chosen angle α the structure can be extended in angular direction up to the number of $360°/\alpha$ elements in the inner row. Further extension would cause overlapping. In radial direction the unfolded tessellation is endlessly extendable, but a continuation of the pattern further as depicted in Figure 4 can lead to intersections of elements in the folded state.

3.2. Effects on geometry

The first and very basic step in the analysis of effects of variations is changing the parameters of elements. Varying the edge length influences characteristics like:

- area or diameter in the unfolded state,
- base area, diameter or enclosed space in the partially folded state,
- area of the cross section in the flat folded state.

The same characteristics also can be influenced by changes of the trapezium's angle. Additionally reducing the trapezium's angle makes the cut angle rise. Assuming that in the unfolded state the structure should form a closed surface the open space in the cut angle must be filled by angular extension. So there are influences on:

- cut angle in the unfolded state,
- number of elements (if extended),
- number of folds (if extended).

In a design study characteristics of the depicted variations are revealed and their respective benefits in terms of suitability for a given task are verified. Furthermore the results of the study give a perspective on which variations could match other use cases. In the following designs with $3 \leq n \leq 10$ segments are examined. Their trapezium's angles $\alpha = 360°/2n$ and the used representing symbols are given in Figure 8.

To enable a comparison a uniform task needs to be set and all variations have to be adjusted to fulfill it. Assuming that in the partially folded state the given pattern might be used as a temporary building the chosen task is to offer a useable space with a ground area of at least $12\,\mathrm{m}^2$ and a height of at least $2.2\,\mathrm{m}$. This task is fulfilled when a cylinder with the given area and height can be integrated inside as shown in Figure 7. The fact that a part of the area is not covered by a ceiling is neglected.

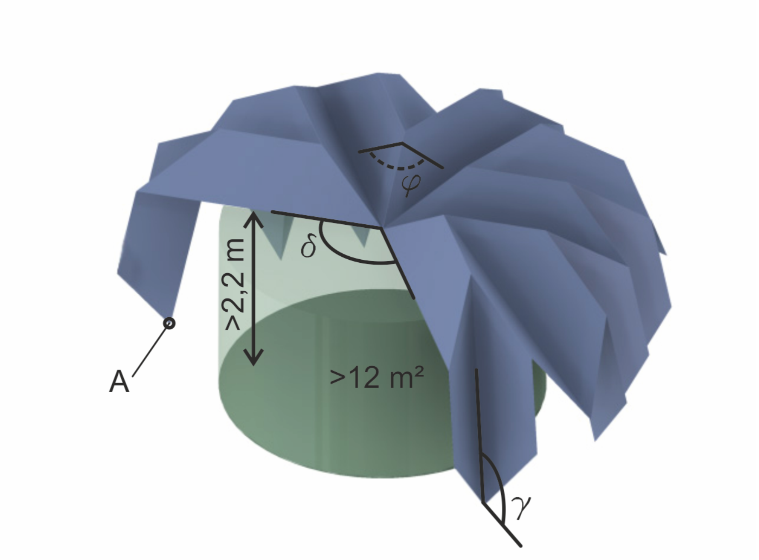


Figure 7: Given task

To meet the requirements there is only one free geometrical parameter (the edge length), but since the regarded state 'partially folded' is not exactly defined yet, the inner folding angle φ between the two plates in the first orbital row of a segment is still free to choose.

Therefore, as a second condition, the opening angle in the cut δ has to accommodate an integral number of additional segments with the same folding angle as the partially folded structure. Since this condition still allows several variations it is defined that the 3-segmented pattern has to accommodate one additional segment. With any higher number of segments in the structure also the number of segments for the gap is raised by one. In this way the opening angle can be expressed according to Eq. (1), whereby n is the number of segments again. From the set of constraints completed now the folding angle of the partially folded status can be determined as given in Eq. (2).

$$\delta = \frac{360°}{2n-2}(n-2) \tag{1}$$

$$\varphi = 2\sin^{-1}\left(\frac{\sin(\frac{360°-\delta}{2n})}{\sin\varphi}\right) \tag{2}$$

The wall angle γ, also depicted in Figure 7, is not part of the condition set but a measure for the wall's inclination. The value of γ can be derived from the folding angle φ by use of Eq. (3), which results from the spherical geometry as described in [5].

$$\gamma = \cos^{-1}\left[1 - \sin^2\alpha\left(\frac{\cos(\varphi) - K - K^2\cos(\varphi) + K^2}{1 - 2\cos(\varphi)K + K^2}\right)\right] \text{ with } K = \frac{1+\cos^2\alpha}{\sin^2\alpha}. \tag{3}$$

The chosen set of conditions leads to the patterns and structures depicted in Figure 8. The scaling necessary to enclose the cylinder given in Figure 7 was performed by manual adjustment of the edge lengths.

Regarding the angles, the first outcomes are that for patterns with high numbers of segments the values of trapezium's angles α as well as the values of folding angles φ, opening angles δ and wall angles γ hardly differ from pattern to pattern. In contrast the differences between patterns with small numbers of segments are distinct.

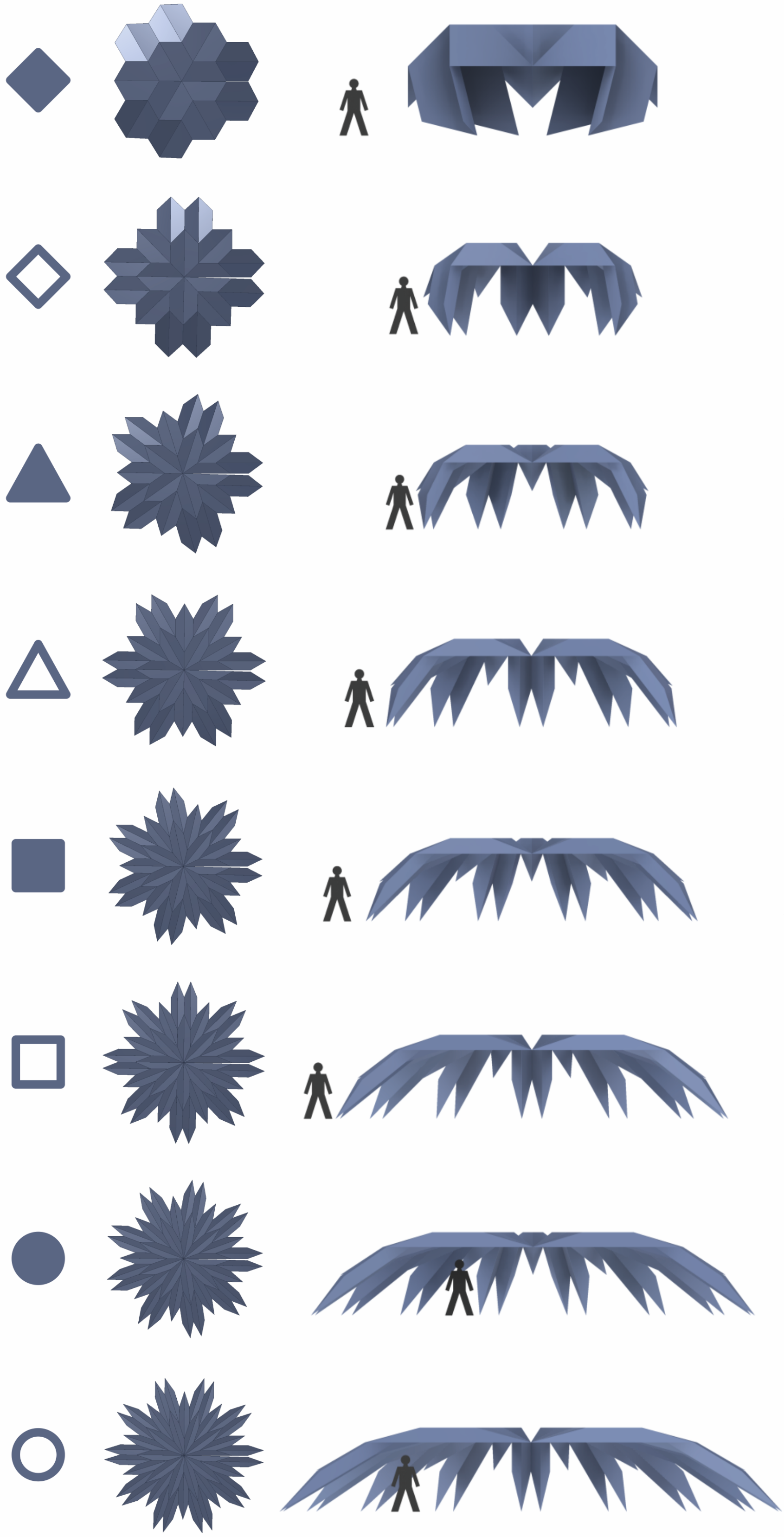
The interpretation of characteristics resulting from the determined angles requires a detailed consideration of the use case. Due to the chosen set of conditions for the partially folded state high segmented patterns leave a bigger sector uncovered. Depending on the use case this open space may be regarded as a benefit or as a disadvantage. Another aspect is the necessary motion range of hinges which depends on the folding angle φ and the initial state of the structure. Starting from the unfolded state (Figure 5 A) low segmented patterns allow one to use hinges with a smaller motion range. But if the folded state (Figure 5 C) is chosen as starting position high segmented patterns are beneficial for construction.

Another characteristic angle is γ, which is between the ground and the walls in the partially folded state. As shown in Figure 8, the walls of the 4-segmented pattern stop close to the vertical position. Patterns with more segments display larger angles. Accordingly they cover a bigger ground area and offer more space; however it is not useable due to the low height. The 3-segmented pattern shows an anomaly. Here the walls pass the right angle and do not stop before reaching about 70°. At first glance this only means that for the chosen conditions the area of the floor is smaller than the area of the ceiling which is not necessarily a problem. But there are also consequences to the folding sequence in use and to the structures' stability.

As already mentioned in the introduction, one key aspect for the choice of a pattern is its complexity. Under this term different characteristics are concentrated, including the numbers of elements and folds. The fewer elements and folds are necessary, the lower the complexity and the more beneficial the structure is with regard to assembly or failure risk. Since the patterns structurally differ only by the number of segments, the numbers of elements and folds increase equally spaced by $+8$ elements and $+12$ folds per additional segment.

Also the use of different element shapes or different joints would increase complexity, however the regarded patterns consist of only one type of elements and only one type of joints respectively. In a later design phase detailed engineering may require different joints, e.g., caused by adaption to loads or necessary motion ranges.

Another aspect is the geometry of elements required to fulfill the given task. Large-area, heavy elements complicate the assembly of a structure. Further they are sensitive to deformation and thereby may impede the folding motion. The altitudes of elements are related to the trapezias' angles which directly depend on the patterns' number of segments. While between the altitudes of the 3-segmented and the 4-segmented pattern there is a factor of about two, patterns with higher numbers of segments have little differences between their trapezias' angles so their altitudes are more alike. The increase of lengths is related to the partially folded state. Patterns with a high number of segments reach the required state with less change to the folding angles. That means that the coiling up of the outer elements is not so advanced and thus the elements need to be longer to achieve the required height of the cupola. The outlier of the three-segmented pattern is due to the fact that here the angle γ between ground and side wall is about 70° and thus the inclination of the wall requires longer elements than at the 4-segmented pattern with an angle of approx. 100°.



Number of Segments $n = 3$
Trapezium‘s Angle $\alpha = 60°$
Opening Angle $\delta = 90°$
Folding Angle $\varphi \approx 109{,}5°$
Wall Angle $\gamma \approx 70{,}5°$

Number of Segments $n = 4$
Trapezium‘s Angle $\alpha = 45°$
Opening Angle $\delta = 120°$
Folding Angle $\varphi = 90°$
Wall Angle $\gamma \approx 101{,}5°$

Number of Segments $n = 5$
Trapezium‘s Angle $\alpha = 36°$
Opening Angle $\delta = 135°$
Folding Angle $\varphi \approx 81{,}2°$
Wall Angle $\gamma \approx 118{,}3°$

Number of Segments $n = 6$
Trapezium‘s Angle $\alpha = 30°$
Opening Angle $\delta = 144°$
Folding Angle $\varphi \approx 76{,}3°$
Wall Angle $\gamma \approx 129{,}0°$

Number of Segments $n = 7$
Trapezium‘s Angle $\alpha \approx 25{,}7°$
Opening Angle $\delta = 150°$
Folding Angle $\varphi \approx 73{,}2°$
Wall Angle $\gamma \approx 136{,}4°$

Number of Segments $n = 8$
Trapezium‘s Angle $\alpha = 22{,}5°$
Opening Angle $\delta \approx 154{,}3°$
Folding Angle $\varphi \approx 71{,}1°$
Wall Angle $\gamma \approx 141{,}9°$

Number of Segments $n = 9$
Trapezium‘s Angle $\alpha = 20°$
Opening Angle $\delta = 157{,}5°$
Folding Angle $\varphi \approx 69{,}6°$
Wall Angle $\gamma \approx 146{,}1°$

Number of Segments $n = 10$
Trapezium‘s Angle $\alpha = 18°$
Opening Angle $\delta = 160°$
Folding Angle $\varphi \approx 68{,}4°$
Wall Angle $\gamma \approx 149{,}5°$

Figure 8: Variations of the structure

According to the results depicted in Figure 9, the 4-, 5- and 6-segmented patterns seem to offer manageable areas and suitable ratios of lengths and altitudes of elements. Detailed consideration may be reasonable, when there are concrete space requirements for transport or storage.

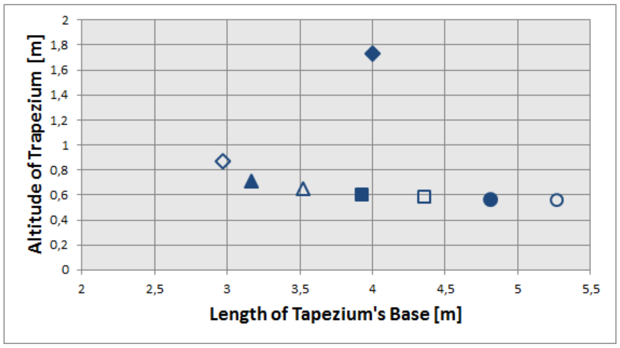


Figure 9: Geometry of elements

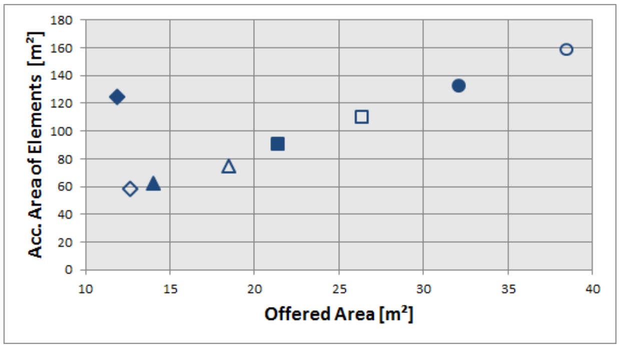


Figure 10: Material efficiency

Regarding sustainability a high material efficiency is aspired. In Figure 10 the offered areas of the structures and the accumulated areas of their elements are given, whereby the distinction between covered and open area is still neglected. Generally the result is that material usage is more efficient in structures with little numbers of segments, but due to the size of its elements the 3-segmented structure forms an exceptional case once again. So the most efficient variation is the 4-segmented one. Patterns with more segments require more material but due to their longer elements they also offer more useable space than required by the given task. Therefore they may be beneficial for use cases with higher space demands.

For the comparison of motion sequences the path of the outmost point of the structure, vertex 'A' depicted in Figure 7, is regarded. The results displayed in the following originate from simulations performed in a CAD-tool. Due to restrictions given by the software neither the unfolded nor the flat folded state can be reached exactly. However for the aspired application oriented design study they are of secondary importance because for constructional and kinematical reasons a realized mechanism is not to be operated in a dead point or in a fully collapsed position. Therefore the recorded courses start from the nearly unfolded state ($\varphi = 179°$) and continue until the simulation terminates close to the flat folded state ($\varphi \approx 0°$). The use-states are marked by the representing symbols of the patterns placed at the correspondent positions of 'A'.

Figure 11 shows the path of 'A' on a moving vertical plane containing 'A' itself and the structure's central vertical axis whereby the central vertex of the structure is fixed in the point of origin. Recording the motion path on this rotating plane reveals the effective radii instead of their projections. The particular path of the 3-segmented structure is caused by its bigger elements and its wall angle $\gamma < 90°$. The bigger elements affect the higher radius at the beginning of the motion while the wall angle makes a downswing below the final value of 2.2 m necessary.

The top view on the motion of 'A' is given in Figure 12. Here again the structure's central vertex is fixed in the point of origin, further the flat folded structure would aim in positive x-direction. Accordingly the upper parts of the paths represent the unfolding from the (nearly) flat folded state, while the lower parts of paths comply with the motion beginning from the unfolded state. The paths of patterns with five or more segments differ primarily by their scaling. The 3- and 4-segmented patterns also show different courses. Here the changes of radii are stronger when the motion starts from the unfolded state. That means that the space required to fold the structure into the use state is significantly bigger than in the use state itself. This outcome can be seen as a disadvantage of these two patterns, but if the structure can be set up by unfolding from the flat folded state this disadvantage carries no weight.

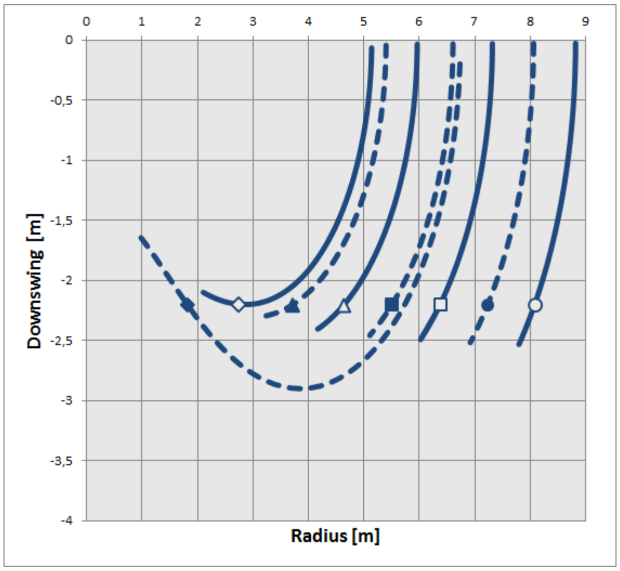


Figure 11: Path of ‘A’ in vertical plane

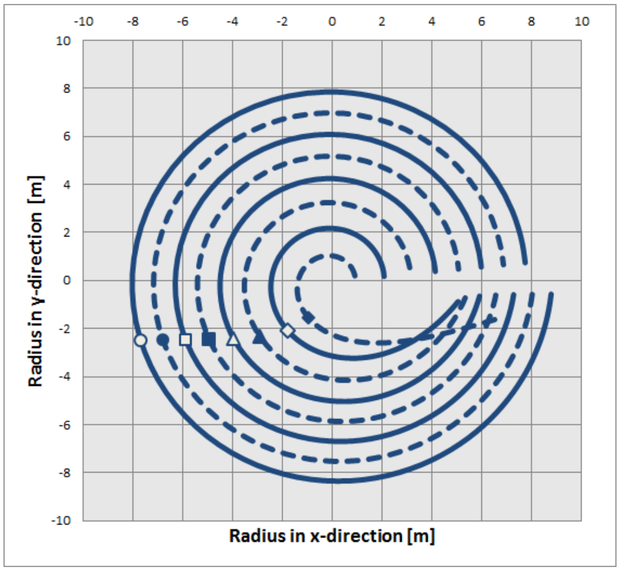


Figure 12: Path of ‘A’ in horizontal plane

Summing up, the patterns with fewer segments seem to be more suitable for the chosen task. A perhaps unexpected exception is the poor result of the 3-segmented pattern. For this structure the required ratio of height and span width is unfavorable. In use cases requiring a covered area in the unfolded state this structure could show its benefits like low complexity or high compressibility. Patterns with higher numbers of elements offer a wider span and should be taken into account for applications where a bigger useable space is required.

4. Application studies

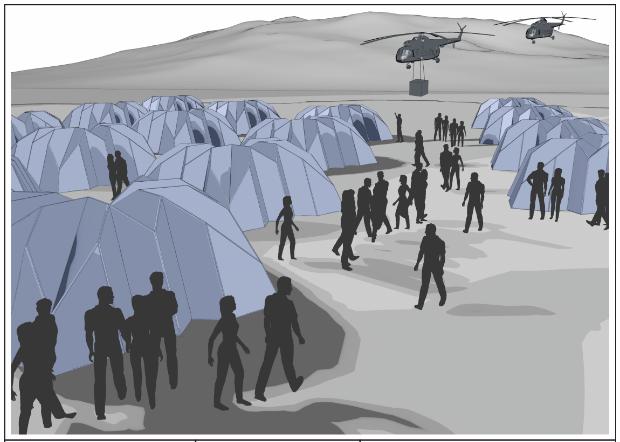
Miura-Ori based cupola forming folding patterns allow a grand variety of possible applications. The field ranges from small-scaled pavilions to large-scaled deployable roofs, from mobile to immobile structures and from using the unfolded state to using a folded state. The chosen examples give a selected overview of the various options these folding patterns can cope with. Furthermore it is indicated which of the states introduced in Figure 5 are relevant for each example.

Figure 13 shows the pattern applied as a temporarily used small pavilion, for example as kiosk on a festival or in a park. This application shows the cupola forming foldable structure with its purest characteristics. Out of a flat folded plate structure, it forms a roof structure in its functionality comparable to simple tents, but with much greater impression compared to an attractive design object. Although this structure is heavier and probably more expensive than competitive products it is more robust and on this account it can fulfill more tasks.

In the next example the structure is utilized as an emergency shelter (Figure 14). The dimensions are comparable to the pavilion, but this time demands to functionality are higher. As a shelter has to provide protection against weathering and protection of privacy, it has to be a closed space that the basic patterns do not offer; extra elements have to be added.

unfolded	✓	starting state
partially folded	✓	use state
flat folded	(poss.)	(transport / storage)
alternatively folded	(poss.)	(transport / storage)

Figure 13: Pavilion



unfolded	--	--
partially folded	✓	use state
flat folded	✓	starting state
alternatively folded	(poss.)	(transport / storage)

Figure 14: Shelter

The folded state is now the first option for transport and storage. Even more advantageous would be using the alternatively folded state (Figure 5 D) because of less stacked layers and therefore optimized space-saving.

The challenges are different when the cupola forming pattern is used in a larger scale as visualized in Figure 15. As an immobile but deployable roof structure for a stage, it can fulfill its function as weather protection and supporting structure for technical elements during an event. Afterwards it is foldable to a space-saving configuration. In any case it works as an interesting landmark. As shown in Figure 15 the structure also can be rotated by 90°. The structural behavior might be more challenging, but the big advantage of using it this way is that no extra closing elements are necessary.

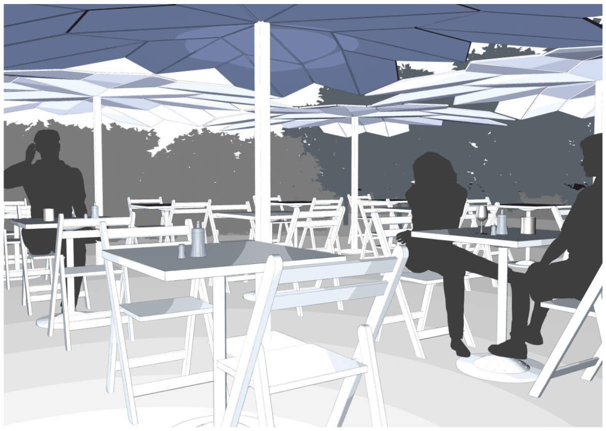
Figure 16 shows the foldable structure as a sunshade element. This works assembled horizontally as well as vertically or in other angles. Since the unfolded state is the use state, large wing-spreads are not possible due to little effective depths. Hence this is an application idea for innovative small-scaled architectural elements. If not in use, the folded structure is optimal for saving space. As this structure also consists of only one module it represents a promising alternative to conventional systems.

The four shown examples are meant to inspire, the whole field of application possibilities is much more versatile.

If an application of the cupola forming Miura-Ori-based structure or any other deployable folded plate structure is to be designed and realized, certain specific requirements have to be fulfilled. Even if these vary from project to project, the following ones are elementary and at the same time the most challenging ones: storage, transport and assembly concept, concept for repetitive motion, rain, and weather protection as well as safety arrangements.

unfolded	--	--
partially folded	✓	use state
flat folded	✓	non-use state
alternatively folded	--	--

Figure 15: Deployable roof



unfolded	✓	use state
partially folded	✓	interstage
flat folded	✓	non-use state
alternatively folded	(poss.)	(transport / storage)

Figure 16: Sunshade

5. Conclusions

The use of deployable folded plate structures provides innovative solutions for technical tasks. Thereby the folding pattern Miura-Ori as the most reduced transformable folding pattern forms the starting point for the investigation of application possibilities in architecture and engineering. The cupola forming Miura-Ori based foldable structure which is analyzed in this paper shows once more the geometric diversity and variability of the folding pattern Miura-Ori. By varying the angles of the trapezia arranged in a staggered manner numerous shapes and thus a wide range of possible applications are generated.

Based on the requirements for a small pavilion the suitability of some variations of the pattern is analyzed. The most surprising conclusion is the poor performance of the simplest case, the 3-segmented pattern. Apart from that most adequate solutions ensue from patterns with low numbers of segments. They show benefits concerning geometrical, kinematical and constructional aspects like smaller and easier to handle elements, lower installation and motion spaces as well as lower complexities. Although generally the simplest modular solutions should be aspired this is only one singular aspect and the preferable structure always depends on particular needs. For other applications, e.g., if a wider span is required, more differentiated patterns could be expedient. So that further design studies are necessary in any case.

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